Isospectral shapes with Neumann and alternating boundary conditions

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The isospectrality of a well-known pair of shapes constructed from two arrangements of seven congruent right isosceles triangles with the Neumann boundary condition is verified numerically to high precision. Equally strong numerical evidence for isospectrality is presented for the eigenvalues of this standard pair in new boundary configurations with alternating Dirichlet and Neumann boundary conditions along successive edges. Good agreement with theory is obtained for the corresponding spectral staircase functions. Strong numerical evidence is also presented for isospectrality in an example of a different pair of shapes whose basic building-block triangle is not isosceles. Some possible confirmatory experiments involving fluids are suggested.

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I. INTRODUCTION

As the most accessible realization of the negative answer to Kac’s [1] question “Can one hear the shape of a drum?,” the pair of isospectral shapes discovered by Gordon et al. [2] (termed, respectively, “bilby” and “hawk” in Fig. 1) have subsequently been investigated from a variety of mathematical, numerical, and experimental viewpoints. Chapman [3] showed how domain eigenfunctions can be mapped from the constituent triangles of one shape to the second to prove isospectrality by transplantation, and described a proof by paper-folding. More recently, Okada and Shudo [4] have investigated isospectrality through a technique of successive unfolding of fundamental building-block shapes and transplantation of eigenfunctions. Wu et al. [5] achieved a proof by an explicit mode-matching method.

The numerical problem concerning the eigenvalues corresponds to solving an eigenvalue problem for the two-dimensional Helmholtz equation subject to the Dirichlet boundary condition (DBC). Wu et al. [5] verified isospectrality numerically by an extrapolated mode-matching method, apparently to about eight significant figures, tabulated for the first 25 modes. The “analytical” 9th and 21st modes there, corresponding to known simple modes of the underlying triangles, were not computed but were taken at their exact values. Subsequently, Driscoll [6], using a much more accurate modified domain-decomposition method, verified isospectrality numerically to 12 significant figures for the first 25 modes, including the two “analytical” modes for which the computation was more or less exact. This work showed that the computed results in Ref. [5] were actually accurate to about four to five significant figures.

On the experimental side, Sridhar and Kudrolli [7] performed measurements on thin microwave cavities of the appropriate shapes, utilizing the correspondence with a two-dimensional Helmholtz equation in the electromagnetic formulation. Then Even and Pieranski [8] constructed actual shaped small “drums”—membranes made from liquid crystal smectic films—and measured their vibrations.

In this paper, we investigate numerical aspects of the isospectrality of the two standard bilby and hawk shapes, as well as other shapes, when Neumann boundary conditions (NBC) are present, and make suggestions for possible experimental verification.

II. NEUMANN BOUNDARY CONDITIONS

The most commonly encountered boundary condition is the Dirichlet BC \(\psi = 0\) on the boundary. This corresponds to the standard “drum” condition for a vibrating membrane with fixed edges, as well as to the boundary condition for quantum billiards [9]. The Neumann BC \(\partial\psi/\partial n = 0\) also has important manifestations [9], especially in acoustics, where the pressure satisfies the Helmholtz equation with NBC at a rigid boundary, and for water surface waves. In electromagnetism also, the magnetic field of the transverse electric (TE) mode in a cavity has NBC. The NBC corresponds to the vibrational modes of a drum with stress-free edges, as discussed by Hobiki et al. [10], who numerically investigated such a situation for fractal boundary shapes. Russ et al. [11] also considered fractal resonators with NBC numerically, remarking that this situation could represent transverse acoustical phonons of a two-dimensional (2D) irregular crystallite. In the field of quantum billiards, Gneimaud and Jain [12] considered rational and irrational rhombus billiards with NBC. Kohler and Blumel [13] considered ray-splitting billiards including NBC. Wiersig [14] has used the fact that, for

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significant figures, for the two standard isospectral shapes of Fig. 1 (basic side two units), with Neumann boundary condition. In their announcement of DBC isospectrality showed isospectrality for Dirichlet and for Neumann boundary conditions and corresponding exact eigenvalues for some NBC the same shapes with NBC were isospectral; Okada and et al. also stated that already established isospectrality to about three significant figures. The exact modes mentioned above, however, serve as benchmarks in any investigation of NBC isospectrality for these shapes.

A numerical verification of isospectrality of the two standard shapes of Fig. 1 for NBC, such as has been done previously for DBC [5,6], does not seem to have been carried out before. For this paper, the earlier work of Driscoll [6] has been adapted to the NBC case. In [6], a candidate eigenfunction near a corner with interior angle $\pi/\alpha$ is expanded in local polar coordinates as $\sum_{n=1}^{M} c_n J_n(r \sqrt{E}) \sin(n \alpha \theta).$ Then one finds an eigenvalue $E$ by matching different expansions along the interfaces of a domain decomposition; numerically, this becomes a minimization of the result of a matrix eigenvalue problem. For NBC we replace the sine by a cosine and start the summation at $n=0$.

We have verified isospectrality in the NBC configuration for the first 30 (nonzero) modes to 12 significant figures. The results for both shapes are given in Table I. With fundamental length unit 2, the analytical modes described above have eigenvalues given by $E_{m,n} = (\pi^2/4)(m^2 + n^2)$; $m \leq n = 0,1,2,\ldots$. The cases $(m,n) = (0,1), (1,1), (0,2), (1,2), (2,2)$ corresponding to the five analytical modes described above, together with their readily identifiable nodal patterns [17], are essentially recovered exactly. The nodal patterns of the fourth nontrivial mode (which is nonanalytical) are shown in Fig. 2.

It may be noted that many of the investigations concerned with quantum chaotic spectral statistics [9] deal with very large numbers of very high levels. The accuracy on these typically was $10^{-2}$ of the mean level spacing for earlier works, and more recently of the order of $10^{-4}$. By contrast, we are concerned here with the first few dozen eigenvalues, computed to very high accuracy, of the order of $10^{-12}$ of the mean level spacing or better.

The spectral staircase (number-counting) function for these systems is $N(E) = \sum_{j=1}^{\infty} \Theta(E - E_j)$, where $\Theta$ is the Heaviside unit step function. This is related to the spectral or trace function $\Phi(t) = \sum_{j=1}^{\infty} \exp(-E_j t)$ via a Laplace

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FIG. 2. Nodal lines for the fourth nontrivial Neumann BC modes.

FIG. 3. Spectral staircase for the bilby/hawk shapes with Neumann BC.
transform. Based on the work of several authors [18–21], with earlier work discussed extensively in Baltes and Hilf [22], the (smoothed) spectral staircase function (corresponding to the trace function appearing in [17]) for polygons is given by

\[
N(E) = \frac{A}{4\pi} E + \frac{(L_N - L_D)}{4\pi} E^{(1/2)} + \left(\sum_{DD,NN} - \sum_{DN,ND}\right) \left(\frac{\pi^2 - \theta^2}{24\pi\theta}\right),
\]

where \(A\) is the area, \(L_D (L_N)\) is the length of that part of the perimeter having Dirichlet (Neumann) boundary condition, and the sums are over corner angles \(\theta\) subtended by pairs of sides with boundary conditions as indicated. For basic unit length 2 for the two isospectral shapes of Fig. 1 (which have the same area, perimeter lengths, and corner angles), this reads, for the NBC case, \(N_N(E) = 1.1141E + 1.6302E + 0.4167\). In Fig. 3, we plot \(N(E)\) for the first 31 modes for NBC (including the zero mode). The agreement with this graph is good, demonstrating the need for inclusion of the zero mode and the plus sign for the second term in the case of NBC compared with the minus sign for the DBC case, as was plotted in [6].

To our knowledge, no experiments involving isospectral shapes with Neumann boundary conditions have been performed, in contrast to reported experiments for the DBC case [7,8]. Some such NBC experiments could be envisaged, however, based on acoustics and wave propagation in liquids [9, Sec. 2.1], where NBC's are involved. For instance, Blumel et al. [23] reported on the nodal patterns of surface waves formed by agitating a tank with circular or stadium-shaped cylindrical walls. Chinnery et al. [24,25] used a schlieren technique to visualize resonances in sonified water cavities with stadium and circular cross sections. Hebert et al. [26] made an experimental study of resonances of a fractal acoustic cavity.

It seems likely that these experimental techniques could be applied to cross-sectional shapes as in Fig. 1 to investigate their isoperpectrality. Independent checks on the accuracy of such experiments would be available through the sequential mode numbers, eigenvalue ratios, and nodal patterns of the analytical modes pictured in [17], along with the results in Table I and plots of Fig. 2 in the present work.
III. ALTERNATING BOUNDARY CONDITIONS

There has been some work done on 2D systems with a mixture of Dirichlet and Neumann BC’s on different parts of the boundary. Baltes and Hilf [22, p. 47] show the appearance of a minus sign in the third (constant, corner-angle) term of the spectral number counting function for a rectangle whose sides successively alternate DBC and NBC [cf. Eq. (1) above]. In quantum billiards, there has been recent work where parts of a rectangular boundary have DBC and parts have NBC, for ray-splitting [13] and barrier [14] billiards. Thus it is important and timely to consider systems with both types of boundary conditions in detail.

Having verified, above, the accuracy of our modified domain-decomposition method for computationally handling Dirichlet or Neumann BC’s in the case of the two standard provably isospectral shapes, we turn to the case of shapes with “alternating boundary conditions” (ABC’s) in which each side is successively DBC or NBC as one moves around the perimeter.

A. Standard “bilby” and “hawk” shapes

The isospectrality of the two standard shapes in the ABC configuration has not been proved mathematically so far, and does not seem immediately amenable to the standard forms of proof. For instance, the transplantation method for unfolded domains described by Okada and Shudo [4] does not work here because a DBC edge, upon folding, would yield a DBC rather than an NBC external edge as desired. Our aim here is to present strong numerical evidence for isospectrality in this new ABC configuration. It can be seen that \( A, L_D, L_N \) and the \( \theta \)'s in Eq. (1) are the same for both shapes, so the coefficients of the three terms in Eq. (1) are equal for both, a necessary condition for isospectrality. In fact, there are two such distinct isospectral pair configurations. We denote by ADNBC (ANDBC) the situation for which the longest side in the alternating boundary condition configuration in each shape is chosen to have DBC (NBC).

Our numerical method now uses \( \sin[(n+1/2)\alpha \theta] \) or \( \cos[(n+1/2)\alpha \theta] \) in the Fourier-Bessel corner expansions, whichever conforms to the local BC. The computed eigenvalues for the first 30 modes of the two shapes (bilby and hawk) in the two ABC configurations are given in Tables II(a) and II(b). In either case, the results for both shapes in the same configuration agree to at least 12 significant figures. The nodal lines for the tenth modes for the two shapes in both configurations are shown in Fig. 4. Unlike the cases of pure DBC [5,17] or pure NBC [17], we have been unable to construct any exact “analytical” modes or to identify particularly simple nodal patterns in the computed eigenfunction plots. Thus independent checks as for the pure DBC and NBC cases do not seem to be available here.

The corresponding spectral staircase functions are plotted and compared with the graphs of Eq. (1) in Fig. 5. For basic side length \( h, A = (7/2)h^2, |L_N-L_D| = \sqrt{2}h, \) and the corner angle term has value \(-5/12\). Thus, for \( h=2, \) Eq. (1) becomes \( N_{ADN}(E) = 1.1141E-0.2251/E-0.4167, \) with a plus sign for the second term in \( N_{AND}(E) \). The plots show good agreement and confirm the minus sign for the third term in these cases of alternating boundary conditions for this pair of shapes.

B. Other isospectral pairs

Inspection of the building scheme as utilized by Even and Pieranski [8] for constructing isospectral pairs (with DBC) from a basic building-block triangle shows that in general a nine-sided shape results; this would not support alternating boundary conditions. Their special case of an isosceles right-angled (90-45-45) triangle produces the standard, eight-sided shapes. The question arises whether the bilby/hawk pair constitute the unique ABC configuration isospectral pair, or whether there are other eight-sided shape pairs with the same property. In the notation of [8], it can be noted that the multiple-angle vertex \( 4\gamma \) appears just once in the construction circuit, so if \( \gamma = \pi/4 \) the number of sides is reduced by 1 from nine to eight. Such even-sided shapes are now amenable to alternating boundary conditions.

Further inspection of the shapes in Fig. 1 of Ref. [8] shows that (in their notation) the multiple angles \( 3\alpha \) and \( 3\beta \) each appear twice, so they could not contribute to a change of parity of the number of sides. This may nevertheless suggest the possibility even of six-sided shapes. However, if \( \alpha \) (or \( \beta \)) = \( \pi/3 \), then for alternating sides the equality of the
length differences $L_N - L_D$ for the pair of shapes as required by Eq. (1) leads to an inconsistency in the geometry of the fundamental building-block triangle.

Thus (in the notation of [8]) with $\gamma = 45^\circ$, and neither of $\alpha$ and $\beta$ equal to $60^\circ$ or $45^\circ$, we get pairs of eight-sided new shapes with, for a given pair, the same areas, the same corner angles, and the same $L_N - L_D$. We have chosen a nonisosceles building-block triangle with angles $\alpha = 65^\circ$, $\beta = 70^\circ$, $\gamma = 45^\circ$, and side length $c = 1$. These two new shapes are depicted in Fig. 6. For either ADNBC or ANDBC pairs, the eigenvalues within a pair were computed using the method described above. Convergence was less rapid than for the bilby/hawk shapes, but we are confident in the agreement of the first 15 eigenvalues to at least seven significant figures, as shown in Table III. Furthermore, even though these regions appear to be nearly mirror images, the nodal line patterns shown in Fig. 7 for the 11th modes of the DN, and the ND, cases show significant topological differences. Altogether the numerical evidence of ABC isospectrality beyond the standard bilby/hawk pair is compelling.

While the range of available examples is more restricted than in the pure Dirichlet or Neumann cases, it is remarkable that isospectrality persists with alternating boundary conditions for regions constructed according to rules that no longer provide a rigorous explanation of the phenomenon.

**TABLE III.** Eigenvalues of the first 15 modes, to seven significant figures, for the two new nonstandard isospectral shapes of Fig. 6, with (a) alternating Dirichlet (longest side)/Neumann boundary conditions; (b) alternating Neumann (longest side)/Dirichlet boundary conditions.

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**FIG. 7.** Nodal lines of the 11th modes for the regions of Fig. 6, in the ADNBC (top) and ANDBC boundary configurations. While the two regions are nearly mirror images, their mode patterns in both these instances are quite different.

**IV. CONCLUSION**

The known isospectrality of the two standard shapes (Fig. 1) with Neumann boundary condition has been confirmed numerically to a high degree of accuracy, and good agreement with theory for the spectral staircase function was obtained.

We have presented numerical evidence that is the first and indeed strong indication of the isospectrality of these two standard shapes in the new boundary condition configurations with alternating Dirichlet and Neumann conditions on successive sides. A pair of nonstandard isospectral shapes (Fig. 6) was similarly dealt with.

It is suggested that some experimental work involving fluids may illustrate the NBC case, and that electromagnetic cavities might be relevant for the case of alternating boundary conditions.

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