

Exam 3 Fall 2006

① $a_n = \cos(\frac{2}{n})$

$$\lim_{n \rightarrow \infty} \cos\left(\frac{2}{n}\right) = \cos(0) = 1 \leftarrow \text{yes it converges since the limit exists.}$$

② a) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$

Alternating, b_n is positive, decreasing and

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{n}{n^2+1} = 0$$

By alt. series test it converges.

b) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}(n-1)} = \sum_{n=2}^{\infty} \frac{1}{n^{3/2}-n^{1/2}}$

Limit comparison: $b_n = \frac{1}{n^{3/2}}$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\frac{1}{n^{3/2}-n^{1/2}}}{\frac{1}{n^{3/2}}} = \frac{n^{3/2}}{n^{3/2}-n^{1/2}} = 1 > 0$$

therefore by the limit comparison test both series converge, since $\frac{1}{n^{3/2}}$ is a p-series with $p > 1$.

③ Find the sum

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

$$= e^{-1} \quad \text{since} \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (\text{put in } -1 \text{ for } x)$$

④ Radius and interval of convergence for $\sum_{n=0}^{\infty} \sqrt{n} (x-2)^n$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\sqrt{n+1} (x-2)^{n+1}}{\sqrt{n} (x-2)^n} \right| = \lim_{n \rightarrow \infty} \left| \sqrt{\frac{n+1}{n}} (x-2) \right|$$

$$= \lim_{n \rightarrow \infty} \left| \sqrt{1 + \frac{1}{n}} (x-2) \right| = |x-2| < 1$$

$$\Rightarrow -1 < x-2 < 1$$

$$\Rightarrow 1 < x < 3$$

$$\Rightarrow \boxed{R=1}$$

Check endpoints:

$$x=1: \sum_{n=0}^{\infty} (-1)^n \sqrt{n} \leftarrow \text{diverges (test for divergence)}$$

$$x=3: \sum_{n=0}^{\infty} \sqrt{n} \leftarrow \text{diverges (test for divergence)}$$

$$\Rightarrow \boxed{I = (1, 3)}$$

$$(5) \quad \frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n$$

$$\Rightarrow \frac{x}{1+x^3} = x \cdot \frac{1}{1+x^3} = x \cdot \frac{1}{1-(-x^3)} = x \cdot \sum_{n=0}^{\infty} (-x^3)^n$$

$$= x \cdot \sum_{n=0}^{\infty} (-1)^n x^{3n}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n x^{3n+1}}$$

$$(6) \quad f(x) = \sqrt{x}, \quad a = 4$$

$$f(4) = 2$$

$$f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow f'(4) = \frac{1}{4}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \Rightarrow f''(4) = -\frac{1}{32}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} \Rightarrow f'''(4) = \frac{3}{256}$$

So $T_3(x)$ for $f(x) = \sqrt{x}$ is

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

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$$\Rightarrow 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$$