NAME SOLUTIONS

MATH341-011, Spring 2007
Exam 1: March 9

Write all solutions on these sheets. Please clearly erase or cross out irrelevant work; otherwise it will be part of the graded material. **You must justify answers to receive full credit.** You may not use calculators or notes.

1. (15 points) Find the position function $x = x(t)$ given the acceleration $a(t) = 6 + \cos t$, and $v(0) = 2$, $x(0) = -1$.

$$v(t) = \int a(t)\,dt = 6t + \sin t + C$$

$$2 = v(0) = C$$

$$x(t) = \int v(t)\,dt = 3t^2 - \cos t + 2t + B$$

$$-1 = x(0) = -1 + B \Rightarrow B = 0$$

$$x = 3t^2 - \cos t + 2t$$

2. (15 points) For each initial-value problem, determine whether the existence and uniqueness theorem guarantees a local solution.

(a) $\dfrac{dy}{dx} = x^{-1}y^{1/2}, \quad y(1) = 1$

(b) $\dfrac{dy}{dx} = x^{-1}y^{1/2}, \quad y(1) = 0$

(c) $\dfrac{dy}{dx} = x^{-1}y^{1/2}, \quad y(0) = 1$

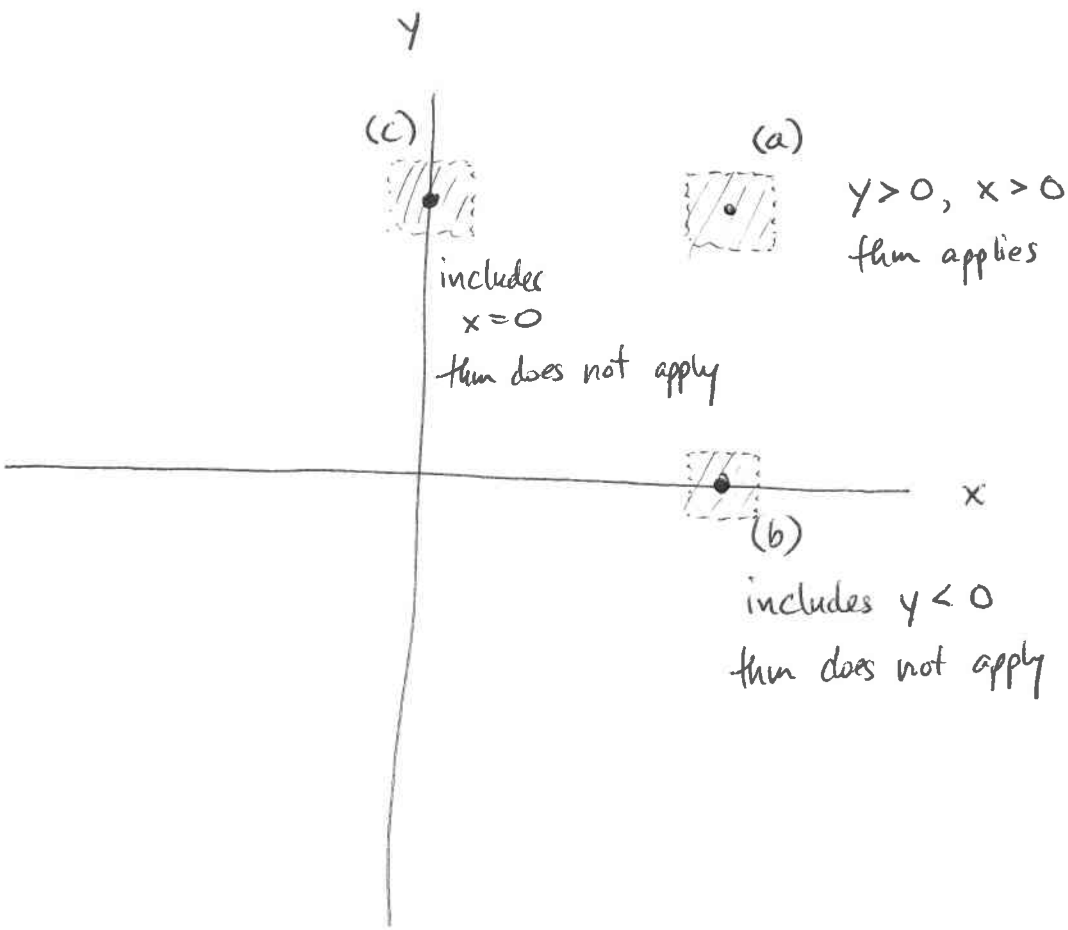
$$f(x,y) = \frac{\sqrt{y}}{x}$$

fails to exist if $y < 0$ or $x = 0$

continuous elsewhere

$$\frac{\partial f}{\partial y} = \frac{1}{2}\,\frac{1}{x\sqrt{y}}$$

fails for $y \le 0$ or $x = 0$



3. (20 points) Find the general solution of $y\dfrac{dy}{dx} = x(y^2+1)$.

$$\int \frac{y}{y^2+1}\,dy = \int x\,dx$$

$$\frac{1}{2}\log|y^2+1| = \frac{1}{2}x^2 + C$$

$$|y^2+1| = e^{2C}e^{x^2} = Ae^{x^2} \qquad (A>0)$$

$$y^2 = -1 + Be^{x^2} \qquad (B \neq 0)$$

We do not need to consider $B=0$, since $y^2=-1$ is not a valid solution (real-valued).

4. (15 points) Use the substitution $v = y/x$ to change $xy' = \sqrt{x^2 + y^2}$ into a separable, first-order equation for v. Show clearly that the equation for v is separable, but **do not try to solve the new equation.**

$y = vx$, $y' = xv' + v$

$\therefore \quad x(xv' + v) = \sqrt{x^2 + v^2 x^2} = x\sqrt{1+v^2}$

$xv' = \sqrt{1+v^2} - v$

$\dfrac{dv}{\sqrt{1+v^2} - v} = \dfrac{dx}{x}$ <u>separable</u>

5. (20 points) Solve $xy' - 3y = x^3, \quad y(1) = 10.$

$$y' - \frac{3}{x}y = x^2$$ linear

I.F. $\rho(x) = \exp\left[\int -\frac{3}{x}\,dx\right] = e^{-3\log x} = x^{-3}$

$$\left[x^{-3}y\right]' = x^{-3} \cdot x^2 = x^{-1}$$

$$x^{-3}y = \log x + C$$ (note $x > 0$ initially)

$$y = x^3(\log x + C)$$

$$10 = 1^3(\log 1 + C) = C$$

$$y = x^3(\log x + 10)$$

6. (15 points) Suppose the fish population $P(t)$ in a lake is afflicted with a disease starting at $t = 0$, such that the birth rate β is zero and the death rate δ is inversely proportional to $\sqrt{P}$. Show that the fish population dies out in a finite amount of time.

$$\frac{dP}{dt} = \beta P - \delta P = 0 - \frac{k}{\sqrt{P}} P = -kP^{1/2}$$

$(k>0)$

$$\frac{dP}{P^{1/2}} = -k\,dt$$

$$2P^{1/2} = -kt + C$$

At $t=0$, $2P_0^{1/2} = C$

$$P = \left(-\frac{k}{2}t + P_0^{1/2}\right)^2$$

$P=0$ when $t = \dfrac{2\sqrt{P_0}}{k}$