FORM B

MATH 241, Spring 2009
Exam 2: April 15

NAME Solutions Discussion section time ________

1	2	3	4	5	6	7	8	Total

Arrange your work as clearly and neatly as possible, and cross out incorrect work. **Unless otherwise noted, you must justify all answers to receive full credit.** You may not use calculators, notes, or any other kinds of aids.

1. (12 points) A particle moves in a straight line for $t \geq 0$ according to the position function $x = \sqrt{k^2 + t^2}$, where k is a constant. Find the velocity and acceleration of the particle.

velocity $= x' = \frac{1}{2}(k^2+t^2)^{-1/2} \frac{d}{dt}(k^2+t^2) = t\,(k^2+t^2)^{-1/2}$

accel $= x'' = (1)(k^2+t^2)^{-1/2} + (t)(-\tfrac{1}{2})(k^2+t^2)^{-3/2}(2t)$

$= (k^2+t^2)^{-1/2}\left[1 - t^2(k^2+t^2)^{-1}\right]$

$= (k^2+t^2)^{-3/2}\left((k^2+t^2)-t^2\right)$

$= \dfrac{k^2}{(k^2+t^2)^{3/2}}$

2. (8 points each) Find y'.

(a) $y = e^{-1/x^2}$

$$y' = \left(e^{-1/x^2}\right) \frac{d}{dx}\left(-\frac{1}{x^2}\right) = e^{-1/x^2} \frac{d}{dx}\left(-x^{-2}\right)$$

$$= e^{-1/x^2} (-1)(-2)x^{-3}$$

$$= \frac{2e^{-1/x^2}}{x^3}$$

(b) $y = \ln(\tan(2x))$

$$y' = \frac{1}{\tan(2x)} \frac{d}{dx}(\tan(2x))$$

$$= \cot(2x) \sec^2(2x) \frac{d}{dx}(2x)$$

$$= 2\cot(2x)\sec^2(2x)$$

$$= \frac{2}{\cos(2x)\sin(2x)}$$

3. (12 points) Find y' if $y = x^{\sin x}$. (No need to simplify very much.)

$$\ln y = \ln\left(x^{\sin x}\right) = (\sin x)(\ln x)$$

$$\frac{y'}{y} = (\cos x)(\ln x) + (\sin x)\left(\frac{1}{x}\right)$$

$$y' = x^{\sin x}\left[(\cos x)(\ln x) + \frac{\sin x}{x}\right]$$

4. (12 points) Find the line tangent to the ellipse $x^2 - xy + 3y^2 = 5$ at the point $(2, 1)$.

$\frac{d}{dx}$ both sides: $\quad 2x - [(1)(y) + (x)(y')] + 6yy' = 0$

Plug in $x=2$, $y=1$:

$$4 - 1 - 2y' + 6y' = 0 \Rightarrow y' = -\frac{3}{4}$$

$= \text{slope}$

$$(y-1) = -\frac{3}{4}(x-2)$$

5. (12 points) A bacteria culture grows with a constant relative growth rate. Initially, there are 50 cells in the culture. After two hours, there are 2000. At what time will there be 50,000 cells in the culture? (Express your answer using logs. No need to simplify, but check that your answer is greater than 2 hours.)

$$P(t) = P_0 e^{kt}$$

$P_0 = 50$ given.

$2000 = P(2) = 50e^{2k}$ given.

$$40 = e^{2k}$$

$$k = \frac{1}{2}\ln(40)$$

Find t so that $50000 = P(t) = 50 e^{\frac{t}{2}\ln(40)}$

$$1000 = e^{\frac{t}{2}\ln(40)}$$

$t = 2 \cdot \dfrac{\ln(1000)}{\ln(40)} > 2$ since $\ln(x)$ is always increasing

6. (12 points) Use a linearization to estimate $\sqrt{3.96}$.

$f(x) = \sqrt{x} = x^{1/2}$ @ $x = 3.96$

choose $a = 4 \Rightarrow f(a) = 4^{1/2} = 2$

$f'(x) = \frac{1}{2} x^{-1/2} \Rightarrow f'(a) = \frac{1}{2} 4^{-1/2} = \frac{1}{4}$

$L(x) = f(a) + f'(a)(x-a)$

$L(3.96) = 2 + \frac{1}{4}(3.96 - 4)$

$= 2 + \frac{1}{4}(-0.04) = 1.99$

7. (12 points) The circumference C of a sphere is measured to be 12 cm, with a relative error of up to 2%. If this measurement is used to compute the surface area A of the sphere, estimate the maximum relative error in A. (For a sphere of radius r: $C = 2\pi r$, $A = 4\pi r^2$, $V = \frac{4}{3}\pi r^3$.)

$$A = 4\pi r^2 = 4\pi\left(\frac{C}{2\pi}\right)^2 = \frac{C^2}{\pi}$$

$$dA = \left(\frac{dA}{dC}\right) dC = \left(2\frac{C}{\pi}\right) dC$$

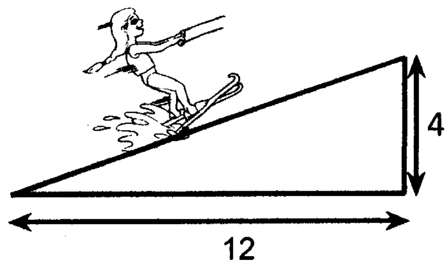
$$\text{relative error in } A = \frac{\Delta A}{A} \approx \frac{dA}{A}$$

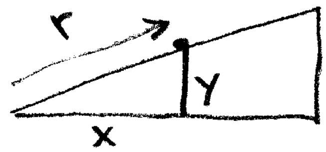
$$= \frac{\frac{2C}{\pi}\, dC}{\frac{C^2}{\pi}}$$

$$= \frac{2\, dC}{C}$$

$$= 2\cdot(0.02) = 0.04 = 4\%$$

8. (12 points) A waterskiier travels along a ramp at 20 ft/sec. How fast is her height above the water changing when she reaches the end of the ramp?





Given $\frac{dr}{dt} = 20$,

find $\frac{dy}{dt}$ @ $y=4$

$x^2+y^2 = r^2$ (right triang.) and $\frac{x}{y} = \frac{12}{4} = 3$ (similar triang.)

$$(3y)^2 + y^2 = r^2$$

$$10y^2 = r^2 \implies @\ y=4,\ 160 = r^2,\ r = 4\sqrt{10}$$

$$20y\frac{dy}{dt} = 2r\frac{dr}{dt}$$

$$80\frac{dy}{dt} = 8\sqrt{10}\cdot 20$$

$$\frac{dy}{dt} = 2\sqrt{10} \text{ ft/sec}$$