1. Let $A \in \mathbb{C}^{m \times m}$, $b \in \mathbb{C}^m$, and $k$ be a positive integer. To solve the linear system $A^k x = b$, you could form the matrix $C = A^k$ and then use pivoted LU factorization on $C$. Describe another algorithm starting with pivoted LU factorization on $A$. (You may express your algorithm as MATLAB code, but do not have to.) Which of these two methods is faster?

$$A = P^T L U \quad \leftrightarrow \quad A^{-1} = U^{-1} L^{-1} P$$

1. Factor.
2. $x = b$
3. for $j = 1, \ldots, k$
   
   $x = P x$
   
   Solve $L y = x$ for $y$
   
   Solve $U x = y$ for $x$

end

$k-1$ matrix multiplications to get $A^k \approx (k-1)(2m^3)$ flops

$$= O(km^3) \text{ flops}$$

This is the slower method.
2. (a) Prove that if $X$ is a hermitian positive definite $m \times m$ matrix such that $X^2 = I$, then $X = I$. (Hint: Use an SVD.)

(b) Prove that if $A$ is any real SPD matrix, then there is another real SPD matrix $X$ such that $X^2 = A$. (Hint: Use an SVD.)

(a) Write $X = U \Sigma U^*$ as SVD. Then

$$I = X^*X = U \Sigma U^* U \Sigma U^* = U \Sigma^2 U^*$$

This is an SVD of $I$, so $\Sigma^2 = I$. Since $X$ is pos. def., $\Sigma = I$

(b) Let $A = R^T R$ be a Cholesky factorization. Write an SVD, $R = U \Sigma V^*$. Then $A = V \Sigma U^* U \Sigma V^* = V \Sigma^2 V^*$

$$= V \Sigma V^* V \Sigma V^* = X^2$$

if $X = V \Sigma V^*$. This is an eigenvalue decomposition, showing that $X$ is positive def.
3. Suppose $A$ is a real symmetric matrix with eigenvalues $-4$, $-1$, $2$, $12$. In each column below are eigenvalue estimates that result from running one of these four iterations on $A$: power iteration, inverse iteration, shifted inverse iteration, or Rayleigh quotient iteration. In each case state which iteration was used, explaining quantitatively why your answer is the most reasonable one.

<table>
<thead>
<tr>
<th></th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
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<td>-3.999999999996177</td>
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</tbody>
</table>

(a) errors are about $3.7$, $1.4$, $0.041$, $7 \times 10^{-8}$, $< 10^{-16}$.
This is clearly superlinear, and consistent with cubic convergence: **R.Q.I.**

(b) errors are $6.6 \times 10^{-3}$, $7.4 \times 10^{-4}$, $8.1 \times 10^{-5}$, $9.1 \times 10^{-6}$, etc.
They decrease by a factor of about $9$ at each step, which is equal to $\left| \frac{\lambda_2}{\lambda_1} \right|^2 = \left| \frac{12}{-4} \right|^2 = 3^2$. **Power iter.**

(c) Converge to an eigenvalue other than largest or smallest in magnitude. Error decreases by a factor of about 1000 at each step, which is linear convergence. **Shifted inverse iter.**
4. Let $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

(a) Use symmetric pivoting to find a symmetric tridiagonal $T$ that is unitarily similar to $A$.
(The standard Hessenberg reduction method is not necessary.)

(b) Describe what happens when the “pure” (unshifted) QR iteration is applied to $A$. Explain this convergence behavior in terms of eigenvalues.

\((a)\) Swap rows 2 and 3, then columns 2 and 3.

\[
PAP^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} P^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = T
\]

\((b)\) $A$ is orthogonal, so $A=A^T$ is the QR factorization, and $I=A=A$ again. No change! (Same for $T$.)

Eigenvalues are \{1, 1, -1, -1\}; all have magnitude 1, so no progress for power iteration.