MATH 342-010 — Differential Equations with Linear Algebra II
Prof. D. A. Edwards — Due: Oct. 24, 2007

Homework Set 6

Read Edwards and Penney, sections 8.3 and 8.4.

Section E8.3(a)

1. Find two linearly independent solutions for the following Euler equations:
   (a)
   $$x^2y'' - 2xy' + 2y = 0.$$
   (b)
   $$x^2y'' - 2y = 0.$$
   (c)
   $$x^2y'' - xy' + y = 0.$$
2. Consider the equation
   $$x^2y'' + xy' - \epsilon^2 y = 0. \tag{6.1}$$
   (a) Find two linearly independent solutions $x^{\lambda_\pm}$ of (6.1) for $\epsilon \neq 0$.
   (b) Explain why
   $$y_*(x) = \frac{x^{\lambda_+} - x^{\lambda_-}}{\lambda_+ - \lambda_-}$$
   also solves (6.1) for $\epsilon \neq 0$.
   (c) Find two linearly independent solutions of (6.1) for $\epsilon = 0$.
   (d) Calculate
   $$\lim_{\epsilon \to 0} y_*(x)$$
   and compare it to your answer for (c).

Section E8.3(b)

3. Sometimes one would also like to characterize the "point" at $x = \infty$. We do this by letting $\xi = 1/x$ in our equation, for example

$$P(x)y'' + Q(x)y' + R(x)y = 0. \tag{6.2a}$$

to obtain the following equation in ξ:

$$\frac{d^2Y}{d\xi^2} + p(\xi)\frac{dY}{d\xi} + q(\xi)Y = 0, \qquad Y(\xi) = y(x). \tag{6.2b}$$

(a) What conditions must hold on (6.2b) for $x = \infty$ to be considered an ordinary point of (6.2a)? A regular singular point of (6.2a)?

(b) Make the actual substitution $\xi = 1/x$ in (6.2a), and show that $x = \infty$ is an ordinary point if is an ordinary point if

$$\frac{1}{P(1/\xi)}\left[\frac{2P(1/\xi)}{\xi} - \frac{Q(1/\xi)}{\xi^2}\right] \text{ and } \frac{R(1/\xi)}{\xi^4 P(1/\xi)} \tag{6.3}$$

are analytic at $\xi = 0$, and $x = \infty$ is a regular singular point if the expressions in (6.3) are not analytic at $\xi = 0$, but

$$\frac{\xi}{P(1/\xi)}\left[2P(1/\xi) - \frac{Q(1/\xi)}{\xi}\right] \text{ and } \frac{R(1/\xi)}{\xi^2 P(1/\xi)}$$

are.

4. Find and classify all *finite* singular points of the following equations:

(a) $(2x - x^3)y'' - y' - 6xy = 0$

(b) $\cos^2 xy'' + (x - \pi/2)y' + y = 0$

5. For the equation

$$2x(1-x)y'' + (1+x)y' - y = 0 \tag{6.4}$$

find and classify *all* singular points. (Include the point at infinity using your answer to #3.)

6. For equation (6.4), calculate a series for the general solution near $x = 0$. What is the radius of convergence of this series?

7. Consider the differential equation

$$x^2y'' + (2x^2 - 1)y = 0.$$

Find the indicial equation, and describe the behavior of the two linearly independent solutions near $x = 0$.

Section E8.4

8. Consider the differential equation

$$x^3y'' + \alpha xy' + \beta y = 0. \tag{6.5}$$

   (a) Show that $x = 0$ is an irregular singular point.

   (b) Show that the indicial equation for (6.5) is linear, and hence there is only one solution of Frobenius form.

   (c) Show that if β/α is an integer greater than -2, the formal series solution terminates. Show that if β/α does not have these special values, the series does not converge.

9. Consider the differential equation

$$xy'' + (x + m + 1)y' + \left(k + 1 + \frac{m}{2}\right) y = 0.$$

   Find all values of m and k for which *two* linearly independent solutions of the form

$$y = x^\lambda \sum_{n=0}^{\infty} a_n x^n \tag{6.6}$$

   exist. (*Hint: When two solutions of the form (6.6) do* not *exist, what does that imply about the roots* λ *of the indicial equation?*)

10. Consider the differential equation

$$xy'' + y' - 4xy = 0$$

   Show (by calculating the series, *not* by direct substitution) that the first several terms in the series for the two linearly independent solutions about $x = 0$ are given by

$$y_1(x) = 1 + x^2, \tag{6.7a}$$

$$y_2(x) = (1 + x^2)\log x + 1 + 0 \cdot x + 0 \cdot x^2. \tag{6.7b}$$

   (*Hint: Ignore any terms that are proportional to* $x^3 \log x$ *in your calculation of* y_2.)