Homework Set 3

Read sections 2.2, 2.4, 3.1, 3.2.

Section 2.2

1. (BH) page 80, exercise 26

2. (BH) Let

\[
A = \begin{pmatrix}
1 & -2 & 0 & 2 \\
2 & -3 & -1 & 5 \\
1 & 3 & 2 & 5 \\
1 & 1 & 0 & 2
\end{pmatrix}.
\]

(a) Find matrices \(B\) and \(C\) in row echelon form that are row equivalent to \(A\).
(b) Find a matrix \(D\) in reduced row echelon form that is row equivalent to \(A\).

3. (MP) Solve the linear system with the given augmented matrix, or show it has no solution.

(a): \[
\begin{pmatrix}
1 & 2 & 3 & 1 & | & 8 \\
1 & 3 & 0 & 1 & | & 7 \\
1 & 0 & 2 & 1 & | & 3
\end{pmatrix}
\]

(b): \[
\begin{pmatrix}
1 & 1 & 3 & -3 & | & 0 \\
0 & 2 & 1 & -3 & | & 3 \\
1 & 0 & 2 & -1 & | & -1
\end{pmatrix}
\]

4. (BH) Consider the following system:

\[
\begin{align*}
2x_1 + ax_2 &= 3, \\
x_1 + x_2 &= b.
\end{align*}
\]

(a) Write (3.1) as an augmented matrix.
(b) Write your answer to (a) in reduced row echelon form.
(c) For which value(s) of \(a\) and \(b\) will (3.1) have
   (i) more than one solution?
   (ii) exactly one solution?
   (iii) no solution?
Section 2.4

5. (BH) page 114, number 8

6. page 115, number 16. Do part (a) with Mathematica and the rest by hand.

7. (BH) You are given three alloys of the following composition:
   Alloy A: 5 parts (by weight) gold, 2 silver, 1 lead
   Alloy B: 2 parts gold, 5 silver, 1 lead
   Alloy C: 3 parts gold, 1 silver, 4 lead
   (a) How much of each metal is in one ounce of Alloy A?
   Suppose we want to make 27 ounces of a new alloy containing equal quantities (by weight) of gold, silver, and lead.
   (b) Write the system of equations needed to solve for the amounts of A, B, and C necessary to make this new alloy.
   (c) Solve the system.

8. Suppose that the following facts are true:
   (1) Of the number of people $x_s$ who start a year living on the East Coast, 80% stay in and 20% move out during the course of that year.
   (2) Of the number of people $y_s$ who start a year living off the East Coast, 90% stay out and 10% move in during the course of that year.
   (3) The number of births and deaths cancel one another out (so you don’t have to worry about them).
   Let $x_e$ and $y_e$ be the number of people living on and off the East Coast, respectively, at the end of the year.
   (a) (BH) Write down equations needed to solve for $x_e$ and $y_e$ as functions of $x_s$ and $y_s$.
   The system in (a) is an example of a Markov process.
   (b) (MP) If $y_e = 100$ million and $x_e = 75$ million, find $x_s$ and $y_s$.
   (c) (BH) If $x_s = x_e$ and $y_s = y_e$, find the ratio $y_e/x_e$.
   In the case given by (c), $(x_s, y_s)$ is called an eigenvector of the linear system. We will study this concept in great detail later.

Sections 3.1/3.2

9. (BH) Let $A \in \mathcal{R}^{m \times n}$, $x, y \in \mathcal{R}^n$. Using the definition of matrix-vector multiplication in terms of the entries, show that $A(2x - y) = 2Ax - Ay$. 
10. (BH) Let $A \in \mathbb{R}^{n \times n}$, $z \in \mathbb{R}^n$. Consider the equation $Az = \lambda z$.

(a) Using the definition of matrix-vector multiplication in terms of the entries, show that this equation is equivalent to $(A - \lambda I)z = 0$.

Now consider the special case where

$$A = \begin{pmatrix} 4 & -3 \\ 2 & -3 \end{pmatrix}.$$

(b) Show that in order for $z \neq 0$, $\lambda$ must be one of two special values, and calculate those values.

These special values of $\lambda$ are called *eigenvalues*, and they will become very important later on.