1. (20 points) Prove true, or give a counterexample: For any $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^n$, $\|uv^*\|_F = \|u\|_F \|v\|_F$.

2. (20 points) Show that the solution $x$ of the least squares problem $\min \|b - Ax\|_2$ for $A \in \mathbb{C}^{m \times n}$ and $b \in \mathbb{C}^m$ satisfies

$$\begin{bmatrix} I & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$ 

What are the dimensions of each symbol appearing in this system?

3. For this problem consider only real numbers. A Givens rotation $G(i, j; a, b)$ for $1 \leq i < j \leq m$ is an $m \times m$ matrix that equals the identity except for the four elements $g_{ii} = g_{jj} = c$, $g_{ij} = -g_{ji} = s$, where

$$c = \frac{a}{\sqrt{a^2 + b^2}}, \quad s = \frac{b}{\sqrt{a^2 + b^2}}.$$ 

For any vector $x$, the vector $G(i, j; x_i, x_j)x$ is zero in the $j$th row.

(a) (15 points) Show that $G(i, j; a, b)$ is orthogonal.

(b) (15 points) This algorithm sketches how to use Givens rotations to reduce $A$ orthogonally to upper triangular $R$:

```
for k from 1 to n do
  for i from k + 1 to m do
    Compute the c and s of $G(k, i; a_{kk}, a_{ik})$.
    $A_{[k, i], k:n} := \begin{bmatrix} c & s \\ -s & c \end{bmatrix} A_{[k, i], k:n}$
  end do
end do
```

Find an asymptotic flop count for the algorithm.

4. (a) (10 points) Find the relative 2-norm condition number for computing $x^2 - y^2$ for real $x$ and $y$.

(b) (10 points) If $x = 1 + 10^{-6}$ and $y = 1$, about how many accurate decimal digits can you expect when computing $x^2 - y^2$ in IEEE double precision? (Answer to the nearest integer.)

(c) (10 points) Suppose now $x$ and $y$ are any floating point numbers. Which computer algorithm is more accurate,

$$(x \otimes x) \oplus (y \otimes y) \quad \text{or} \quad (x \oplus y) \otimes (x \ominus y)?$$ 

Justify your response mathematically.