Enumeration of Orthogonal Buekenhout Unitals

R. D. Baker, G. L. Ebert[*], and K. L. Wantz

August 25, 2009

Abstract

In this paper we develop general techniques for enumerating orthogonal Buekenhout unitals embedded in two-dimensional translation planes. We then apply these techniques in the regular nearfield planes, the odd-order Hall planes, and the odd-order flag-transitive affine planes. Stabilizers of the resulting unitals also are computed.

1 Introduction

A *unital* is any 2–$(n^3+1, n+1, 1)$ design, for some integer $n > 2$. The classical example is a Hermitian curve in the square order Desarguesian plane $PG(2, q^2)$, where we take $n = q$ and the blocks become the intersections of the Hermitian curve with its "secant" lines. This example is often called the *classical unital.*

If we restrict to the family of translation planes which are two-dimensional over their kernels, and hence arise from line spreads of $\Sigma = PG(3, q)$, there are some general techniques for constructing unitals embedded in such planes. These were developed by Buekenhout [9], and use the Bruck-Bose [7, 8] representation. Namely, let $\mathcal{S}$ be a spread in $\Sigma = PG(3, q)$, and embed Σ as a hyperplane at infinity in $\bar{\Sigma} = PG(4, q)$. Then the two-dimensional (affine) translation plane of order q^2 corresponding to $\mathcal{S}$ is the incidence structure whose points are the points of $\bar{\Sigma} \setminus \Sigma$, whose lines are the planes of $\bar{\Sigma}$ which

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[*] Research partially supported by NSA grant H98230-09-1-0074

meet the hyperplane Σ in a line of the spread $\mathcal{S}$, and whose incidence is inherited from $\bar{\Sigma}$. This affine plane is completed to a projective plane $\pi(\mathcal{S})$ by adding the spread lines of $\mathcal{S}$ as the points at infinity. The plane $\pi(\mathcal{S})$ is Desarguesian (that is, isomorphic to $PG(2, q^2)$) if and only if the spread $\mathcal{S}$ is regular (see [8]).

Buekenhout gave two configurations in this Bruck-Bose model that represent unitals in the associated two-dimensional translation plane. One configuration is a nonsingular (parabolic) quadric in $\bar{\Sigma}$ that meets Σ in a regulus of the spread $\mathcal{S}$. In this case the resulting unital, which we call a *nonsingular Buekenhout unital*, meets the line at infinity in $q+1$ points. This construction is valid only for those two-dimensional translation planes whose associated spread contains at least one regulus. In [5] such unitals are carefully studied and completely enumerated in the regular nearfield planes and the Hall planes of odd order.

The other Buekenhout configuration is an ovoidal cone of $\bar{\Sigma}$ (that is, the point cone over some 3-dimensional ovoid) that meets Σ in a line of $\mathcal{S}$. The resulting unital, which we call an *ovoidal Buekenhout unital*, meets the line at infinity in one point. If the ovoidal cone is an orthogonal cone (with an elliptic quadric as base), then we call the unital an *orthogonal Buekenhout unital*. In this paper we analyze the orthogonal Buekenhout unitals embedded in odd-order regular nearfield planes, Hall planes, and flag-transitive affine planes.

It should be noted that the only nonsingular Buekenhout unital embedded in the Desarguesian plane $PG(2, q^2)$ is the classical unital (see [6]), and the orthogonal Buekenhout unitals in $PG(2, q^2)$ have been enumerated in [3] and [12], where the full stabilizers of these unitals are also computed. No previous attempts, other than those mentioned above, have been made at enumerating Buekenhout unitals embedded in various families of two-dimensional translation planes.

2 General Results

Using the Bruck-Bose model, we let $(X_0, X_1, X_2, X_3, X_4)$ denote homogeneous coordinates for $\bar{\Sigma} = PG(4, q)$, where the hyperplane at infinity, $\Sigma \cong PG(3, q)$, has equation $X_0 = 0$ and contains the given spread $\mathcal{S}$. Points are always represented by row vectors, and we freely identify actions on $\pi(\mathcal{S})$ with the associated action on $\bar{\Sigma}$. Throughout this paper $q = p^n$ will be an odd prime power, and we always denote a primitive element of $\mathrm{GF}(q)$ by ω. For any matrix (or row vector) A, we let A^T denote the transpose of A.

We represent $Aut(\pi(\mathcal{S}))$ as in [5]. Briefly, any automorphism of $\pi(\mathcal{S})$ is represented by a field automorphism applied to the $\bar{\Sigma}$-coordinates followed by right multiplication of a 5×5 "normalized" nonsingular matrix $\bar{M}$ over $\mathrm{GF}(q)$, where the first column of $\bar{M}$ may be assumed to be $(1, 0, 0, 0, 0)^T$ and the 4×4 lower right submatrix M induces a collineation of Σ which stabilizes the spread $\mathcal{S}$. The four components of the first row of $\bar{M}$, other than the first component, are arbitrary elements of $\mathrm{GF}(q)$, indicating the q^4 translations of $\pi(\mathcal{S})$. If these four components are all zero, then $\bar{M}$ represents an element in the *linear translation complement* of $\pi(\mathcal{S})$. In any case, one sees that $|Aut(\pi(\mathcal{S}))| = q^4(q-1)|Aut(\mathcal{S})|$. It is important to note that replacing the lower right submatix M by kM, for some $k \in \mathrm{GF}(q)^* = \mathrm{GF}(q) \setminus \{0\}$ with $k \neq 1$, produces a new collineation in $Aut(\pi(\mathcal{S}))$ even though M and kM induce the same collineation of Σ.

Consider an orthogonal Buekenhout unital embedded in $\pi(\mathcal{S})$, and thus represented by some ovoidal cone in $\bar{\Sigma}$ that meets Σ in a spread line of $\mathcal{S}$, necessarily one of the generators of the cone. We refer to this line as the *generator spread line* of the cone. In particular, the vertex of the cone lies on this spread line. Moreover, since q is odd, the ovoidal cone is necessarily an orthogonal cone. We begin by showing that there is a convenient collection of certain orthogonal cones which are representative, up to equivalence, of the set of all orthogonal cones with a given vertex on a given generator spread line. This equivalence is obtained by using only the translation subgroup, and hence is valid in any (two-dimensional) translation plane.

Theorem 2.1. *Let q be any odd prime power, and let $\mathcal{S}$ be any spread of $\Sigma = \mathrm{PG}(3, q)$. Let V and P be any two distinct points on some spread line, and let $\mathcal{C}$ be an orthogonal cone of $\bar{\Sigma} = \mathrm{PG}(4, q)$ with vertex V that meets Σ in the line PV. Let $\mathcal{B} = \{\mathbf{b_0}, \mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}, \mathbf{b_4}\}$ be a basis for the underlying vector space of $\bar{\Sigma}$, chosen such that $\mathbf{b_3}$ represents V, $\mathbf{b_4}$ represents P, and Span$\{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}, \mathbf{b_4}\}$ represents Σ. Then $\mathcal{C}$ is translation-equivalent to an orthogonal cone whose associated quadratic form has Gram matrix with* *respect to $\mathcal{B}$ given by*

$$\bar{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 2a & b & 0 & 0 \\ 0 & b & 2c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{1}$$

for some $a, b, c \in \mathrm{GF}(q)$ with $b^2 - 4ac \in \not\square_q$.

Proof. Consider some quadratic form representing $\mathcal{C}$, and let $\perp$ denote the orthogonal complement with respect to the (degenerate) symmetric bilinear form associated with this quadratic form. Since V is the vertex of $\mathcal{C}$, all $\mathbf{b_3}$-row and $\mathbf{b_3}$-column entries of the Gram matrix are zero. Since $P \in \mathcal{C}$, the $(\mathbf{b_4}, \mathbf{b_4})$-entry is also zero. Let Σ' be some hyperplane which contains a base of the cone, and suppose Σ' meets PV in the point P'. Then $\pi = \Sigma \cap \Sigma'$ is a plane of Σ' meeting this base elliptic quadric only in P', and thus π is the tangent plane to this elliptic quadric at P'. Since Σ contains this tangent plane π as well as the point $V \notin \pi$, we see that $\Sigma = (P')^\perp$. Now both the vertex V and P' are orthogonal to all points of Σ, and hence $P \in \Sigma^\perp$ as $P \in P'V'$. This shows that the Gram matrix with respect to $\mathcal{B}$ of the quadratic form for $\mathcal{C}$ looks like

$$A = \begin{bmatrix} s_0 & s_1 & s_2 & 0 & s_4 \\ s_1 & 2a & b & 0 & 0 \\ s_2 & b & 2c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ s_4 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Moreover, since none of the points of *Span*$\{\mathbf{b_1}, \mathbf{b_2}\}$ lie in $\mathcal{C}$, necessarily $f(X_1, X_2) = aX_1^2 + bX_1X_2 + cX_2^2$ is an irreducible binary quadratic form over $\mathrm{GF}(q)$, and hence $a, b, c \in \mathrm{GF}(q)$ satisfy $b^2 - 4ac \in \not\square_q$. Moreover, $s_4 \neq 0$ since $P^\perp = \Sigma$ and $\langle \mathbf{b_0} \rangle \notin \Sigma$.

Now consider a translation $\tau_{\bar{M}}$ whose matrix representation with respect to the basis $\mathcal{B}$ is of the form

$$\bar{M} = \begin{bmatrix} 1 & t_1 & t_2 & 0 & t_4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The image $\mathcal{C}'$ of $\mathcal{C}$ under $\tau_{\bar{M}}$ has Gram matrix with respect to $\mathcal{B}$ equal to $\bar{M}^{-1}A(\bar{M}^{-1})^T$. In particular, if we let $t_1 = \frac{-2cs_1+bs_2}{b^2-4ac}$, $t_2 = \frac{bs_1-2cs_2}{b^2-4ac}$ and $t_4 = \frac{s_0(b^2-4ac)+2(cs_1^2-bs_1s_2+as_2^2)}{2s_4(b^2-4ac)}$, then the Gram matrix of $\mathcal{C}'$ with respect to $\mathcal{B}$ is

$$A' = \begin{bmatrix} 0 & 0 & 0 & 0 & s_4 \\ 0 & 2a & b & 0 & 0 \\ 0 & b & 2c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ s_4 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since we may replace the form by any nonzero scalar multiple of it, we may instead use the Gram matrix $\bar{A} = -s_4^{-1}A'$, which completes the proof. □

When dealing with orthogonal Buekenhout unitals, we always assume that the underlying vector space for $\bar{\Sigma}$ has a basis $\mathcal{B}$ as described in Theorem 2.1, and we let $\mathcal{C}_{a,b,c}$ denote the orthogonal cone of $\bar{\Sigma}$ whose Gram matrix respect to $\mathcal{B}$ is the matrix given in the statement of this theorem. Thus, once a vertex and its generator spread line are given, we have a family $\Omega = \{\mathcal{C}_{a,b,c} \colon a, b, c \in \mathrm{GF}(q) \mid b^2 - 4ac \in \not\square_q\}$ of representative orthogonal cones from which we may sort out equivalence. It is important to note that all cones in this family contain the affine "origin" $P_0 = \langle \mathbf{b_0} \rangle$.

Sorting out equivalences will become easier if we can identify a natural subgroup of $Aut(\pi(\mathcal{S}))$ that acts on the orthogonal cones in this collection Ω. For instance, suppose that in addition to the generator spread line PV, the stabilizer H in $Aut(\pi(\mathcal{S}))$ of the vertex V fixes another spread line ℓ. In the notation of Theorem 2.1, we may then select $\mathbf{b_1}, \mathbf{b_2}$ as representatives of distinct points P_1, P_2 on this spread line ℓ, and let H_0 be the subgroup of H whose translations are only in the "direction" of the vertex $V = \langle \mathbf{b_3} \rangle$. That is, the 5×5 matrix representation $\bar{M}$ with respect to $\mathcal{B}$ for any linear collineation in H_0 will have its first row of the form $(1, 0, 0, t, 0)$ for some $t \in \mathrm{GF}(q)$, and will have its lower right 4×4 submatrix M block diagonal with 2×2 blocks. Moreover, the $(\mathbf{b_3}, \mathbf{b_4})$-entry of $\bar{M}$ will be zero, and the $(\mathbf{b_4}, \mathbf{b_4})$-entry of $\bar{M}$ will be nonzero. Straightforward matrix computations then show that H_0 acts on the set Ω. In fact, we can say a bit more.

Corollary 2.2. *Suppose that the stabilizer H in $Aut(\pi(\mathcal{S}))$ of the vertex V fixes some spread line ℓ in addition to the generator spread line PV. Let H_0 denote the subgroup of H with all translations in the direction of V as defined above. If $\tau \in Aut(\pi(\mathcal{S}))$ is any collineation that maps some orthogonal cone*

$C_{a,b,c} \in \Omega$ to another (possibly the same) cone $C_{a',b',c'} \in \Omega$, then $\tau \in H_0$. In particular, the stabilizer of $C_{a,b,c}$ in $Aut(\pi(\mathcal{S}))$ is a subgroup of H_0.

Proof. Let τ be any collineation of $Aut(\pi(\mathcal{S}))$ which maps $\mathcal{C}_{a,b,c}$ to $C_{a',b',c'}$. Since τ must fix V (and $\mathcal{S}$), necessarily $\tau \in H$. Choose a basis $\mathcal{B}$ for the underlying vector space as described in the above paragraph, and let P_0' be the image of $P_0 = \langle \mathbf{b_0} \rangle \in \mathcal{C}_{a,b,c}$ under τ. Then, since $P_0' \notin \Sigma$, we know P_0' has homogeneous coordinates $\mathbf{b_0} + t_1\mathbf{b_1} + t_2\mathbf{b_2} + t_3\mathbf{b_3} + t_4\mathbf{b_4}$, for some $t_1, t_2, t_3, t_4 \in \mathrm{GF}(q)$. From the quadratic form for $\mathcal{C}_{a,b,c}$, we know that P_0 is orthogonal to all points of the line P_1P_2. Moreover, by the assumption on H and the choice of the basis $\mathcal{B}$, this line is fixed by τ. Since the orthogonal complement of P_0 (respectively, P_0') consists of all generators and tangent lines of $\mathcal{C}_{a,b,c}$ (respectively, $\mathcal{C}_{a',b',c'}$) through this point, we see that P_0' must also be orthogonal to P_1P_2. Straightforward matrix computations using the Gram matrix for $\mathcal{C}_{a',b',c'}$ now show that $t_1 = 0 = t_2$, since the determinant for the resulting linear system of equations is $4ac - b^2 \neq 0$. As $P_0' \in \mathcal{C}_{a',b',c'}$, it follows that $f'(t_1, t_2) - t_4 = 0$, where $f'(X_1, X_2) = a'X_1^2 + b'X_1X_2 + c'X_2^2$, and hence $t_4 = 0$. We conclude P_0' has homogeneous coordinates $\mathbf{b_0} + t_3\mathbf{b_3}$, and therefore $\tau \in H_0$ by definition. $\square$

We now apply these general results to some well-known families of two-dimensional translation planes.

3 Regular Nearfield Planes

We model two-dimensional regular nearfield planes exactly as in [5]. That is, we utilize the quadratic extension field $\mathrm{GF}(q^2)$ of $\mathrm{GF}(q)$ whenever convenient. Let β denote a primitive element of $\mathrm{GF}(q^2)$, so that $\omega = \beta^{q+1}$ is a primitive element of $\mathrm{GF}(q)$. Using the ordered basis $\{1, \epsilon = \beta^{\frac{q+1}{2}}\}$ for $\mathrm{GF}(q^2)$ over $\mathrm{GF}(q)$, we identify ordered pairs (s_1, s_2) from $\mathrm{GF}(q)$ with the element $s = s_1 + s_2\epsilon$ in $\mathrm{GF}(q^2)$.

Let $\mathcal{S}_0 = \{\ell_\infty\} \cup \{\ell_s \mid s \in \mathrm{GF}(q^2)\}$, where $\ell_\infty = \langle (0,0,0,1), (0,0,1,0) \rangle$ and

$$\ell_s = \ell_{(s_1,s_2)} = \langle (1, 0, s_1, s_2), (0, 1, \omega s_2, s_1) \rangle.$$

Then $\mathcal{S}_0$ is a regular spread of $PG(3, q)$, and $\{\mathcal{R}_t \mid t \in \mathrm{GF}(q)^*\}$ is a linear set of $q - 1$ mutually disjoint reguli in $\mathcal{S}_0$ with carriers ℓ_0 and ℓ_∞, where

$\mathcal{R}_t = \{\ell_s \mid s^{q+1} = t\}$. Straightforward computations show that $\mathcal{R}_t^{\rm opp} = \{m_s \mid s^{q+1} = t\}$ is the opposite regulus to $\mathcal{R}_t$, where

$$m_s = m_{(s_1,s_2)} = \langle (1, 0, s_1, s_2), (0, 1, -\omega s_2, -s_1) \rangle.$$

Hence

$$\begin{aligned} \mathcal{S} &= \Big(\mathcal{S}_0 \setminus \cup_{t \in \square_q} \mathcal{R}_t\Big) \cup \Big(\cup_{t \in \square_q} \mathcal{R}_t^{\rm opp}\Big) \\ &= \{\ell_0, \ell_\infty\} \cup \{\ell_s \mid s^{q+1} \in \not\square_q\} \cup \{m_s \mid s^{q+1} \in \square_q\} \end{aligned}$$

is a regular nearfield spread of Σ (see [13]), unique up to equivalence.

We now describe our model for $Aut(\mathcal{S})$. For $q = p^n \geq 5$ it is well known (see [1], for instance) that $|Aut(\mathcal{S})| = 4n(q^2 - 1)(q + 1)$. It should be noted that for $q = 3$, the regular nearfield spread and the Hall spread of PG$(3, 3)$ are equivalent. The automorphism group of this spread has order 1920, and it acts transitively on the 10 lines and 10 reguli of the spread. The corresponding translation plane is called the *exceptional* nearfield plane. We thus assume for the rest of this section that $q \geq 5$.

If σ is an automorphism of GF(q^2), say given by $x \mapsto x^r$, where $r = p^i$ for some $0 < i \leq 2n$, we let ψ_σ (or ψ_r for emphasis) denote the map which takes (X_1, X_2, X_3, X_4) to $(X_1^r, \omega^{\frac{r-1}{2}} X_2^r, X_3^r, \omega^{\frac{r-1}{2}} X_4^r)$. Then ψ_σ is an automorphism of $\mathcal{S}$ and

$$\left.\begin{array}{lcl} \psi_\sigma & : & \ell_s \mapsto \ell_{s^\sigma}, \\ \psi_\sigma & : & m_s \mapsto m_{s^\sigma}, \end{array}\right\}$$

for any choice of σ.

For each $e = e_1 + e_2\epsilon$, $f = f_1 + f_2\epsilon \in$ GF(q^2), with $ef \neq 0$, let $\phi_{e,f}$ denote the linear collineation of Σ induced by right multiplication (on row vectors) by the matrix

$$M_{e,f} = \begin{bmatrix} e_1 & e_2\delta_{e,f} & 0 & 0 \\ \omega e_2 & e_1\delta_{e,f} & 0 & 0 \\ 0 & 0 & f_1 & f_2 \\ 0 & 0 & \omega f_2 & f_1 \end{bmatrix},$$

where $\delta_{e,f}$ is the quadratic character of ef in GF(q^2); namely

$$\delta_{e,f} = \begin{cases} 1, & \text{if } ef \in \square_{q^2}, \\ -1, & \text{if } ef \in \not\square_{q^2}. \end{cases}$$

As shown in [5], if $ef \in \square_{q^2}$, then

$$\left.\begin{array}{lcl} \phi_{e,f} & : & \ell_s \mapsto \ell_{\frac{f}{e}s}, \\ \phi_{e,f} & : & m_s \mapsto m_{\frac{f}{e^q}s}, \end{array}\right\} \tag{2}$$

while if $ef \in \not\square_{q^2}$, then

$$\left.\begin{array}{lcl} \phi_{e,f} & : & \ell_s \mapsto m_{\frac{f}{e}s}, \\ \phi_{e,f} & : & m_s \mapsto \ell_{\frac{f}{e^q}s}. \end{array}\right\} \tag{3}$$

Moreover, one sees that $\psi_\sigma \phi_{e,f} \psi_\sigma^{-1} = \phi_{e^{\sigma^{-1}}, f^{\sigma^{-1}}}$, and thus

$$K = \{\psi_\sigma \phi_{e,f} \mid \sigma \in Aut(\mathrm{GF}(q^2)) \text{ and } e, f \in \mathrm{GF}(q^2)^*\}$$

is a subgroup of $Aut(\mathcal{S})$ which stabilizes ℓ_∞ (and ℓ_0). The description of $Aut(\mathcal{S})$ is completed by defining ν to be the linear collineation of Σ induced by the mapping

$$(X_1, X_2, X_3, X_4) \mapsto (X_3, X_4, X_1, X_2).$$

Then

$$\left.\begin{array}{lcl} \nu & : & \ell_s \mapsto \ell_{\frac{1}{s}}, \\ \nu & : & m_s \mapsto m_{\frac{1}{s^q}} \end{array}\right\}$$

for any $s \neq 0$, and ν interchanges ℓ_0 and ℓ_∞. One checks that $\nu\psi_\sigma\phi_{e,f}\nu = \psi_\sigma\phi_{f,e}$ when $ef \in \square_{q^2}$, and $\nu\psi_\sigma\phi_{e,f}\nu = \psi_{q\sigma}\phi_{f^q,e^q}$ when $ef \in \not\square_{q^2}$.

Elementary counting shows that $G = K \rtimes J$ is the full automorphism group of $\mathcal{S}$, where J is the cyclic subgroup generated by the involution ν. Moreover, the previously described group actions show that $Aut(\mathcal{S})$ has two orbits on the lines of $\mathcal{S}$, namely $\{\ell_0,\ \ell_\infty\}$ and $\mathcal{S} \setminus \{\ell_0,\ \ell_\infty\}$.

Next $Aut(\pi(\mathcal{S}))$ is obtained from $Aut(\mathcal{S})$ as described in Section 2. Field automorphisms are accounted for by

$$\bar{\psi}_r : (X_0, X_1, X_2, X_3, X_4) \mapsto (X_0^r, X_1^r, \omega^{\frac{r-1}{2}} X_2^r, X_3^r, \omega^{\frac{r-1}{2}} X_4^r),$$

which lifts the action of ψ_r on Σ. Defining linear collineations $\bar{\phi}_{e,f}$ and $\bar{\nu}$ in a similar fashion (bordering the associated matrices with a 1 in the $(1,1)$-position and 0's elsewhere in the first row and column), we see that every element of the translation complement can be written uniquely as $\bar{\psi}_r\bar{\phi}_{e,f}$ or as $\bar{\psi}_r\bar{\phi}_{e,f}\bar{\nu}$, for some $e, f \in \mathrm{GF}(q^2)^*$ and some integer $r = p^i$ with $0 < i \leq 2n$. In particular, the translation complement of $\pi(\mathcal{S})$ has order $4n(q^2-1)^2$ since $\bar{\phi}_{e,f} \neq \bar{\phi}_{e',f'}$ whenever $(e,f) \neq (e',f')$.

We now turn our attention to the enumeration of the orthogonal Buekenhout unitals embedded in the regular nearfield plane $\pi(\mathcal{S})$, and the computation of their stabilizers. As discussed in Section 2, these unitals are represented by orthogonal cones in $\bar{\Sigma}$ that meet Σ in a spread line of $\mathcal{S}$, necessarily one of the generators of the cone. Thus the vertex of the cone lies on this generator spread line.

Recall that $Aut(\mathcal{S})$ has two orbits on its lines, $\{\bar{\ell}_0,\ \bar{\ell}_\infty\}$ and $\mathcal{S} \setminus \{\bar{\ell}_0,\ \bar{\ell}_\infty\}$. Moreover, the subgroup $\{\bar{\phi}_{e,e} \colon e \in \mathrm{GF}(q^2)^*\}$ of order $q+1$ in $Aut(\pi(\mathcal{S}))$ acts (sharply) transitively on the points of each line $\bar{\ell}_s$, while the subgroup $\{\bar{\phi}_{e,e^q} \colon e \in \mathrm{GF}(q^2)^*\}$ of order $q+1$ in $Aut(\pi(\mathcal{S}))$ acts (sharply) transitively on the points of each line m_s. Thus it suffices only to consider cones $\mathcal{C}$ with vertex $V = (0,0,0,1,0)$ and cones $\mathcal{C}'$ with vertex $V' = (0,1,0,1,0)$.

First we consider orthogonal cones with vertex $V = (0,0,0,1,0)$, thus having $\bar{\ell}_\infty = \langle (0,0,0,1,0), (0,0,0,0,1) \rangle$ as the uniquely determined generator spread line. Choose $\mathbf{b_0} = (1,0,0,0,0)$, $\mathbf{b_1} = (0,1,0,0,0)$, $\mathbf{b_2} = (0,0,1,0,0)$, $\mathbf{b_3} = (0,0,0,1,0)$, $\mathbf{b_4} = (0,0,0,0,1)$ and apply Theorem 2.1 with basis $\mathcal{B} = \{\mathbf{b_0}, \mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}, \mathbf{b_4}\}$. Since $\mathcal{B}$ happens to be the standard basis used in describing $\pi(\mathcal{S})$ above, we may restrict our attention to the orthogonal cones in

$$\Omega = \{\mathcal{C}_{a,b,c} \colon a,b,c \in \mathrm{GF}(q) \mid b^2 - 4ac \in \not\square_q\}$$

as explicitly given in the statement of Theorem 2.1, where the associated Gram matrix of $\mathcal{C}_{a,b,c}$ is $\bar{A}_{a,b,c}$, which is labelled as $\bar{A}$ in (1). A semi-linear collineation $(\sigma, \bar{M})$ maps the orthogonal cone $\mathcal{C}_{\bar{a},\bar{b},\bar{c}}$ to the orthogonal cone $\mathcal{C}_{a,b,c}$ (where possibly $\mathcal{C}_{a,b,c} = \mathcal{C}_{\bar{a},\bar{b},\bar{c}}$) if and only if $\bar{M}\bar{A}_{a,b,c}\bar{M}^T$ is a nonzero scalar multiple of $(\bar{A}_{\bar{a},\bar{b},\bar{c}})^\sigma$.

Since $\{\bar{\ell}_0,\ \bar{\ell}_\infty\}$ is an orbit of $G = Aut(\pi(\mathcal{S}))$, the group $H = Stab_G(V)$ leaves invariant $\bar{\ell}_0 = \langle \mathbf{b_1}, \mathbf{b_2} \rangle$, and thus Corollary 2.2 applies. The subgroup H_0 of H consists of semi-linear collineations $\bar{\psi}_r\tau_{\bar{M}}$, where $r = p^i$ for some $0 < i \leq 2n$ and $\bar{M}$ has the form

$$\bar{M} = \begin{bmatrix} 1 & 0 & 0 & t & 0 \\ 0 & e_1 & e_2\delta_{e,f} & 0 & 0 \\ 0 & \omega e_2 & e_1\delta_{e,f} & 0 & 0 \\ 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & f \end{bmatrix}, \tag{4}$$

for some $t \in \mathrm{GF}(q)$, $f \in \mathrm{GF}(q)^*$, and $e \in \mathrm{GF}(q^2)^*$. In particular, we see that $|H_0| = 2nq(q-1)(q^2-1)$.

By Corollary 2.2, determining the equivalence classes of cones in Ω is accomplished by the action of H_0 on Ω. Let $\bar{M}_\sigma$ denote the companion matrix of the semi-linear map $\bar{\psi}_r$, namely $Diag\{1, 1, \omega^{\frac{r-1}{2}}, 1, \omega^{\frac{r-1}{2}}\}$. The collineation $\bar{\psi}_r \tau_{\bar{M}}$ maps $\mathcal{C}_{\bar{a},\bar{b},\bar{c}}$ to $\mathcal{C}_{a,b,c}$, if and only if $\bar{M}_\sigma \bar{M} \bar{A}_{a,b,c} \bar{M}^T \bar{M}_\sigma^T = k(\bar{A}_{\bar{a},\bar{b},\bar{c}})^\sigma$ for some $k \in \mathrm{GF}(q)^*$. Straightforward computations show that this occurs if and only if the following conditions are satisfied:

$$k = \omega^{\frac{r-1}{2}} f \tag{5}$$

$$k\bar{a}^r = ae_1^2 + b\delta e_1 e_2 + ce_2^2 \tag{6}$$

$$k\bar{b}^r = \omega^{\frac{r-1}{2}} \big(b\delta(e_1^2 + \omega e_2^2) + 2(a\omega + c)e_1 e_2 \big) \tag{7}$$

$$k\bar{c}^r = \omega^{r-1}(a\omega^2 e_2^2 + b\omega\delta e_1 e_2 + ce_1^2). \tag{8}$$

The absence of t in this system indicates that t can take on any value in $\mathrm{GF}(q)$.

In all the following manipulations it is important to remember that $b^2 - 4ac \in \not\square_q$ and hence neither a nor c can be zero. Now the system given by (5)–(8) is equivalent to the system obtained from $(8) - \omega^r \cdot (6)$, $(8) + \omega^r \cdot (6)$, and (7) by replacing k by its value from (5). This system is

$$f(\bar{c} - \bar{a}\omega)^r = \omega^{\frac{r-1}{2}}(c - a\omega)(e_1^2 - \omega e_2^2) \tag{9}$$

$$f(\bar{c} + \bar{a}\omega)^r = \omega^{\frac{r-1}{2}} \big[(c + a\omega)(e_1^2 + \omega e_2^2) + 2b\delta\omega e_1 e_2 \big] \tag{10}$$

$$f\bar{b}^r = b\delta(e_1^2 + \omega e_2^2) + 2(c + a\omega)e_1 e_2 \tag{11}$$

We can combine the two equations (11) and (10) over $\mathrm{GF}(q)$ into the single equation $\epsilon^r \cdot (11) + (10)$ over $\mathrm{GF}(q^2)$. Using $\epsilon^r = \omega^{\frac{r-1}{2}}\epsilon$, we get

$$f\big[\bar{b}\epsilon + (\bar{c} + \bar{a}\omega)\big]^r = \omega^{\frac{r-1}{2}} \big[b\delta\epsilon + (c + a\omega) \big] \big[(e_1^2 + \omega e_2^2) + 2e_1 e_2 \epsilon \big]$$

Noting that $(e_1^2 + \omega e_2^2) + 2e_1 e_2 \epsilon = (e_1 + e_2\epsilon)^2$ this simplifies to

$$f\big[\bar{b}\epsilon + (\bar{c} + \bar{a}\omega)\big]^r = \omega^{\frac{r-1}{2}} \big[b\delta\epsilon + (c + a\omega) \big] (e_1 + e_2\epsilon)^2. \tag{12}$$

Thus we see that $\mathcal{C}_{\bar{a},\bar{b},\bar{c}}$ and $\mathcal{C}_{a,b,c}$ are H_0-equivalent if and only if equations (9) and (12) hold.

We now obtain a quantity that serves as a type of "discriminant" of $\mathcal{C}_{a,b,c}$. The equations obtained from $(9)^2$ and $\omega^r \cdot (11)^2 - (10)^2$ are

$$f^2 \big[(\bar{c} - \bar{a}\omega)^2 \big]^r = \omega^{r-1}(c - a\omega)^2 (e_1^2 - \omega e_2^2)^2 \tag{13}$$

$$f^2 \big[\bar{b}^2\omega - (\bar{c} + \bar{a}\omega)^2 \big]^r = \omega^{r-1} \big[b^2\omega - (c + a\omega)^2 \big] (e_1^2 - \omega e_2^2)^2, \tag{14}$$

respectively. Notice that $\bar{b}^2\omega-(\bar{c}+\bar{a}\omega)^2=0$ if and only if $b^2\omega-(c+a\omega)^2=0$. If the sides of (14) are nonzero, we divide (13) by (14) to obtain

$$\left[\frac{(\bar{c}-\bar{a}\omega)^2}{\bar{b}^2\omega-(\bar{c}+\bar{a}\omega)^2}\right]^\sigma=\frac{(c-a\omega)^2}{b^2\omega-(c+a\omega)^2}.$$

Define $\Delta_{a,b,c}=(c-a\omega)^2/\left[b^2\omega-(c+a\omega)^2\right]$, often calling it Δ for brevity. We conclude that the discriminant $\Delta_{a,b,c}$ of the image cone $\mathcal{C}_{a,b,c}$ under the collineation $\bar{\psi}_r\tau_{\bar{M}}$ is $\left(\Delta_{\bar{a},\bar{b},\bar{c}}\right)^\sigma$.

To assist in the computations that follow, we define

$$\alpha=b\epsilon+(c+a\omega) \tag{15}$$

and note that $\Delta=-(c-a\omega)^2/\alpha^{q+1}$. We similarly define $\bar{\alpha}=\bar{b}\epsilon+(\bar{c}+\bar{a}\omega)$ and $\tilde{\alpha}=b\delta\epsilon+(c+a\omega)$, where δ is 1 or -1 as ef is square or nonsquare in $\mathrm{GF}(q^2)$, to simplify Equation (12).

In the event that $\alpha=0$ (hence $b=0$ and $c=-a\omega$), we see that (10) and (11) are trivial, and (9) will have solutions for e,f given any value of r. If we use the convention $\Delta=\infty$ in this case, then $\Delta_{\bar{a},\bar{b},\bar{c}}=\infty$ as well. Therefore, for any $\mathcal{C}_{a,b,c},\mathcal{C}_{\bar{a},\bar{b},\bar{c}}\in\Omega$ (hence $\Delta_{a,b,c},\Delta_{\bar{a},\bar{b},\bar{c}}\in\mathrm{GF}(q)\cup\{\infty\}$), a necessary condition for $\mathcal{C}_{a,b,c}$ to be the image of $\mathcal{C}_{\bar{a},\bar{b},\bar{c}}$ is that $\Delta_{a,b,c}=\left(\Delta_{\bar{a},\bar{b},\bar{c}}\right)^\sigma$ where σ is an automorphism of $\mathrm{GF}(q^2)$.

Notice that the numerator and denominator of Δ cannot both be zero since $(c-a\omega)^2-\alpha^{q+1}=\omega(b^2-4ac)\in\Box_q$. Moreover, in the event that $c-a\omega\neq 0$ and $\alpha\neq 0$,

$$\Delta(\Delta+1)=\frac{(c-a\omega)^2\omega(b^2-4ac)}{\left[\omega b^2-(c+a\omega)^2\right]^2}\in\Box_q$$

Let $D=\{\Delta\in\mathrm{GF}(q)^*:\Delta(\Delta+1)\in\Box_q\}$. Consider the equivalence relation defined on D via $\Delta\sim\Delta'$ if and only if $\Delta'=\Delta^\sigma$ for some $\sigma\in Aut(\mathrm{GF}(q^2))$. Additionally consider the equivalence relation induced on $\Omega_D=\{\mathcal{C}_{a,b,c}\in\Omega:\Delta_{a,b,c}\in D\}$, which we also denote by $\sim$, whereby $\mathcal{C}_{a,b,c}\sim\mathcal{C}_{\bar{a},\bar{b},\bar{c}}$ if and only if $\Delta_{a,b,c}\sim\Delta_{\bar{a},\bar{b},\bar{c}}$. Finally, consider the equivalence relation on Ω whereby $\mathcal{C}_{a,b,c}$ and $\mathcal{C}_{\bar{a},\bar{b},\bar{c}}$ are said to be H_0-equivalent if and only if there is a collineation of H_0 mapping $\mathcal{C}_{a,b,c}$ to $\mathcal{C}_{\bar{a},\bar{b},\bar{c}}$. From the discussion above we see that if $\mathcal{C}_{a,b,c},\mathcal{C}_{\bar{a},\bar{b},\bar{c}}\in\Omega_D$ are H_0-equivalent, then $\Delta_{a,b,c}\sim\Delta_{\bar{a},\bar{b},\bar{c}}$ and $\mathcal{C}_{a,b,c}\sim\mathcal{C}_{\bar{a},\bar{b},\bar{c}}$.

Lemma 3.1. *Let D and Ω_D be defined as above.*

1. *For each $\Delta \in D$, there exists some $\mathcal{C}_{a,b,c} \in \Omega_D$ such that $\Delta_{a,b,c} = \Delta$.*
2. *If $\mathcal{C}_{a,b,c}, \mathcal{C}_{\bar{a},\bar{b},\bar{c}} \in \Omega_D$ such that $\Delta_{a,b,c} = \Delta_{\bar{a},\bar{b},\bar{c}}$, then $\mathcal{C}_{a,b,c}$ and $\mathcal{C}_{\bar{a},\bar{b},\bar{c}}$ are H_0-equivalent.*

Proof. Given $\Delta \in D$, let α be any of the $q+1$ solutions of $\alpha^{q+1}\Delta = -\omega^2$, which must exist as $\Delta, -\omega^2 \in \mathrm{GF}(q)^*$. By setting $\alpha = b\epsilon + (c + a\omega)$ as in (15) and "normalizing" by setting $c - a\omega = \omega$, a system is obtained to uniquely determine a triple (a, b, c). Notice that $\Delta_{a,b,c} = \frac{\omega^2}{-\alpha^{q+1}} = \Delta$ and $b^2 - 4ac = \omega\frac{\Delta(\Delta+1)}{\Delta^2} \in \not\square_q$, showing that there is a cone $\mathcal{C}_{a,b,c}$ with discriminant Δ and therefore proving the first statement. Indeed, we have shown that for a given $\Delta \in D$ there are precisely $q+1$ normalized cones according to the value of α, and any two of these values for α differ by a factor which is a $(q+1)^{st}$-root of unity.

Now consider any $\mathcal{C}_{a,b,c} \in \Omega_D$. By setting $r = q^2$, $f = 1$ and choosing e such that $e^{q+1} = \frac{\omega}{c - a\omega}$, it follows from (9) and (12) that $\mathcal{C}_{a,b,c}$ is H_0-equivalent to a normalized cone $\mathcal{C}_{\bar{a},\bar{b},\bar{c}}$ with the same discriminant; that is, $\Delta_{a,b,c} = \Delta_{\bar{a},\bar{b},\bar{c}}$ and $\bar{c} - \bar{a}\omega = \omega$.

To complete the argument it now suffices to show that any two of the $q+1$ normalized cones with a fixed value of Δ are H_0-equivalent. Thus we may assume that the triple (a, b, c) has been normalized, and observe as in the previous paragraph that this implies α satisfies $\alpha^{q+1}\Delta = -\omega^2$. We must now find e, f, r so that (9) and (12) imply Δ and $c - a\omega$ are fixed, while α cycles through the $q+1$ possible values.

From (9) we see that choosing $r = q$ and $f = -e^{q+1}$ implies that we preserve the normalization $c - a\omega = \omega$. We then look at (12) to determine the effect these choices make on α, which is $\bar{\alpha}^q = \frac{-e^2}{f}\tilde{\alpha} = \frac{1}{e^{q-1}}\tilde{\alpha}$. Hence by further choosing $e = \beta$, so that $s = e^{q-1}$ is a primitive $(q+1)^{st}$–root of unity and $e \in \not\square_{q^2}$, we obtain $\bar{\alpha}^q = \frac{1}{s}\alpha^q$ and thus $\bar{\alpha} = \frac{1}{s^q}\alpha$. Since $\frac{1}{s^q} = \frac{1}{s^{-1}}$ is also a primitive $(q+1)^{st}$–root of unity, this shows that all possible solutions for α belong to the same H_0-equivalence class.

□

Theorem 3.2. *Let $\mathcal{C}_{a,b,c}, \mathcal{C}_{\bar{a},\bar{b},\bar{c}} \in \Omega_D$. Then $\mathcal{C}_{a,b,c}$ and $\mathcal{C}_{\bar{a},\bar{b},\bar{c}}$ are H_0-equivalent if and only if $\mathcal{C}_{a,b,c} \sim \mathcal{C}_{\bar{a},\bar{b},\bar{c}}$.*

Proof. If $\mathcal{C}_{a,b,c} \sim \mathcal{C}_{\bar{a},\bar{b},\bar{c}}$, then $\Delta_{a,b,c} = \Delta^{\sigma}_{\bar{a},\bar{b},\bar{c}}$ for some $\sigma \in Aut(\mathrm{GF}(q^2))$. Now, the image of $\mathcal{C}_{\bar{a},\bar{b},\bar{c}}$ under $\bar{\psi}_\sigma$ is H_0-equivalent to $\mathcal{C}_{\bar{a},\bar{b},\bar{c}}$ and has discriminant

$\Delta^{\sigma}_{\bar{a},\bar{b},\bar{c}} = \Delta_{a,b,c}$. By Lemma 3.1, $\left(\mathcal{C}_{\bar{a},\bar{b},\bar{c}}\right)^{\bar{\psi}_\sigma}$ is therefore H_0-equivalent to $\mathcal{C}_{a,b,c}$. We conclude $\mathcal{C}_{a,b,c}$ and $\mathcal{C}_{\bar{a},\bar{b},\bar{c}}$ are H_0-equivalent. The converse was proved in the discussion prior to Lemma 3.1. $\square$

We now turn our attention to computing the stabilizer of $\mathcal{C}_{a,b,c}$ in H_0 for each $\mathcal{C}_{a,b,c} \in \Omega$. We have seen in the previous discussion that a necessary condition for a collineation $\bar{\psi}_r\tau_{\bar{M}}$ in H_0 to fix $\mathcal{C}_{a,b,c}$ is that $\Delta \in D \cup \{0, \infty\}$ must be fixed by $\sigma : x \mapsto x^r$. Cases will be sorted by $\Delta = \infty$, $\Delta = 0$, and $\Delta \in D$.

Lemma 3.3. *Let $q = p^n \geq 5$ be an odd prime power. Then, for any $a \in \mathrm{GF}(q)^*$, the stabilizer in H_0 of $\mathcal{C}_{a,0,-a\omega}$ has order $2nq(q^2-1)$.*

Proof. This is the case when $\alpha = 0$, whence $\Delta = \infty$. As was previously noted, the field automorphism σ is arbitrary for $\Delta = \infty$. This gives $2n$ collineations of the type $\bar{\psi}_r$, based on the dimension of $\mathrm{GF}(q^2)$ over the prime field $\mathrm{GF}(p)$. We count collineations $\bar{\psi}_r\tau_{\bar{M}}$ that fix $\mathcal{C}_{a,b,c}$ using conditions (9), (10) and (11) with $(\bar{a}, \bar{b}, \bar{c}) = (a, b, c) = (a, 0, -a\omega)$. In this case conditions (10) and (11) are trivial. Moreover, for any field automorphism σ and for any nonzero element $e = e_1 + e_2\epsilon$ of $\mathrm{GF}(q^2)$, we see that $f \in \mathrm{GF}(q)$ is uniquely determined by (9) since $c - a\omega \neq 0$. Recalling that $t \in \mathrm{GF}(q)$ is arbitrary, we conclude that the stabilizer of $\mathcal{C}_{a,0,-a\omega}$ has order $2nq(q^2-1)$. $\square$

Although we find it useful to take advantage of $\mathrm{GF}(q^2)$ arithmetic, it should be noted that we are dealing with $\mathrm{GF}(q)$-semi-linear transformations. In particular, the associated $\mathrm{GF}(q)$-field automorphism for $\bar{\psi}_r$ and $\bar{\psi}_{qr}$ is the same. The next case we consider has $\alpha \neq 0$ and $c - a\omega = 0$, hence $-\alpha^{q+1} = (c - a\omega)^2 - \alpha^{q+1} = \omega(b^2 - 4ac) \in \square_q$. This further implies that $\frac{\alpha}{\epsilon} \in \not\square_{q^2}$ since $\left(\frac{\alpha}{\epsilon}\right)^{q+1} = \frac{-\alpha^{q+1}}{\omega} \in \not\square_q$.

Lemma 3.4. *Let $q = p^n \geq 5$ be an odd prime power. Then, for any $a \in \mathrm{GF}(q)^*$, the stabilizer in H_0 of $\mathcal{C}_{a,b,a\omega}$ has order $4nq(q-1)$.*

Proof. This is the case when $c - a\omega = 0$. Thus $\Delta = 0$, which is fixed by every $\sigma \in Aut(\mathrm{GF}(q^2))$. As condition (9) is now trivial, we must count solutions for $\bar{\psi}_r\tau_{\bar{M}}$ to (12) where $(\bar{a}, \bar{b}, \bar{c}) = (a, b, c) = (a, b, a\omega)$. For a given f, each solution e of (12) is a square root of $f\alpha^r\tilde{\alpha}^{-1}\omega^{\frac{1-r}{2}}$. Recall $\tilde{\alpha} = b\delta\epsilon + (c + a\omega)$, so that $\tilde{\alpha}$ is either α or α^q, depending on whether δ is 1 or -1. But δ is the quadratic character of $ef \in \mathrm{GF}(q^2)$, and hence is the quadratic character of

e since $f \in \mathrm{GF}(q)^* \subseteq \square_{q^2}$. Since $\omega^{\frac{r-1}{2}} \in \mathrm{GF}(q)$, there exists $\gamma \in \mathrm{GF}(q^2)$ such that $\gamma^2 = \omega^{\frac{1-r}{2}}$. Indeed, we may assume γ is chosen so that $\alpha^{\frac{r-1}{2}}\gamma = (\alpha\epsilon^{-1})^{\frac{r-1}{2}}$.

If $\delta = 1$, the possible pairs (e, f) may be parametrized by $s \in \mathrm{GF}(q)^*$ as

$$f = s^2 \qquad e = (\alpha\epsilon^{-1})^{\frac{r-1}{2}} s \tag{16}$$

$$f = \omega s^2 \qquad e = (\alpha\epsilon^{-1})^{\frac{r-1}{2}} s\epsilon. \tag{17}$$

On the other hand, if $\delta = -1$, the possible pairs are

$$f = s^2 \qquad e = (\alpha\epsilon^{-1})^{\frac{r-1}{2}} \alpha^{\frac{1-q}{2}} s \tag{18}$$

$$f = \omega s^2 \qquad e = (\alpha\epsilon^{-1})^{\frac{r-1}{2}} \alpha^{\frac{1-q}{2}} s\epsilon. \tag{19}$$

Direct computations show that the $q-1$ pairs (e, f) in (16) are indeed solutions to (12) if and only if $(\alpha\epsilon^{-1})^{\frac{r-1}{2}} \in \square_{q^2}$. Moreover, the $q-1$ pairs (e, f) in (17), (18), and (19) are solutions to (12) if and only if $(\alpha\epsilon^{-1})^{\frac{r-1}{2}}\epsilon \in \square_{q^2}$, $(\alpha\epsilon^{-1})^{\frac{r-1}{2}}\alpha^{\frac{1-q}{2}} \in \not\square_{q^2}$, and $(\alpha\epsilon^{-1})^{\frac{r-1}{2}}\alpha^{\frac{1-q}{2}}\epsilon \in \not\square_{q^2}$, respectively.

Since $q = p^n$ is an odd prime power and $r = p^i$ for some $0 < i \leq 2n$, we certainly know that either $r \equiv 1 \pmod 4$ or $r \equiv 3 \pmod 4$. If $\epsilon \in \not\square_{q^2}$, straightforward computations show that for each possible r, one obtains $2(q-1)$ pairs (e, f) so that (12) is satisfied. Using the fact that $t \in \mathrm{GF}(q)$ is arbitrary, we see that the stabilizer of $\mathcal{C}_{a,b,a\omega}$ has order $2n \cdot q \cdot 2(q-1) = 4nq(q-1)$ in this case.

On the other hand, if $\epsilon \in \square_{q^2}$, then direct computations that there are no pairs (e, f) that satisfy (12) when $r \equiv 3 \pmod 4$ and $4(q-1)$ pairs (e, f) that satisfy (12) when $r \equiv 1 \pmod 4$. Since $\epsilon \in \square_{q^2}$, we know that $q \equiv 3 \pmod 4$ and thus necessarily $p \equiv 3 \pmod 4$. Hence there are n choices for $r \equiv 1 \pmod 4$ (i must be even), and we again obtain precisely $4nq(q-1)$ collineations in the stabilizer of $\mathcal{C}_{a,b,a\omega}$, proving the result. $\square$

The last case has both $\alpha \neq 0$ and $c - a\omega \neq 0$. As a piece of notation, we denote the smallest subfield of $\mathrm{GF}(q)$ containing the element x by $\mathbb{F}_p\langle x\rangle$.

Lemma 3.5. *Let $\mathcal{C}_{a,b,c} \in \Omega_D$, so that $\Delta_{a,b,c}(\Delta_{a,b,c} + 1) \in \square_q$. Then the stabilizer in H_0 of $\mathcal{C}_{a,b,c}$ has order $2n_0q(q-1)$, where $n_0 = \dim_F(\mathrm{GF}(q))$ and $F = \mathbb{F}_p\langle\Delta_{a,b,c}\rangle$.*

Proof. Since σ must fix Δ, we see that there are only $2n_0$ possibilities for r, where n_0 is defined as in the statement of the lemma. Moreover, σ must map

$(c-a\omega)/\alpha^{\frac{q+1}{2}}$ (a square root of $-\Delta$) to itself or its additive inverse. In other words, clearing denominators, we have

$$\alpha^{\frac{q+1}{2}}(c-a\omega)^r = u_r(c-a\omega)\big(\alpha^{\frac{q+1}{2}}\big)^r,$$

for some well-defined sign $u_r = \pm 1$. Raising this to the q^{th} power, we see

$$\alpha^{\frac{q^2+q}{2}}(c-a\omega)^{qr} = u_r(c-a\omega)^q\big(\alpha^{\frac{q+1}{2}}\big)^{qr},$$

and hence

$$\alpha^{\frac{q^2-1}{2}}\alpha^{\frac{q+1}{2}}(c-a\omega)^{qr} = u_r(c-a\omega)\big(\alpha^{\frac{q+1}{2}}\big)^{qr}.$$

Thus $u_{qr} = \alpha^{\frac{q^2-1}{2}}u_r$.

The enumeration proceeds as in the proof of Lemma 3.4, except we have condition (9) to consider as well as condition (12). We solve $\tilde{\alpha}\cdot(12)$ for $(\tilde{\alpha}e)^2$ and substitute in $\tilde{\alpha}^{q+1}\cdot(9)$ to obtain $\tilde{\alpha}^{q+1}f(c-a\omega)^r = \omega^{\frac{r-1}{2}}(c-a\omega)\big[f\alpha^r\tilde{\alpha}\omega^{\frac{1-r}{2}}\big]^{\frac{q+1}{2}}$, which simplifies to

$$\tilde{\alpha}^{\frac{q+1}{2}}(c-a\omega)^r = f^{\frac{q-1}{2}}(-1)^{\frac{r-1}{2}}(c-a\omega)\big(\alpha^{\frac{q+1}{2}}\big)^r.$$

Thus, if (16) or (17) provide solutions, we know that $\delta = 1$, implying $\tilde{\alpha} = \alpha$ and hence $u_r = f^{\frac{q-1}{2}}(-1)^{\frac{r-1}{2}}$. The situation when (18) or (19) provide solutions, where $\delta = -1$, is similar. When $\delta = -1$, we know that $\tilde{\alpha} = \alpha^q$, which implies $\tilde{\alpha}^{\frac{q+1}{2}} = \alpha^{\frac{q^2-1}{2}}\alpha^{\frac{q+1}{2}}$ and hence $u_r = f^{\frac{q-1}{2}}\alpha^{\frac{q^2-1}{2}}(-1)^{\frac{r-1}{2}}$. Direct computations show that the converse is true; that is, if one of these series of solutions to (12) satisfies the equation for u_r, then it also satisfies (9). We now split into four cases, according to the quadratic character of α and of ϵ.

First suppose $\alpha, \epsilon \in \square_{q^2}$. Neither (18) nor (19) provide any solutions as $ef \in \square_{q^2}$. Moreover, (16) are solutions when $u_r = (-1)^{\frac{r-1}{2}}$ and (17) are solutions when $u_r = -(-1)^{\frac{r-1}{2}}$. Thus for each of the $2n_0$ choices for r, the sign u_r must satisfy exactly one of these equations and hence precisely $q-1$ pairs (e, f) provide solutions to (12) and (9). As $t \in \mathrm{GF}(q)$ is arbitrary, we see the stabilizer of $\mathcal{C}_{a,b,c}$ has order $2n_0q(q-1)$ in this case.

Next suppose $\alpha \in \square_{q^2}$ and $\epsilon \in \not\square_{q^2}$. The pairs (e, f) in (16) are solutions when $r \equiv 1 \pmod 4$ and $u_r = (-1)^{\frac{r-1}{2}} = 1$. The pairs (e, f) in (17) are solutions when $r \equiv 3 \pmod 4$ and $u_r = -(-1)^{\frac{r-1}{2}} = 1$. The pairs (e, f) in (18) are solutions when $r \equiv 3 \pmod 4$ and $u_r = (-1)^{\frac{r-1}{2}} = -1$. The pairs

(e, f) in (19) are solutions when $r \equiv 1 \pmod 4$ and $u_r = -(-1)^{\frac{r-1}{2}} = -1$. Thus for any r and u_r exactly one series yields solutions. As t is arbitrary, there are again $2n_0q(q-1)$ collineations in the stabilizer.

Next assume $\alpha \in \not\square_{q^2}$ and $\epsilon \in \square_{q^2}$. Thus $q \equiv 3 \pmod 4$ and hence $p \equiv 3 \pmod 4$. In order for (16)–(19) to be solutions, it is necessary that $r \equiv 1 \pmod 4$ in all four series. Hence there are only n_0 possibilities for r in this case. Direct computations show that when $u_r = 1$, (16) and (19) provide solutions, and when $u_r = -1$, (17) and (18) provide solutions. Thus we get $2(q-1)$ pairs (e, f) for each such r. As t is arbitrary, there are again $n_0 \cdot 2q(q-1) = 2n_0q(q-1)$ collineations in the stabilizer.

Finally consider $\alpha, \epsilon \in \not\square_{q^2}$. Neither (17) nor (18) are solutions as $(\frac{\alpha}{\epsilon})^{\frac{r-1}{2}}\epsilon \in \not\square_{q^2}$ and $(\frac{\alpha}{\epsilon})^{\frac{r-1}{2}}\alpha^{\frac{1-q}{2}} \in \square_{q^2}$, respectively. Moreover, (16) and (19) are solutions when $u_r = (-1)^{\frac{r-1}{2}}$. Notice that since $\alpha \in \not\square_{q^2}$ we know $u_{qr} = -u_r$ on the one hand, while $(-1)^{\frac{qr-1}{2}} = (-1)^{\frac{r-1}{2}}$ as $q \equiv 1 \pmod 4$ on the other hand. So again we obtain $2(q-1)$ solutions each for half of the possible values of r. As t is arbitrary, in this final case the stabilizer of $\mathcal{C}_{a,b,c}$ once again has order $2n_0q(q-1)$, thus proving the result.

$\square$

We summarize the results of these lemmas in the following theorem.

Theorem 3.6. *Let $q = p^n \geq 5$ be an odd prime power and suppose $\mathcal{C}_{a,b,c}$ is an orthogonal cone in Ω. Let $U_{a,b,c}$ be the corresponding orthogonal Buekenhout unital in the regular nearfield plane of order q^2 which meets the line at infinity in a point of the short orbit. Then the stabilizer of $U_{a,b,c}$ has order*

1. *$2nq(q^2-1)$ when $(a, b, c) = (a, 0, -a\omega)$,*
2. *$4nq(q-1)$ when $(a, b, c) = (a, b, a\omega)$, and*
3. *$2n_0q(q-1)$ in all other cases,*

where $n_0 = \dim_F(\mathrm{GF}(q))$ and $F = \mathbb{F}_p\langle\Delta_{a,b,c}\rangle$.

Next we address the enumeration of such unitals. Rather than relying solely on the Orbit-Stabilizer Theorem, we take a somewhat indirect approach.

Theorem 3.7. *Let $q = p^n \geq 5$ be an odd prime power. The number of inequivalent orthogonal Beukenhout unitals in the regular nearfield plane $\pi(\mathcal{S})$*

of order q^2 *which meet the line at infinity in some point of the short orbit is the same as the number of Beukenhout-Metz unitals in* $PG(2, q^2)$*, which is explicitly determined in [3].*

Proof. We claim that the partition of Ω into H_0-equivalence classes is completely determined by $\Delta \in D \cup \{0, \infty\}$. Using $|H_0|$ and Lemma 3.3, the Orbit-Stabilizer Theorm implies that there is a unique H_0-equivalence class for $\Delta = \infty$ $(\alpha = 0)$. Similarly, there is a unique class when $\Delta = 0$, as shown by Lemma 3.4, since the index of the stabilizer of any such unital is the same as the number of triples $(a, b, a\omega)$ with $b^2 - 4\omega a^2 \in \not\square_q$, namely $\frac{q^2-1}{2}$.

For $\Delta \in D$, we know from Theorem 3.2 that H_0-equivalence classes of the unitals in Ω are induced by the equivalence classes of D under the relation $\sim$. The number of inequivalent unitals is then $N_0 + 2$, where N_0 is the number of equivalence classes of D under $\sim$. This number is determined in [3] using Mobiüs inversion, where it is further shown that $N_0 + 2$ is precisely the number of inequivalent Beukenhout-Metz unitals in $PG(2, q^2)$. $\square$

Remark 3.8. *When q is prime, there is a simple formula for the number of inequivalent unitals of the type described in Theorem 3.7, namely* $(q + 1)/2$*. For proper (odd) prime powers the formula given by Mobiüs inversion in [3] is not so simple. On the other hand, for a given value of q, it is easy to count the equivalence classes of D under* $\sim$ *since they are orbits of the group of field automorphisms acting on D. For* $q = 9$ *the number of such unitals in the regular nearfield plane of order* 81 *is* 4*, for* $q = 25$ *the number is* 9*, and for* $q = 27$ *the number is* 6*, for* $q = 81$ *the number is* 14*, and for* $q = 125$ *the number is* 23*.*

We now turn our attention to the cones with vertex $V' = (0, 1, 0, 1, 0)$ on the generator spread line $\bar{m}_1 = \bar{m}_{(1,0)}$. From our previous computations we know that $\bar{m}_1$ is fixed by $\bar{\nu}$ as well as $\bar{\psi}_r$, for any choice of r. However, $\bar{\phi}_{e,f}$ fixes $\bar{m}_1$ precisely when $\frac{f}{e^q} = 1$ (see (2) and (3)). Moreover, $\bar{\nu}$ and $\bar{\psi}_r$ both fix V', while $\bar{\phi}_{e,e^q}$ fixes V' only for $e \in \mathrm{GF}(q)$. Thus the subgroup H' of $Aut(\pi(\mathcal{S}))$ fixing the flag $(V', \bar{m}_1)$ is the collection of maps of the form $\bar{\psi}_r\bar{\phi}_{e,e}$ or $\bar{\psi}_r\bar{\phi}_{e,e}\bar{\nu}$, for any choice of $e \in \mathrm{GF}(q)$ and for any $r = p^i$, $0 < i \leq 2n$. We also observe that every element of H' fixes the line $\bar{m}_{-1} = \bar{m}_{(-1,0)}$, and so we apply Corollary 2.2 with the basis $\mathcal{B}' = \{\mathbf{b_0'} = (1, 0, 0, 0, 0), \mathbf{b_1'} = (0, 1, 0, -1, 0), \mathbf{b_2'} = (0, 0, 1, 0, 1), \mathbf{b_3'} = (0, 1, 0, 1, 0), \mathbf{b_4'} = (0, 0, 1, 0, -1)\}$. Thus we may restrict our attention to the family of orthogonal cones

$$\Omega' = \{\mathcal{C}'_{a,b,c} \colon a, b, c \in \mathrm{GF}(q) \mid b^2 - 4ac \in \not\square_q\},$$

where the usual change of basis algorithm shows that the associated Gram matrix of $\mathcal{C}'_{a,b,c}$ with respect to the standard basis is

$$\bar{A}'_{a,b,c} = \frac{1}{4}\begin{bmatrix} 0 & 0 & -2 & 0 & 2 \\ 0 & 2a & b & -2a & b \\ -2 & b & 2c & -b & 2c \\ 0 & -2a & -b & 2a & -b \\ 2 & b & 2c & -b & 2c \end{bmatrix}. \tag{20}$$

Thus the maps in H'_0 are of the form $\bar{\psi}_r\tau_{\bar{M}}$ with $\bar{M}$ either eI or eN bordered; that is, $\bar{M}$ is either

$$\bar{M}_1 = \begin{bmatrix} 1 & t & 0 & t & 0 \\ 0 & e & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 \\ 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix} \text{ or } \bar{M}_2 = \begin{bmatrix} 1 & t & 0 & t & 0 \\ 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & e \\ 0 & e & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 \end{bmatrix}$$

where $t, e \in \mathrm{GF}(q)$, $e \neq 0$. Observe the order of the group H'_0 is $4nq(q-1)$, where $q = p^n$.

Once again, determining the equivalence classes of cones in Ω' is accomplished by sorting out the action of H'_0 on these cones. Defining $\bar{M}_\sigma$ as before, the collineation $\bar{\psi}_r\tau_{\bar{M}}$ maps $\mathcal{C}'_{\bar{a},\bar{b},\bar{c}}$ to $\mathcal{C}'_{a,b,c}$ if and only if $\bar{M}_\sigma\bar{M}\bar{A}'_{a,b,c}\bar{M}^T\bar{M}^T_\sigma = k(\bar{A}'_{\bar{a},\bar{b},\bar{c}})^\sigma$ for some $k \in \mathrm{GF}(q)^*$. Straightforward computations show that this occurs if and only if

$$(\bar{a}^\sigma, \bar{b}^\sigma, \bar{c}^\sigma) = (\pm ae\omega^{-\frac{r-1}{2}}, be, \pm ce\omega^{\frac{r-1}{2}}), \tag{21}$$

where one takes the plus sign in the first and last coordinates when using $\bar{M}_1$ and takes the minus sign when using $\bar{M}_2$.

We now describe a simple way to enumerate the orthogonal Buekenhout unitals embedded in the regular nearfield plane that are associated with these cones.

Theorem 3.9. *Let $q = p^n \geq 5$ be an odd prime power. The number of inequivalent orthogonal Beukenhout unitals in the regular nearfield plane $\pi(\mathcal{S})$ of order q^2 which meet the line at infinity in some point of the long orbit is equal to $\frac{N+N_e}{2}$, where N is the number of monic irreducible factors of $P(x) = \frac{x^{q^2}-x}{x^q+x}$, treated as polynomial over the prime field $\mathbb{F}_p$, and N_e is the number of such irreducible polynomials that represent even functions.*

Proof. We must determine the number of H_0'-equivalence classes of the cones in Ω', where the equivalence relation is described in Equation (21). Letting σ be the identity automorphism and using $\bar{M} = \bar{M}_1$, we see that a triple (a, b, c) and any nonzero GF(q)-scalar multiple of it represent equivalent cones in Ω'. Since the only condition on (a, b, c) is that $f(x) = ax^2 + bx + c$ be irreducible over GF(q), we may take all monic irreducible quadratics to represent the (not necessarily distinct) equivalence classes of cones. If we did not have to consider the matrix $\bar{M}_2$ or non-identity field automorphisms, we would simply count the number of factors of $P'(x) = \frac{x^{q^2}-x}{x^q-x}$ in its unique factorization into monic irreducible (quadratic) factors over GF(q). Allowing for $\bar{M} = \bar{M}_2$ introduces the minus signs in (21), and hence (with appropriate scaling) means that the irreducible quadratics $ax^2 + bc + c$ and $ax^2 - bx + c$ should be considered equivalent. Thus these polynomials are paired, noting that any polynomial representing an even function is paired with itself, and we choose one polynomial from each pair. Hence, when q is prime, we see that the formula stated in the theorem does indeed give the number of distinct H_0'-equivalence classes, except that N now represents the number of monic irreducible (quadratic) factors of $P'(x)$.

To account for non-trivial field automorphisms σ, for each irreducible factor of $P'(x)$ (necessarily a quadratic polynomial) we chose one of its roots, say γ, in GF(q^2). It does not matter which of the two roots is chosen, as the field automorphism $x \mapsto x^q$ will send one to the other. We then let this factor correspond to the element $\frac{\gamma}{\epsilon}$ in GF(q^2). Straightforward computations, using Equation (21), show that γ is a root of $\bar{a}x^2 + \bar{b}x + \bar{c}$ if and only if $(\frac{\gamma}{\epsilon})^\sigma \epsilon$ is a root of $ax^2 \pm bx + c$. Thus $\frac{\gamma}{\epsilon}$ corresponds to $\bar{a}x^2 + \bar{b}x + \bar{c}$ if and only if $(\frac{\gamma}{\epsilon})^\sigma$ corresponds to $ax^2 \pm bx + c$ (depending upon which sign is used in (21)). Since orbits under the Frobenious map $x \mapsto x^p$ (which generates $Aut(\mathrm{GF}(q^2))$) are the roots of irreducible polynomials, we replace x by $\frac{x}{\epsilon}$ in $P'(x)$ to obtain the monic polynomial $P(x) = \frac{x^{q^2}-x}{x^q+x}$ and then let N denote the number of (monic) irreducible factors of this polynomial. Of course, these irreducible factors are not necessarily quadratic. The result now follows from our work in the previous paragraph. $\square$

Remark 3.10. *Using the above result, one quickly computes that the number of mutually inequivalent orthogonal Buekenhout unitals in the regular nearfield plane of order q^2 that meet the line at infinity in a point of the long orbit is* 11, 80, 62, 413, 1306, 2954, 24422, *and* 22155 *for q =* 9, 25, 27, 81, 125, 243, 625, *and* 729, *respectively. A formula could be developed using*

Mobiüs inversion, but we will not do so here. In the case when q is prime, a particularly nice formula is easily obtained, as we now show.

Corollary 3.11. *Let $q = p \geq 5$ be a prime. Then the number of inequivalent orthogonal Beukenhout unitals passing through a point of the long orbit on the line at infinity in the regular nearfield plane $\pi(\mathcal{S})$ of order q^2 is $\frac{q^2-1}{4}$. Thus the total number of mutually inequivalent orthogonal Buekenhout unitals in this plane is $\frac{(q+1)^2}{4}$.*

Proof. As indicated in the first part of the proof of Theorem 3.9, when q is prime we simply let N be the number of monic irreducible (quadratic) factors of $P'(x) = \frac{x^{q^2}-x}{x^q-x}$. Exactly $\frac{q-1}{2}$ of these factors are even, namely those of the form $x^2 - c$ for some $c \in \not\Box_q$. As the degree of $P'(x)$ is $q^2 - q$, there are $\frac{(q-1)^2}{2}$ remaining (quadratic) factors, none of which is even. Thus the number of mutually inequivalent unitals meeting the line at infinity in a point of the long orbit is

$$\frac{(q-1)^2}{4} + \frac{q-1}{2} = \frac{q^2-1}{4},$$

proving the first statement in the theorem. The second statement now follows from the remark following Theorem 3.7 since

$$\frac{q^2-1}{4} + \frac{q+1}{2} = \frac{(q+1)^2}{4}.$$

□

We close our discussion of the regular nearfield planes by determining the stabilizers of the orthogonal Buekenhout unitals embedded in these planes which meet the line at infinity in a point of the long orbit. Equivalently, we compute the stabilizers in H_0' of the orthogonal cones $\mathcal{C}'_{a,b,c}$ in Ω'. We rewrite Equation (21) to obtain the following necessary and sufficient conditions:

$$\begin{aligned} (\bar{c} - \bar{a}\omega)^r &= \pm(c - a\omega)e\omega^{\frac{r-1}{2}} && (22) \\ (\bar{c} + \bar{a}\omega)^r &= \pm(c + a\omega)e\omega^{\frac{r-1}{2}} && (23) \\ \bar{b}^r &= be && (24) \end{aligned}$$

Theorem 3.12. *Let $q = p^n \geq 5$ be an odd prime power and suppose $\mathcal{C}'_{a,b,c}$ is an orthogonal cone in Ω'. Let $U'_{a,b,c}$ be the corresponding orthogonal Buekenhout unital in the regular nearfield plane of order q^2 which meets the line at infinity in a point of the long orbit. Then the stabilizer of $U'_{a,b,c}$ has order*

1. *$4nq$ when $b=0$ and $c=\pm a\omega$,*

2. *$4n_0q$ when $b=0$ and $c\neq\pm a\omega$, where we define $n_0=\dim_F(\mathrm{GF}(q))$ and $F=\mathbb{F}_p\langle\frac{a\omega}{c}\rangle$,*

3. *$2n_0q$ in all other cases, where again $n_0=\dim_F(\mathrm{GF}(q))$ and*
    (a) *$F=\mathbb{F}_p\langle\frac{a^2\omega}{b^2}\rangle$ if $b\neq 0$ and $c=\pm a\omega$,*
    (b) *$F=\mathbb{F}_p\langle\frac{a\omega}{c},\frac{b^2\omega}{c^2}\rangle$ if $b\neq 0$ and $c\neq\pm a\omega$.*

Proof. Recall that the parameter t in $\bar{M}_1$ and $\bar{M}_2$ is free to assume any value from $\mathrm{GF}(q)$. We partition into cases, recalling that $c+a\omega$ and $c-a\omega$ cannot both be 0 since a and c are not zero.

If $b=0$ and either $c+a\omega=0$ or $c-a\omega=0$, then one of (22) and (23) is trivial, as is (24), and the remaining nontrivial equation uniquely determines e for any σ, both in the $\bar{M}_1$ case and the $\bar{M}_2$ case. Hence the order of the stabilizer of $U'_{a,b,c}$ is $2\cdot 2n\cdot q=4nq$ in such situations, proving the first statement of the theorem.

If $b=0$ and neither $c+a\omega$ nor $c-a\omega$ is zero, then both (22) and (23) are nontrivial. This system is equivalent to the system (23) ± (22), which reduces to

$$\begin{aligned} c^{r-1} &= \pm e\omega^{\frac{r-1}{2}} \\ a^{r-1}\omega^{\frac{r-1}{2}} &= \pm e. \end{aligned}$$

In either the $\bar{M}_1$ or the $\bar{M}_2$ case, one obtains a unique solution for e if and only if $\frac{a\omega}{c}$ is fixed by σ. Thus we obtain $2\cdot 2n_0\cdot q=4n_0q$ elements in the stabilizer of $U'_{a,b,c}$ in this case, where n_0 is the dimension stated in the second part of the theorem.

Finally, we turn to the case when $b\neq 0$, and hence (24) uniquely determines that $e=b^{r-1}$. Hence we must determine for which choices of σ (that is, r) are equations (22) and (23) satisfied with $e=b^{r-1}$. First suppose $c-a\omega=0$, in which case (22) is trivial. When $\bar{M}=\bar{M}_1$, then (23) is satisfied if and only if $\frac{a\epsilon}{b}$ is fixed by σ; when $\bar{M}=\bar{M}_2$, then (23) is satisfied if and only if $\frac{a\epsilon}{b}$ is mapped to its negative by σ. Hence we obtain solutions if and only if $(\frac{a\epsilon}{b})^2=\frac{a^2w}{b^2}$ is fixed by σ, in which case we obtain a solution either for $\bar{M}_1$ or $\bar{M}_2$, but not both. Thus the order of the stabilizer of $U'_{a,b,c}$ in this case is $2n_0q$, where $n_0=\dim_F(\mathrm{GF}(q))$ and $F=\mathbb{F}_p\langle\frac{a^2\omega}{b^2}\rangle$. The situation when $c+a\omega=0$ is exactly the same.

The remaining case is when $b \neq 0$ and $c \neq \pm a\omega$. As in one of the previous cases, we replace equations (22) and (23) by the system (23)$\pm$(22). As shown above, this system is consistent (both for $\bar{M} = \bar{M}_1$ and $\bar{M} = \bar{M}_2$) if and only if σ fixes $\frac{a\omega}{c}$. Furthermore, consistency with $e = b^{r-1}$ from equation (24) is equivalent to σ fixing $\frac{b\epsilon}{c}$ when $\bar{M} = \bar{M}_1$ and is equivalent to σ sending $\frac{b\epsilon}{c}$ to its negative when $\bar{M} = \bar{M}_2$. Hence we obtain solutions if and only if σ fixes both $\frac{a\omega}{c}$ and $(\frac{b\epsilon}{c})^2 = \frac{b^2\omega}{c^2}$, in which case we obtain a solution for $\bar{M}_1$ or $\bar{M}_2$, but not both. Thus in this final situation the stabilizer of $U'_{a,b,c}$ has order $2n_0q$, where $n_0 = \dim_F(\mathrm{GF}(q))$ and $F = \mathbb{F}_p\langle \frac{a\omega}{c}, \frac{b^2\omega}{c^2}\rangle$, completing the proof of the theorem. $\square$

4 Hall planes

We use the same model for the Hall plane as that given in [5]. Using the regular spread $\mathcal{S}_0$ and reguli $\mathcal{R}_t$ defined in the previous section,

$$\begin{aligned} \mathcal{S} &= \Big(\mathcal{S}_0 \setminus \mathcal{R}_1\Big) \cup \Big(\mathcal{R}_1^{\mathrm{opp}}\Big) \\ &= \{\ell_\infty\} \cup \{\ell_s \mid s^{q+1} \neq 1\} \cup \{m_s \mid s^{q+1} = 1\} \end{aligned} \tag{25}$$

is a Hall spread, unique up to projective equivalence.

To describe $Aut(\mathcal{S})$, we let $\phi_{e,f,g,h}$ denote the linear collineation of $PG(3,q)$ stabilizing the regular spread $\mathcal{S}_0$ that is induced by the matrix

$$M_{e,f,g,h} = \begin{bmatrix} e_1 & e_2 & g_1 & g_2 \\ \omega e_2 & e_1 & \omega g_2 & g_1 \\ h_1 & h_2 & f_1 & f_2 \\ \omega h_2 & h_1 & \omega f_2 & f_1 \end{bmatrix},$$

acting on row vectors by right multiplication, where $e, f, g, h \in \mathrm{GF}(q^2)$ with $ef - gh \neq 0$. Here, as before, we use the ordered basis $\{1, \epsilon\}$ for $\mathrm{GF}(q^2)$ over $\mathrm{GF}(q)$, so that $e = e_1 + e_2\epsilon$ and so on. For each $\sigma \in Aut(\mathrm{GF}(q^2))$, we let ψ_σ be the collineation of $PG(3,q)$ defined in the previous section. Then, as shown in [5], every automorphism of the Hall spread $\mathcal{S}$, for $q \geq 5$, may be uniquely written as $\psi_\sigma \phi_{e,f,g,h}$ for some $\sigma \in Aut(\mathrm{GF}(q^2))$ and some $e, f, g, h \in \mathrm{GF}(q^2)$, subject to the conditions

$$e^{q+1} = f^{q+1} \neq g^{q+1} = h^{q+1}, \tag{26}$$

$$f^q g = h^q e. \tag{27}$$

In particular, we have

$$|Aut(\mathcal{S})| = 2nq(q^2-1)^2/(q-1) = 2nq(q^2-1)(q+1),$$

for any $q > 3$. The case $q = 3$ was discussed in the last section. Also note that $Aut(\mathcal{S})$ has two orbits on lines, namely $\mathcal{R}_1^{\text{opp}}$ and $\mathcal{S} \setminus \mathcal{R}_1$.

We now enumerate the orthogonal Buekenhout unitals in the Hall planes. To avoid some of the technical difficulties in this enumeration process that arise from working with field automorphisms, we restrict at this stage to odd primes $q \geq 5$. This restriction still illustrates all the main ideas. Moreover, convenient formulas ore obtained for the number of equivalence classes when q is prime. Since ψ_q is the only non-trivial linear collineation arising from a field automorphism, the automorphism group of the Hall spread $\mathcal{S}$ now consists of the (linear) collineations induced by nonsingular matrices of the form

$$M_{e,f,g,h} = \begin{bmatrix} e_1 & e_2 & g_1 & g_2 \\ \pm\omega e_2 & \pm e_1 & \pm\omega g_2 & \pm g_1 \\ h_1 & h_2 & f_1 & f_2 \\ \pm\omega h_2 & \pm h_1 & \pm\omega f_2 & \pm f_1 \end{bmatrix}, \tag{28}$$

where the conditions on the parameters e, f, g, h are precisely those described in Equation (26) and Equation (27). We border, as previously discussed, any 4×4 matrix associated with a (linear) collineation in $Aut(\mathcal{S})$ to obtain $Aut(\pi(\mathcal{S}))$.

Recall that $Aut(\mathcal{S})$ has two orbits on its lines, namely the $q^2 - q$ lines ℓ_s from $\mathcal{S}_0 \setminus \mathcal{R}_1$ and the $q + 1$ lines m_s of $\mathcal{R}_1^{\text{opp}}$. Moreover, the subgroup $\{\bar{\phi}_{e,e,0,0} \colon e \in \mathrm{GF}(q^2)^*\}$ of order $q+1$ in $Aut(\pi(\mathcal{S}))$ acts (sharply) transitively on the points of each line ℓ_s, while the subgroup $\{\bar{\phi}_{e,e^q,0,0} \colon e \in \mathrm{GF}(q^2)^*\}$ of order $q+1$ in $Aut(\pi(\mathcal{S}))$ acts (sharply) transitively on the points of each line m_s. Thus it suffices only to consider cones $\mathcal{C}$ with vertex $V = (0, 0, 0, 1, 0)$ and cones $\mathcal{C}'$ with vertex $V' = (0, 1, 0, 1, 0)$.

First we consider orthogonal cones with vertex $V = (0, 0, 0, 1, 0)$, thus having $\bar{\ell}_\infty = \langle(0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\rangle$ as the uniquely determined generator spread line. Choose $\mathbf{b_0} = (1, 0, 0, 0, 0)$, $\mathbf{b_1} = (0, 1, 0, 0, 0)$, $\mathbf{b_2} = (0, 0, 1, 0, 0)$, $\mathbf{b_3} = (0, 0, 0, 1, 0)$, $\mathbf{b_4} = (0, 0, 0, 0, 1)$ and apply Theorem 2.1 with basis $\mathcal{B} = \{\mathbf{b_0}, \mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}, \mathbf{b_4}\}$. Since $\mathcal{B}$ is the standard basis, we may restrict our attention to the orthogonal cones in

$$\Omega = \{\mathcal{C}_{a,b,c} \colon a, b, c \in \mathrm{GF}(q) \mid b^2 - 4ac \in \not\square_q\},$$

where the associated Gram matrix $\bar{A}$ of $\mathcal{C}_{a,b,c}$ is given by Equation (1) in the statement of Theorem 2.1.

Letting $G = Aut(\pi(\mathcal{S}))$, straightforward computations using Equation (28) show that $H = Stab_G(V)$ has order $2q^4(q^2-1)$ and consists of those collineations induced by matrices of the form

$$\begin{bmatrix} 1 & t_1 & t_2 & t_3 & t_4 \\ 0 & e_1 & e_2 & 0 & 0 \\ 0 & \pm\omega e_2 & \pm e_1 & 0 & 0 \\ 0 & 0 & 0 & f_1 & 0 \\ 0 & 0 & 0 & 0 & \pm f_1 \end{bmatrix},$$

where $t_1, t_2, t_3, t_4 \in \mathrm{GF}(q), f_1 \in \mathrm{GF}(q)^*, e \in \mathrm{GF}(q^2)$, and $e^{q+1} = f_1^2$. Note that H leaves invariant the spread line $\bar{\ell}_0 = \langle \mathbf{b_1}, \mathbf{b_2} \rangle$ as well as the generator spread line $\bar{\ell}_\infty$, and thus Corollary 2.2 applies. The subgroup H_0 of H consisting of those elements whose translations are in the "direction" of the vertex V has order $2q(q^2-1)$ and is induced by matrices of the form

$$\bar{M} = \begin{bmatrix} 1 & 0 & 0 & t & 0 \\ 0 & e_1 & e_2 & 0 & 0 \\ 0 & \pm\omega e_2 & \pm e_1 & 0 & 0 \\ 0 & 0 & 0 & f_1 & 0 \\ 0 & 0 & 0 & 0 & \pm f_1 \end{bmatrix},$$

where $t \in \mathrm{GF}(q), f_1 \in \mathrm{GF}(q)^*, e \in \mathrm{GF}(q^2)$, and $e^{q+1} = f_1^2$.

As in the previous section, sorting out equivalences among the cones in Ω is accomplished by letting H_0 act on Ω. Routine matrix computations show that $\tau_{\bar{M}} \in H_0$ leaves $\mathcal{C}_{a,b,c}$ invariant if and only if the following conditions are satisfied:

$$\left.\begin{array}{rcl} ae_1^2 + be_1e_2 + ce_2^2 & = & af_1 \\ ce_1^2 + \omega be_1e_2 + a\omega^2 e_2^2 & = & cf_1 \\ be_1^2 + 2(c + a\omega)e_1e_2 + \omega be_2^2 & = & \pm bf_1 \\ e_1^2 - \omega e_2^2 & = & f_1^2 \end{array}\right\}, \tag{29}$$

where the last condition arises from the description of the group H_0 given above. The ± sign in the third equation indicates which sign should be used in the matrix $\bar{M}$. For convenience we will refer to this sign as the "signature" of $\bar{M}$. Thus we must determine the number of choices for t, e_1, e_2, f_1 and the signature to compute the order of $Stab_{H_0}(\mathcal{C}_{a,b,c})$, for a given orthogonal cone $\mathcal{C}_{a,b,c}$. As the entry $t \in \mathrm{GF}(q)$ does not appear in the above system of equations, there will always be q choices for t. We now compute the order of this stabilizer by considering several cases.

Proposition 4.1. *Let $q \geq 5$ be an odd prime, and let $\mathcal{C}_{a,b,c}$ be an orthogonal cone from the set Ω. Let H_0 be the subgroup of $Aut(\pi(\mathcal{S}))$ defined above, where $\mathcal{S}$ is a Hall spread given in Equation (25). Let $K = Stab_{H_0}(\mathcal{C}_{a,b,c})$.*

I. If $c = -a\omega$ and $b = 0$, then $|K| = 2q(q+1)$.

II. If $c = -a\omega$ and $b \neq 0$, then $|K| = 2q$ if $q \equiv 1 \pmod 4$ and $|K| = 4q$ if $q \equiv 3 \pmod 4$.

III. If $c \neq \pm a\omega$ and $b \neq 0$, then $|K| = 2q$ if $(c + a\omega)^2 - \omega b^2 \in \not\Box_q$ and $|K| = 4q$ if $(c + a\omega)^2 - \omega b^2 \in \Box_q$.

IV. In all other cases $|K| = 4q$.

Proof. We break the proof into cases. As stated above, we always have q choices for the entry t in the matrix $\bar{M}$. Also note that the condition $b^2 - 4ac \in \not\Box_q$ on the cone $\mathcal{C}_{a,b,c}$ implies that $a \neq 0$ and $c \neq 0$ in each case.

Case I: Suppose that $c = -a\omega$ and $b = 0$. Note that $b^2 - 4ac = 4a^2\omega \in \not\Box_q$ is satisfied for any nonzero a. In this case the system of equations in (29) reduces to: $e_1^2 - \omega e_2^2 = f_1$, $e_1^2 - \omega e_2^2 = f_1^2$. Thus $f_1^2 = f_1$ and hence $f_1 = 1$ as $f_1 \neq 0$. The only remaining equation to solve is $e_1^2 - \omega e_2^2 = 1$, which has $q+1$ solutions for the pair (e_1, e_2). As there are no restrictions on t and either signature may be used, we obtain $|K| = 2q(q+1)$.

Case II: Suppose that $c = -a\omega$ and $b \neq 0$. The discriminant condition on the orthogonal cone implies that we must have $b^2 + 4a^2\omega \in \not\Box_q$. This time the system of equations in (29) reduces to

$$\begin{aligned} ae_1^2 + be_1e_2 - a\omega e_2^2 &= af_1, \\ -a\omega e_1^2 + \omega be_1e_2 + a\omega^2 e_2^2 &= -a\omega f_1, \\ e_1^2 + \omega e_2^2 &= \pm f_1, \\ e_1^2 - \omega e_2^2 &= f_1^2. \end{aligned}$$

Adding ω times the first equation to the second equation yields $2\omega be_1e_2 = 0$ and hence $e_1 = 0$ or $e_2 = 0$ (but not both as $e \neq 0$).

Suppose first that $e_1 = 0$. Then the above equations become $\omega e_2^2 = -f_1$ and $\omega e_2^2 = -f_1^2$, with the signature determined in the third equation. This forces $f_1 = 1$, which further implies that $\omega e_2^2 = -1$ and hence $-1 \in \not\square_q$. Thus this subcase only occurs when $q \equiv 3 \pmod 4$, and then there will be two choices for e_2 and the usual q choices for t.

Next suppose that $e_2 = 0$. Then the above the equations become $e_1^2 = f_1$ and $e_1^2 = f_1^2$, with the signature again determined in the third equation. Thus $f_1 = 1$ as above, and hence $e_1^2 = 1$. Hence this subcase produces $2q$ choices for elements in K, independent of the parity of q.

Therefore in Case II we have $|K| = 4q$ if $q \equiv 3 \pmod 4$ and $|K| = 2q$ if $q \equiv 1 \pmod 4$.

Case III: Suppose that $c \neq \pm a\omega$ and $b \neq 0$, as well as the usual discriminant condition that $b^2 - 4ac \in \not\square_q$. We get no further reduction for the system of equations in (29). Subtracting c times the second equation from $a\omega^2$ times the first equation enables one to obtain $f_1 = e_1^2 + (\frac{b\omega}{c+a\omega})e_1e_2$ after dividing by $a^2\omega^2 - c^2 \neq 0$ and cancelling $a\omega - c \neq 0$. Then substitution for f_1 in the first equation yields $(\frac{b}{c+a\omega})e_1e_2 + e_2^2 = 0$. Hence either $e_2 = 0$ or $e_2 = (\frac{-b}{c+a\omega})e_1$. If $e_2 = 0$, we obtain $f_1 = e_1^2 = f_1^2$ with the signature being determined as positive in the third equation. This further implies $f_1 = 1$, $e_1^2 = 1$, and we obtain $2q$ elements in K with $e_2 = 0$.

Now suppose that $e_2 = (\frac{-b}{c+a\omega})e_1$. Then substitution for e_2 in the first and fourth equations shows that $f_1^2 = f_1$ and again $f_1 = 1$. The signature is now determined as negative from the reduced first and third equations. Thus we are left with the single equation: $((c + a\omega)^2 - \omega b^2)e_1^2 = (c + a\omega)^2$. Hence we get two choices for e_1 and an additional $2q$ elements in K precisely when $(c + a\omega)^2 - \omega b^2 \in \square_q$ (this expression cannot be zero as ω is a non-square and $b \neq 0$). Thus $|K| = 2q$ if $(c + a\omega)^2 - \omega b^2 \in \not\square_q$ and $|K| = 4q$ if $(c + a\omega)^2 - \omega b^2 \in \square_q$ in Case III.

Case IV: The remaining cases are $b = 0$, $c \neq -a\omega$ and $b \neq 0$, $c = a\omega$. Suppose first that $b = 0$ and $c \neq -a\omega$. The discriminant condition then implies that $-ac \in \not\square_q$. The system of equations in (29) reduces to

$$\begin{aligned} ae_1^2 + ce_2^2 &= af_1, \\ ce_1^2 + a\omega^2 e_2^2 &= cf_1, \\ 2(c + a\omega)e_1e_2 &= 0, \\ e_1^2 - \omega e_2^2 &= f_1^2. \end{aligned}$$

The third equation implies that either $e_1 = 0$ or $e_2 = 0$. If $e_1 = 0$, elimination of f_1 from the first and second equation shows that $c^2 - a^2\omega^2 = 0$, implying that necessarily $c = a\omega$ since $e_2 \neq 0$. This further implies that $-1 \in \square_q$ and hence $q \equiv 1 \pmod 4$ from the discriminant condition. But now the last equation forces $e_2 = 0$, a contradiction. So we must have $e_1 \neq 0$ and $e_2 = 0$, which implies $f_1 = e_1^2 = f_1^2$. Thus $f_1 = 1$ and $e_1^2 = 1$. As there are no restrictions on t or the signature, we see that $|K| = 4q$ when $b = 0$ and $c \neq -a\omega$.

Finally, suppose that $b \neq 0$ and $c = a\omega$, and hence $b^2 - 4a^2\omega \in \not\square_q$ by the discriminant condition. Then the system of equations in (29) reduces to

$$\begin{aligned} ae_1^2 + be_1e_2 + a\omega e_2^2 &= af_1, \\ be_1^2 + 4a\omega e_1e_2 + \omega be_2^2 &= \pm bf_1, \\ e_1^2 - \omega e_2^2 &= f_1^2. \end{aligned}$$

Suppose first we take the plus sign in the second equation above. Then eliminating the square order terms from the first and second equations above, one obtains $(b^2 - 4a^2\omega)e_1e_2 = 0$. Since $b^2 - 4a^2\omega \neq 0$, either $e_1 = 0$ or $e_2 = 0$ (but not both). Computations similar to those above show that we obtain $2q$ elements in the stabilizer with $e_2 = 0$ (as we are assuming the signature is positive). On the other hand, if $e_1 = 0$, then we obtain $\omega e_2^2 = f_1$ and $\omega e_2^2 = -f_1^2$, which implies $f_1 = -1$ and thus $\omega e_2^2 = -1$. This forces $-1 \in \not\square_q$, and thus we obtain an additional $2q$ elements in the stabilizer with $e_1 = 0$ provided $q \equiv 3 \pmod 4$. Therefore, using the positive signature, we find $2q$ elements in K when $q \equiv 1 \pmod 4$ and $4q$ elements in K when $q \equiv 3 \pmod 4$.

We now take the minus sign in the second equation above. Then eliminating the square order terms from the first and second equations yields $(b^2 - 4a^2\omega)e_1e_2 = 2abf_1$ and thus neither e_1 nor e_2 is zero. Solving for bf_1 and substituting back into the second equation yields a quadratic equation in the ratio $\frac{e_1}{e_2}$. Solving this quadratic shows that either $\frac{e_1}{e_2} = \frac{-b}{2a}$ or $\frac{e_1}{e_2} = \frac{-2a\omega}{b}$. In the former case, substitution of e_1 into the third equation above shows that $(b^2 - 4a^2\omega)e_2^2 = 4a^2f_1^2$, contradicting the fact that $b^2 - 4a^2\omega \in \not\square_q$. Thus we must have the latter case, and substitution of e_1 into the third equation yields $\omega(4a^2\omega - b^2)e_2^2 = b^2f_1^2$ and hence $4a^2\omega - b^2 \in \not\square_q$. Combining this with the discriminant condition now shows that we must have $-1 \in \square_q$ and hence $q \equiv 1 \pmod 4$ when the signature is negative. Substitution of e_1 into the first equation shows that $\omega(4a^2\omega - b^2)e_2^2 = b^2f_1$ and hence $f_1^2 = f_1$, implying $f_1 = 1$. One then obtains two choices for e_2, and e_1 is determined from e_2 by the ratio. Thus we obtain an additional $2q$ elements in K using the negative signature, but only when $q \equiv 1 \pmod 4$.

Thus, all told, we have $|K| = 4q$ when $b \neq 0$ and $c = a\omega$, independent of the parity of q. This finishes the proof of Case IV, completing the proof of the proposition. $\square$

We now need a technical lemma involving finite field arithmetic to determine the number of cones in certain cases of the above proposition.

Lemma 4.2. *Let q be an odd prime power, and let ω be some non-square in* $\mathrm{GF}(q)$. *Let*

$$S = \{(a, b, c)\colon a, b, c \in \mathrm{GF}(q) \mid c \neq a\omega,\ b^2 - 4ac \in \not\square_q,\ (c + a\omega)^2 - \omega b^2 \in \not\square_q\}.$$

Then

$$|S| = \begin{cases} \frac{1}{4}(q-1)(q^2-1) & \text{if } q \equiv 1 \pmod 4 \\ \frac{1}{4}(q-3)(q^2-1) & \text{if } q \equiv 3 \pmod 4 \end{cases}.$$

Proof. Let $s = b^2 - 4ac$ for some non-square s in GF(q). Then

$$(c + a\omega)^2 - \omega b^2 = (c - a\omega)^2 - s\omega$$

and hence any ordered triple $(a, b, c) \in S$ must satisfy

$$\frac{(c - a\omega)^2}{\omega(b^2 - 4ac)} = \frac{(c - a\omega)^2}{s\omega} \in (\not\square_q + 1) \cap \square_q.$$

Conversely, suppose we fix some $\lambda \in (\not\square_q + 1) \cap \square_q$. We want to find all ordered triples (a, b, c) of elements in $\mathrm{GF}(q)$ such that $\frac{(c-a\omega)^2}{\omega(b^2-4ac)} = \lambda$. That is, we want to find all ordered triples (a, b, c) such that

$$\omega^2 a^2 - \lambda\omega b^2 + c^2 + 2(2\lambda - 1)\omega ac = 0.$$

Thus we are looking for all points in $\mathrm{PG}(2, q)$ that satisfy the degree two homogenous equation $w^2 X_0^2 - \lambda\omega X_1^2 + X_2^2 + 2(2\lambda - 1)\omega X_0 X_2 = 0$. But this is the equation of a non-degenerate conic in $\mathrm{PG}(2, q)$ since the associated symmetric matrix has determinant $4\omega^3\lambda^2(\lambda - 1)$, which is nonzero by the choice of λ. Thus there are $q + 1$ projective points satisfying this equation, and hence $q^2 - 1$ choices for (a, b, c).

Note that for any such triple (a,b,c) satisfies $c \neq a\omega$ and $b^2 - 4ac \in \not\square_q$. Moreover, since $(c - a\omega)^2 = \lambda\omega(b^2 - 4ac)$, we have

$$(c + a\omega)^2 = \lambda\omega b^2 + 4(1-\lambda)\omega ac$$

and thus

$$(c + a\omega)^2 - \omega b^2 = (\lambda - 1)\omega(b^2 - 4ac),$$

which is a non-square by the choice of λ and our previous observation that necessarily $b^2 - 4ac$ is a non-square. Thus each such triple (a,b,c) is an element of S by definition. Allowing λ to vary, the result now follows using the elementary cyclotomic numbers computed by Dickson [11]. □

Theorem 4.3. *Let $q \geq 5$ be an odd prime, and let $\pi = Hall(q^2)$ be the Hall plane of order q^2. Then the number of mutually inequivalent orthogonal Buekenhout unitals embedded in π that meet the line at infinity of π in a point of its long orbit is $\frac{3}{4}(q-1)$ if $q \equiv 1 \pmod 4$ and $\frac{1}{4}(3q-1)$ if $q \equiv 3 \pmod 4$. Moreover, precisely one of these unitals has full stabilizer in $Aut(\pi)$ of order $2q(q+1)$, while all the others have stabilizers of order $2q$ or $4q$.*

Proof. As previously discussed, it suffices to determine the number of orbits under H_0 in its action on the orthogonal cones in the set Ω. Suppose first that $q \equiv 1 \pmod 4$. Let S denote the set of triples (a,b,c) defined in Lemma 4.2. Since $-1 \in \square_q$ in this case, we see from Proposition 4.1 that the cones $\mathcal{C}_{a,b,c} \in \Omega$ with stabilizers of order $2q$ are precisely those with $(a,b,c) \in S$, and hence there are $\frac{1}{4}(q-1)(q^2-1)$ such cones. Clearly H_0 acts on this subset of cones as the action preserves the order of the stabilizer. Now the Orbit-Stabilizer Theorem implies that all orbits among this subset of cones are of size $q^2 - 1$ since $|H_0| = 2q(q^2-1)$. Thus we see there are precisely $\frac{1}{4}(q-1)$ such orbits. The cones in Ω with stabilizers of order $2q(q+1)$ are those in Case I, and there are exactly $q-1$ such cones. Thus these cones form a single orbit under H_0. As straightforward counting shows that $|\Omega| = \frac{1}{2}q(q-1)^2$, we see that there remain $\frac{1}{4}(q^2-1)(q-3)$ cones in Ω, each of which has a stabilizer of order $4q$ by Proposition 4.1. Thus orbits among these cones all are of size $\frac{1}{2}(q^2-1)$, and there are $\frac{1}{2}(q-3)$ such orbits. Hence we obtain a total of

$$1 + \frac{1}{2}(q-3) + \frac{1}{4}(q-1) = \frac{3}{4}(q-1)$$

orbits when $q \equiv 1 \pmod 4$.

Now suppose that $q \equiv 3 \pmod 4$. This time $-1 \in \not\square_q$ and once again from Proposition 4.1 we see that the cones $\mathcal{C}_{a,b,c} \in \Omega$ with stabilizers of order $2q$ are precisely those with $(a, b, c) \in S$. By Lemma 4.2, there are $\frac{1}{4}(q-3)(q^2-1)$ such cones. As the orbits under H_0 among this subset of cones are of size q^2-1, we see that there are $\frac{1}{4}(q-3)$ such orbits. Just as in the previous case the cones in Ω with stabilizes of order $2q(q+1)$ form a single orbit of size $q-1$. Each of the remaining $\frac{1}{4}(q^2-1)(q-1)$ cones in Ω has a stabilizer of order $4q$ by Proposition 4.1, and thus the orbits among these cones all are of size $\frac{1}{2}(q^2-1)$. Hence there are $\frac{1}{2}(q-1)$ such orbits, and the total number of H_0-orbits on Ω when $q \equiv 3 \pmod 4$ is

$$1 + \frac{1}{2}(q-1) + \frac{1}{4}(q-3) = \frac{1}{4}(3q-1).$$

The result now follows from the stabilizer computations in Proposition 4.1 and Corollary 2.2. $\square$

We now consider the orthogonal cones with vertex $V' = (0, 1, 0, 1, 0)$ and hence generator spread line $\bar{m}_1 = \langle (0, 1, 0, 1, 0), (0, 0, 1, 0, -1) \rangle$, where $\bar{m}_1$ lies in the short orbit on lines of $\mathcal{S}$. Choose $\mathbf{b_0'} = (1, 0, 0, 0, 0)$, $\mathbf{b_1'} = (0, 1, 0, -1, 0)$, $\mathbf{b_2'} = (0, 0, 1, 0, 1)$, $\mathbf{b_3'} = (0, 1, 0, 1, 0)$, $\mathbf{b_4'} = (0, 0, 1, 0, -1)$ and apply Theorem 2.1 with basis $\mathcal{B}' = \{\mathbf{b_0'}, \mathbf{b_1'}, \mathbf{b_2'}, \mathbf{b_3'}, \mathbf{b_4'}\}$. As this is the same basis that was used in the previous section, we know that we may restrict our attention to the orthogonal cones in

$$\Omega' = \{\mathcal{C}'_{a,b,c} \colon a, b, c \in \mathrm{GF}(q) \mid b^2 - 4ac \in \not\square_q\},$$

where $\mathcal{C}'_{a,b,c}$ has as its associated Gram matrix with respect to the standard basis the matrix $\bar{A}'$ given in Equation (20). Straightforward computations using Equation (28) show that $H' = Stab_G(V')$ has order $2q^5(q-1)^2$ and H' leaves invariant the spread line $\bar{m}_{-1} = \langle \mathbf{b_1'}, \mathbf{b_2'} \rangle$ as well as the generator spread line $\bar{m}_1$. In fact, the subgroup H'_0 of H' whose translations are in the "direction" of V' has order $2q^2(q-1)^2$ and is induced by matrices of the form

$$\bar{M}' = \begin{bmatrix} 1 & t & 0 & t & 0 \\ 0 & e_1 & e_2 & g_1 & e_2 \\ 0 & \pm\omega e_2 & \pm e_1 & \pm\omega e_2 & \pm g_1 \\ 0 & g_1 & -e_2 & e_1 & -e_2 \\ 0 & \mp\omega e_2 & \pm g_1 & \mp\omega e_2 & \pm e_1 \end{bmatrix},$$

where $t, e_1, e_2, g_1 \in \mathrm{GF}(q)$ with $e_1^2 \neq g_1^2$.

We now sort out equivalences among the cones in Ω' by letting H'_0 act on Ω'. Choosing the cone $\mathcal{C}'_{1,0,-\omega} \in \Omega'$, more routine, but messy, matrix computations show that $\tau_{\bar{M}'} \in H'_0$ leaves this cone invariant if and only if the following conditions are satisfied:

$$\left.\begin{array}{ccc} e_2 & = 0 & \\ (g_1+e_1)^2 & = & (g_1-e_1)^2 \\ (g_1-e_1)^2 & = & \pm(g_1-e_1) \\ g_1^2 & \neq & e_1^2 \end{array}\right\},$$

where the last condition arises from the description of the group H'_0 given above. Once again note that the entry $t \in \mathrm{GF}(q)$ does not appear in the above system of equations, and thus there are q choices for it. The third and fourth equations imply that $g_1 - e_1 = \pm 1$ and thus $(g_1 + e_1)^2 = 1$. Therefore one obtains four solutions for the pair (e_1, g_1), and $K = Stab_{H'_0}(\mathcal{C}'_{1,0,-\omega})$ has order $4q$. An application of the Orbit-Stabilizer Theorem now shows that H'_0 acts transitively on Ω' as $|\Omega'| = \frac{1}{2}q(q-1)^2$. Thus we have the following results.

Theorem 4.4. *Let $q \geq 5$ be an odd prime, and let $\pi = Hall(q^2)$ be the Hall plane of order q^2. Then there is a unique orthogonal Buekenhout unital embedded in π that meets the line at infinity of π in a point of its short orbit. The full stabilizer in $Aut(\pi)$ of this unital has order $4q$.*

Corollary 4.5. *Let $q \geq 5$ be an odd prime. Then, up to equivalence, the Hall plane π of order q^2 contains exactly $1 + \lfloor \frac{3q}{4} \rfloor$ ovoidal Buekenhout unitals, where $\lfloor x \rfloor$ denotes the floor of x. The orders of the full stabilizers in $Aut(\pi)$ of these unitals may be found in Theorems 4.3 and 4.4.*

5 Conclusion

The techniques developed in this paper may be used to enumerate the orthogonal Buekenhout unitals embedded in any two-dimensional translation plane, provided one has a convenient description of the automorphism group of the associated spread $\mathcal{S}$. For instance, one could consider the odd-order two-dimensional flag-transitive planes (see [2], [4] for a description of these planes in terms of the associated spreads). By definition, the stabilizer of the spread acts transitively on its lines, but one must look at the point orbits on any line under that stabilizer of that line. When q is prime, one can easily show that there are $\frac{q+1}{2}$ such orbits. Computations similar to the ones in the previous two sections show that any flag-transitive two-dimensional plane of order q^2, where $q \geq 5$ is prime, has $\frac{q(q^2-1)}{2}$ mutually inequivalent orthogonal Buekenhout unitals, and the stabilizer of any such unital has order $2q$. It should be noted that when q is prime, there are $\frac{q-1}{2}$ mutually non-isomorphic such planes of order q^2 (the Desarguesian plane is not counted as it is not two-dimensional).

As a final remark, we should mention that the software package Magma [10] played an integral part in the discovery process of several results in this paper by providing data that eventually allowed us to see many general patterns.

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R. D. Baker
Dept. of Math. & C.S.
West Virginia State Univ.
Institute, WV 25112

G. L. Ebert
Dept. of Math. Sci.
Univ. of Delaware
Newark, DE 19716

K. L. Wantz
Dept. of Mathematics
Southern Nazarene Univ.
Bethany, OK 73008