

## Beats

In class we considered the solution of the system

$$\ddot{u} + k^2 u = F_0 \cos \omega t, \quad u(0) = 0, \quad \dot{u}(0) = 0,$$

and found the answer to be

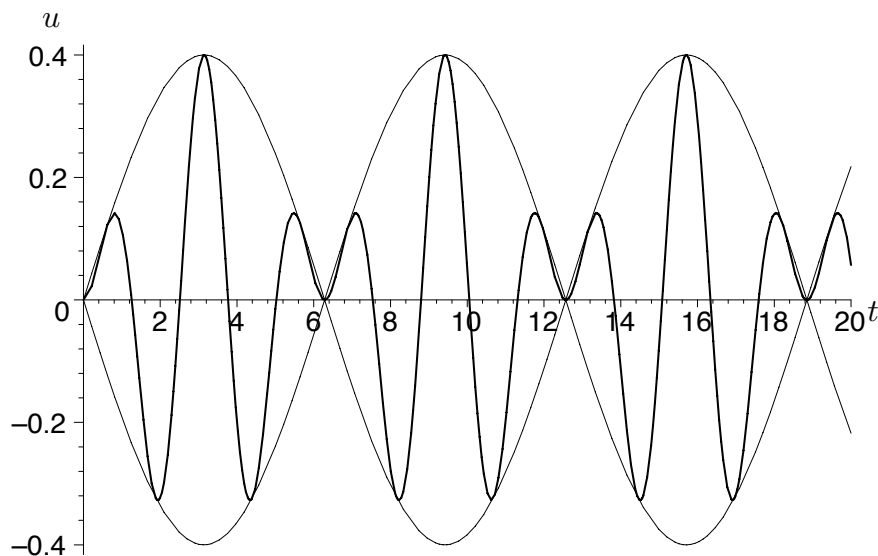
$$u = \frac{2F_0}{k^2 - \omega^2} \sin\left(\frac{(k - \omega)t}{2}\right) \sin\left(\frac{(k + \omega)t}{2}\right), \quad k \neq \omega.$$

We consider the first two factors to be a time-varying amplitude (or envelope).

If  $k = \omega$ , the envelope grows linearly and we have

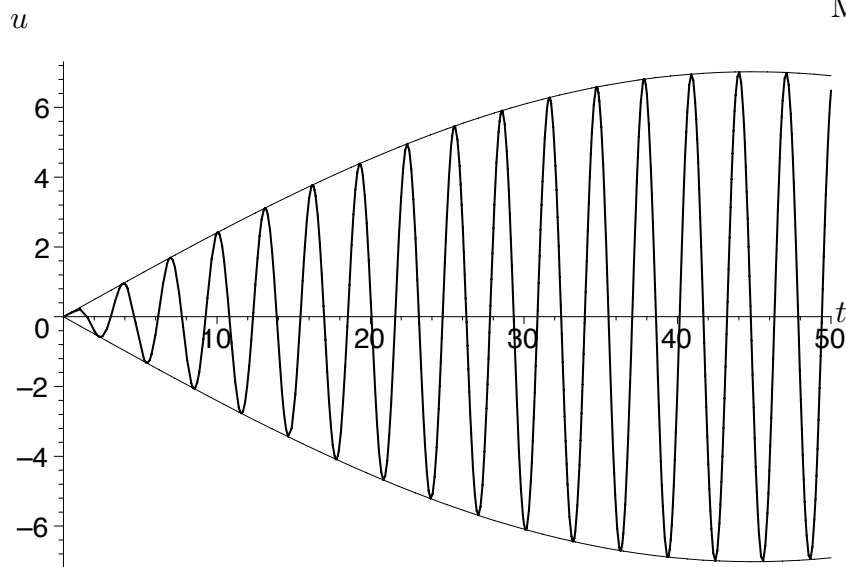
$$u = \frac{F_0 t}{2k} \sin kt.$$

Now take  $k = 2$  and  $F_0 = 1$ . Here are some graphs of the solution for various values of  $\omega$ .



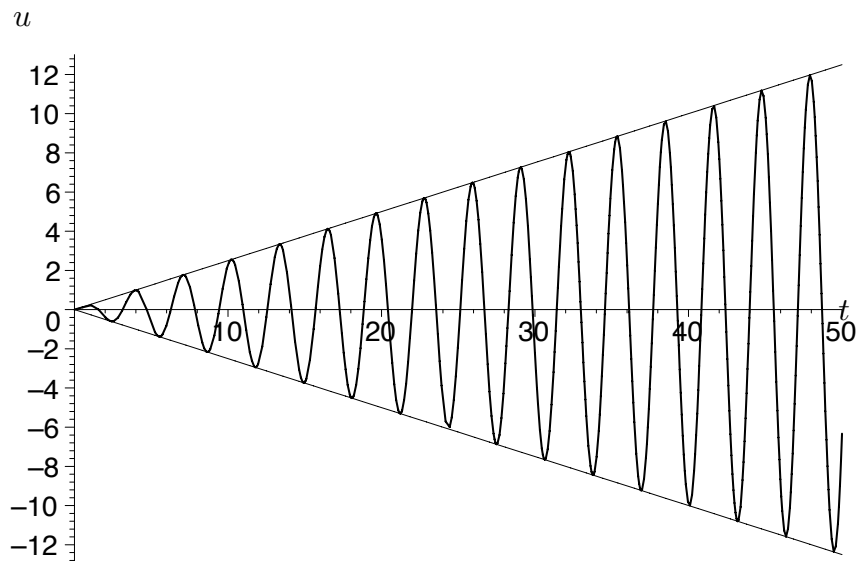
$u(t)$  vs.  $t$  for  $\omega = 3$ . Thin line: envelope. Thick line: solution.

In this case the forcing frequency is far from the natural frequency  $k = 2$ , so we have languid oscillations within the beats.



$u(t)$  vs.  $t$  for  $\omega = 2.07$ . Thin line: envelope. Thick line: solution.

As the forcing frequency nears the natural frequency  $k = 2$ , we see much more rapid oscillations within the beats, but note that the envelope eventually reaches a maximum and will return to zero.



$u(t)$  vs.  $t$  for  $\omega = 2$ . Thin line: envelope. Thick line: solution.

If the forcing frequency is the same as the natural frequency, the envelope increases linearly and so do the amplitude of the oscillations.