1. $m\frac{dv}{dt} = -kv^2$ or $\frac{dv}{dt} = -kv^2$, $k > 0$

Separable: $-\frac{dv}{v^2} = k\,dt \Rightarrow \frac{1}{v} = kt + C \Rightarrow v = \frac{1}{kt+C} \to 0$ as $t \to \infty$

$$\frac{dx}{dt} = \frac{1}{kt+C} \Rightarrow x = A + \frac{1}{k}\log(kt+C) \to \infty \text{ as } t \to \infty$$

2. Linear: $y' - \frac{1}{x}y = 3x$ I.F. $p(x) = e^{\int -\frac{1}{x}dx} = x^{-1}$

$$x^{-1}y' - x^{-2}y = 3 \Rightarrow (x^{-1}y)' = 3$$

$$\Rightarrow y = x(3x + C)$$

$1 = y(1) = 3 + C$, so $C = -2$.

3. $5I'' + 100I' + 1000I = 1000\cos 10t$

or $I'' + 20I' + 200I = 200\cos 10t$

$I_{sp} = a\cos 10t + b\sin 10t$

$$(-100a + 200b + 200a)\cos 10t + (-100b - 200a + 200b)\sin 10t$$

$$= 200\cos 10t$$

$$\left[\begin{array}{cc|c} 100 & 200 & 200 \\ -200 & 100 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} +2/5 \\ 4/5 \end{bmatrix}$$

$I_{sp} = \frac{2}{5}\cos 10t + \frac{4}{5}\sin 10t$ amplitude $= \sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \frac{\sqrt{20}}{5} = \frac{2}{\sqrt{5}}$

4.
(a) $\underline{x}' = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \underline{x}$ $\quad A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ $\quad |A - \lambda I| = \lambda^2 - 2\lambda + 5$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

$$A - \lambda_1 I = \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix}, \quad \underline{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$e^{\lambda_1 t} \underline{v}_1 = e^t (\cos 2t + i \sin 2t) \begin{bmatrix} 1 \\ -i \end{bmatrix} = e^t \left(\begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix} + i \begin{bmatrix} \sin 2t \\ -\cos 2t \end{bmatrix} \right)$$

$$\underline{x}(t) = c_1 e^t \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin 2t \\ -\cos 2t \end{bmatrix} \qquad \left(\text{or use } e^{At} \underline{c}\right)$$

(b) $\underline{x}' = A\underline{x} + e^t \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ $\quad \underline{x}_p = \underline{a} e^t, \quad \underline{x}_p' = \underline{a} e^t$

$$\therefore \underline{a} e^t = A\underline{a} e^t + \begin{bmatrix} 4 \\ -2 \end{bmatrix} e^t \Rightarrow (I - A)\underline{a} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \underline{a} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} \Rightarrow \underline{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

5.
(a) $A^2 = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^4 = 0$

(b) $e^A = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + 0 = \begin{bmatrix} 1 & 2 & 2 & 4/3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

6. Columns of

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -2 \\ 2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

3 lead variables

$\Rightarrow$ all 3 needed for a basis

$\Rightarrow$ dimension = 3

7. Triangular matrix : Eigenvalues on diagonal.

$\lambda_1 = 2$

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \underline{x} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_2 = 1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \qquad \underline{x} = \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{bmatrix}$$

$\lambda_3 = -1$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \underline{x} = \alpha \begin{bmatrix} 1/3 \\ -1 \\ 1 \end{bmatrix}$$

Inverse :

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1/3 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 1/3 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & & \\ & 1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1/3 \end{bmatrix}$$

8. $\det(A) = \det(L)\det(U)$ (product)

$= (1\cdot 1\cdot 1)(2\cdot(-1)\cdot 2)$ (triangular)

$= -4$

$\neq 0$ for all values of c.