Sine and Cosine Series

We derived in class that the Fourier cosine series for the function

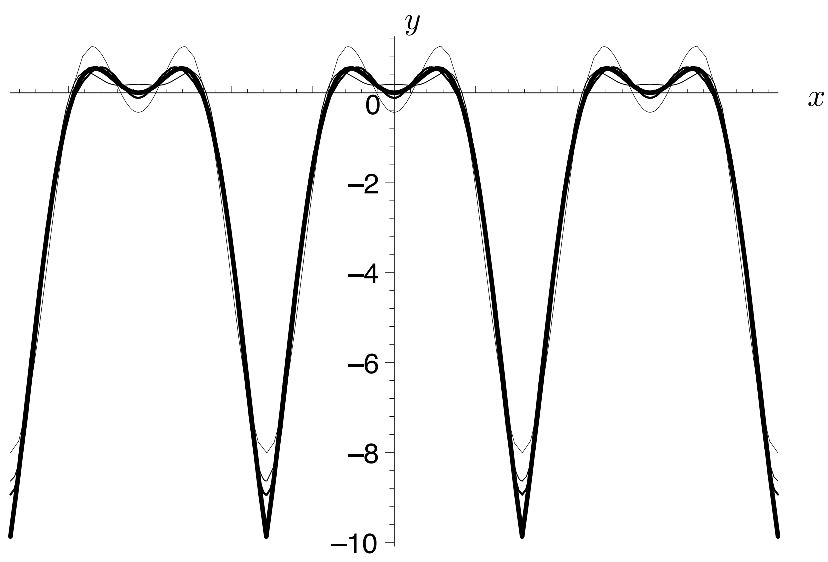
$$f(x) = x^2 \cos x, \qquad x \in (-\pi, \pi)$$

is given by

$$f(x) = -2 + \left(\frac{1}{2} + \frac{\pi^2}{3}\right)\cos x + 2\sum_{n=2}^{\infty}\left[\frac{(-1)^{n+1}}{(n+1)^2} + \frac{(-1)^{n-1}}{(n-1)^2}\right]\cos nx.$$

Note that $f(x)$ is smooth in its domain, but f' is discontinuous at $x = \pm\pi$.

Below are plotted the extension of the function f to the domain $x \in (-3\pi, 3\pi)$ (thickest line), as well as the Fourier series taking the first three, four, and five terms in the sum.

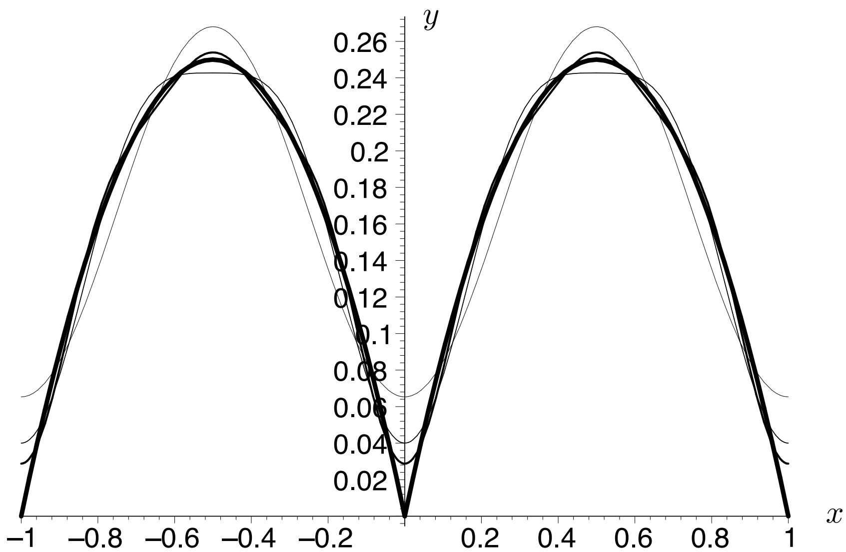


In increasing order of thickness: Fourier series keeping the first three, four, and five terms in the series, as well as $f(x)$ *vs.* x.

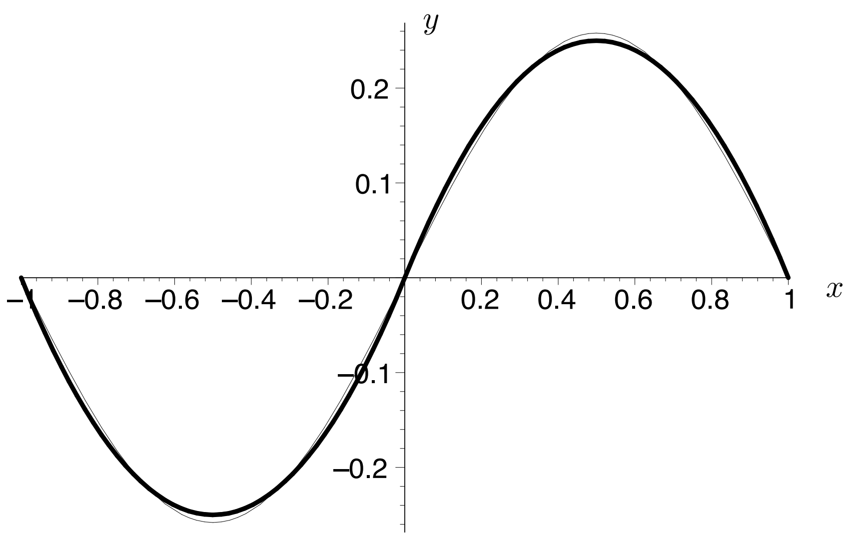
The Fourier cosine series f_c and sine series f_s for the even and odd extensions of the function $f(x) = x(1-x)$, $x \in [0, \pi]$ to the region $x \in [-\pi, \pi]$ are given by

$$f_c(x) = \frac{1}{6} - \frac{2}{\pi^2} \sum_{n=2}^{\infty} \frac{(-1)^n + 1}{n^2} \cos n\pi x, \qquad f_s(x) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \sin n\pi x.$$

Below are plotted the even extension of the function f to the domain $x \in [-1, 1]$ (thickest line), as well as f_c taking the first several terms in the sum. Note that since the extension has a jump in f', we see that the terms decay slowly. In contrast, the odd extension of $f(x)$ (also shown below) is smoother, so the Fourier coefficients decay more quickly.



In increasing order of thickness: f_c keeping the first two, three, and four nonzero terms in the series, as well as the even extension of $f(x)$ *vs.* x.



In increasing order of thickness: f_o keeping the first one, two, and three nonzero terms in the series, as well as the odd extension of $f(x)$ *vs.* x.