Math 242 Homework Set #10
Due: 11/9/07

Section 12.3

5. The function \( f(x) = \frac{1}{3x+1} \) is continuous, positive and decreasing on \([1, \infty)\), so the Integral Test applies. So we can look at the integral of \( f(x) = \frac{1}{3x+1} \):

\[
\int_{1}^{\infty} \frac{1}{3x+1} \, dx = \lim_{b \to \infty} \frac{1}{3} \ln(3x + 1) \bigg|_{1}^{b} = \lim_{b \to \infty} \frac{1}{3} \ln(3b + 1) - \frac{1}{3} \ln 4 = \infty.
\]

Therefore, since the improper integral diverges, so does the series \( \sum_{n=1}^{\infty} \frac{1}{3x+1} \).

6. The function \( f(x) = e^{-x} \) is continuous, positive, and decreasing on \([1, \infty)\), so the Integral Test applies. \( \int_{1}^{\infty} e^{-x} \, dx = \lim_{b \to \infty} \int_{1}^{b} e^{-x} \, dx = \lim_{b \to \infty} [-e^{-x}]_{1}^{b} = \lim_{b \to \infty} [-e^{-b} + e^{-1}] = e^{-1} \),

therefore \( \sum_{n=1}^{\infty} e^{-n} \) converges. Note, this is a geometric series, that converges to

\[
\frac{e^{-1}}{1 - e^{-1}} = \frac{1}{e - 1}.
\]

12. \( 1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + ... = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \). This is a p-series with \( p = \frac{3}{2} > 1 \), so it converges by definition 1.

20. \( f(x) = \frac{\ln x}{x^2} \) is continuous and positive for \( x \geq 2 \), and \( f'(x) = \frac{1 - 2 \ln x}{x^3} < 0 \) for \( x \geq 2 \), so \( f \) is decreasing. \( \int_{1}^{\infty} \frac{\ln x}{x^2} \, dx = \lim_{t \to \infty} \frac{-\ln x}{x} \bigg|_{1}^{2} = \frac{\ln 2 + 1}{2} \). Therefore

\[
\sum_{n=1}^{\infty} \frac{\ln n}{n} = \sum_{n=2}^{\infty} \frac{\ln n}{n} \text{ converges by the Integral Test.}
\]

Section 12.4

3. \( \frac{1}{n^2 + n + 1} < \frac{1}{n^2} \) for all \( n \geq 1 \), so \( \sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1} \) converges by comparison with \( \sum_{n=1}^{\infty} \frac{1}{n^2} \), which converges because it is a p-series with \( p=2>1 \).
6. \( \frac{1}{n - \sqrt{n}} > \frac{1}{n} \) for all \( n \geq 2 \), so \( \sum_{n=2}^{\infty} \frac{1}{n - \sqrt{n}} \) diverges by comparison with the divergent (partial) harmonic series \( \sum_{n=2}^{\infty} \frac{1}{n} \).

9. \( \cos^2 \left( \frac{1}{n} \right) \leq \frac{1}{n^2} \leq \frac{1}{n^2 + 1} \), so the series \( \sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2} \) converges by comparison with the p-series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) (p = 2 > 1).

14. \( \frac{\sqrt{n}}{n - 1} > \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \), so diverges by comparison with the divergent (partial) p-series \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) (p = 1/2 < 1).

19. \( \frac{2^n}{1 + 3^n} < \frac{2^n}{3^n} = \left( \frac{2}{3} \right)^n \) is a convergent geometric series, since 2/3 < 1, so \( \sum_{n=1}^{\infty} \frac{2^n}{1 + 3^n} \) converges by the Comparison Test.

Section 12.5

2. \(-1 + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \ldots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n + 2} \). Here \( a_n = (-1)^n \frac{n}{n + 2} \). Since \( \lim_{n \to \infty} a_n \neq 0 \) (in fact the limit does not exist), the series diverges by the Test for Divergence.

3. \(-\frac{4}{7} + \frac{4}{8} - \frac{4}{9} + \frac{4}{10} - \frac{4}{11} + \ldots = \sum_{n=1}^{\infty} (-1)^n \frac{4}{n + 6} \). Here, \( b_n = \frac{4}{n + 6} > 0 \), \( \{b_n\} \) is decreasing, and \( \lim_{n \to \infty} b_n = 0 \), so the series converges by the Alternating Series Test.

6. \( b_n = \frac{1}{3n - 1} > 0 \), \( \{b_n\} \) is decreasing, and \( \lim_{n \to \infty} b_n = 0 \), so the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n - 1} \) converges by the Alternating Series Test.

13. \( \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} \). \( \lim_{x \to \infty} \frac{1}{\ln x} = \frac{1}{\ln \frac{1}{x}} = \infty \), so the series diverges by the Test for Divergence.
15. \[ \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}. \] \[ b_n = \frac{1}{n^{3/4}} \text{ is decreasing and positive and } \lim_{n \to \infty} \frac{1}{n^{3/4}} = 0, \text{ so the series converges by the Alternating Series Test.} \]