NAME SOLUTIONS

MATH 242, Fall 2007
Final exam

Arrange your work as clearly and neatly as possible, and cross out incorrect work. **Unless otherwise noted, you must justify all answers to receive full credit.** You may not use calculators, notes, or any other kinds of aids.

Each question is worth 20 points, for a total of 200.

1. (a) Find $\dfrac{dy}{dx}$ if $y = \cosh(4^x)$.

   (b) Evaluate $\displaystyle\lim_{x\to\infty} \frac{\tanh(x)}{x}$.

(a) $y' = \sinh(4^x)\,\frac{d}{dx}(4^x) = (\ln 4)\,4^x \sinh(4^x)$

(b) $\displaystyle\lim_{x\to\infty} \frac{\tanh(x)}{x} = \frac{1}{\infty} = 0$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$
$$1 - \sin^2\theta = \cos^2\theta$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)$$
$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta)$$

Logistic equation: $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{K}\right)$,

$P(t) = \dfrac{K}{1 + Ae^{-kt}}$, $\quad A = \dfrac{K - P_0}{P_0}$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

2. Find $\int \frac{e^x}{1+e^x}\,dx$.

$u = 1+e^x, \quad du = e^x\,dx$

$\int \frac{du}{u} = \ln|u| + C = \ln(1+e^x) + C$

3. Evaluate $\lim_{x \to 0} \frac{\tan(2x^2)}{x^2}$.

$\frac{0}{0}$ L'H $\Rightarrow$ $\lim_{x \to 0} \frac{4x \sec^2(2x^2)}{2x} = \lim_{x \to 0} 2\sec^2(2x^2)$

$= 2$

4. Evaluate $\int_0^{\pi/4} 4\sin^4 x\,dx$, or show that it is divergent.

$$\int_0^{\pi/4} 4\left(\tfrac{1}{2}(1-\cos 2x)\right)^2 dx$$

$$= \int_0^{\pi/4} (1 - 2\cos 2x + \cos^2 2x)\, dx$$

$$= \int_0^{\pi/4} \left(1 - 2\cos 2x + \tfrac{1}{2} + \tfrac{1}{2}\cos 4x\right) dx$$

$$= \left[\tfrac{3}{2}x - \sin 2x + \tfrac{1}{8}\sin 4x\right]_0^{\pi/4}$$

$$= \frac{3\pi}{8} - 1$$

5. Evaluate $\int_0^1 \ln(x)\,dx$, or show that it is divergent.

$$\lim_{t\to 0^+} \int_t^1 \ln x \; dx \qquad \begin{bmatrix} u = \ln x & du = dx/x \\ dv = dx & v = x \end{bmatrix}$$

$$= \lim_{t\to 0^+} \left[\left[x \ln x\right]_t^1 - \int_t^1 dx \right]$$

$$= \lim_{t\to 0^+} \left[\left(1 \overbrace{\cancel{\ln 1}}^{0} - t \ln t\right) - (1-t) \right]$$

$$= -1 - \lim_{t\to 0^+} \frac{\ln t}{1/t} \qquad \frac{\infty}{\infty}$$

$$= -1 - \lim_{t\to 0^+} \underbrace{\frac{1/t}{-1/t^2}}_{\to (-t)}$$

$$= -1$$

6. This question is about the curve $x = t^3 - 3t + 3$, $y = 2t - 6$.

(a) Find equations for all of the vertical tangent lines.

(b) Find the equation for the line tangent at the point $(3, -6)$.

(a) $\frac{dx}{dt} = 0 \Rightarrow 3t^2 - 3 = 0$

$\Rightarrow t = 1$ or $t = -1$

If $t = 1$, $(x,y) = (1,-4)$. $x = 1$

$t = -1$, $(x,y) = (5,-8)$ $x = 5$

vertical lines

(b) $(x,y) = (3,-6) \Rightarrow \left.\begin{array}{l} 3 = t^3 - 3t + 3 \\ -6 = 2t - 6 \end{array}\right\} t = 0$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2}{3t^2 - 3}$ @ $t = 0$, $\frac{dy}{dx} = -\frac{2}{3}$

$(y+6) = -\frac{2}{3}(x-3)$, or $y = -\frac{2}{3}x - 4$

7. Convert the polar curve $r = 4\sin\theta$ to cartesian coordinates, and identify it as an ellipse, parabola, or hyperbola.

$r^2 = 4r\sin\theta \quad \Rightarrow \quad x^2 + y^2 = 4y$

$x^2 + (y-2)^2 = 4$

<u>ellipse</u> (circle)

8. Find the Taylor series of $f(x) = \dfrac{1}{(x+1)^2}$ at $a = 0$.

$f(x) = (1+x)^{-2}, \quad f(0) = 1$

$f'(x) = -2(1+x)^{-3}, \quad f'(0) = -2$

$f''(x) = 6(1+x)^{-4}, \quad f''(0) = 3!$

$\vdots$

$f^{(n)}(0) = (-1)^n (n+1)!$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{n!} x^n = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$$

(OR: Differentiate $-\dfrac{1}{1+x} = \displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} x^n$ term by term.)

9. Determine whether $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n!}$ is absolutely convergent, conditionally convergent, or divergent.

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2} \right| = \frac{1}{n+1} \cdot \left(1 + \frac{1}{n}\right)^2$$

$$\to \frac{1}{\infty} \cdot 1 = 0$$

as $n \to \infty$

$\therefore$ Converges absolutely by Ratio Test.

10. A lake with a carrying capacity of 900 fish is stocked with 100 fish. The relative growth rate k is assumed to be equal to $\ln(2)$ per year.

(a) How long will it take for the population to reach 300 fish? (Your answer should be simplified as far as possible.)

(b) What would the answer to (a) be if the carrying capacity were essentially infinite?

(a) $A = \dfrac{900-100}{100} = 8$

$$300 = \frac{900}{1+8e^{-t\ln 2}}$$

$$1+8e^{-t\ln 2} = 3$$

$$-t\ln 2 = \ln\left(\tfrac{1}{4}\right) = -2\ln 2 \Rightarrow t=2$$

(b) $P(t) = P_0 e^{kt} = 100\, e^{t\ln 2}$

$$3 = e^{t\ln 2} \Rightarrow t = \frac{\ln 3}{\ln 2} = \log_2 3$$