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Interpolation in the Limit of Increasingly Flat Radial Basis Functions

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Abstract—Many types of radial basis functions, such as multiquadrics, contain a free parameter. In the limit where the basis functions become increasingly flat, the linear system to solve becomes highly ill-conditioned, and the expansion coefficients diverge. Nevertheless, we find in this study that limiting interpolants often exist and take the form of polynomials. In the 1-D case, we prove that with simple conditions on the basis function, the interpolants converge to the Lagrange interpolating polynomial. Hence, differentiation of this limit is equivalent to the standard finite difference method. We also summarize some preliminary observations regarding the limit in 2-D.

Keywords—Radial basis functions, RBF, PDEs, Singular limit, Interpolation, Lagrange polynomial, Ill-conditioning.

1. INTRODUCTION

During the last few decades, radial basis functions (RBFs) have found increasingly widespread use for functional approximation of scattered data. Given data at nodes $\mathbf{x}_1, \dots, \mathbf{x}_N$ in d dimensions, the basic form for such approximations is

$$s(\mathbf{x}) = \sum_{k=1}^{N} \lambda_k \, \phi\left(\|\mathbf{x} - \mathbf{x}_k\|\right), \tag{1.1}$$

where $\|\cdot\|$ denotes the Euclidean distance between two points, and $\phi(r)$ is some function defined for $r \geq 0$. Given scalar function values $f_i = f(x_i)$, the expansion coefficients λ_k are obtained by

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solving the linear system

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix} \qquad \text{or} \qquad A\boldsymbol{\lambda} = \mathbf{f}, \tag{1.2}$$

where $A_{i,j} = \phi(\|\mathbf{x}_i - \mathbf{x}_j\|)$. This ensures that (1.1) interpolates $f(x)$ at $x_1, \dots, x_N$.

Many of the common choices for $\phi(r)$ fall into one of two categories:

- infinitely smooth and containing a free parameter, such as multiquadrics (MQ, $\phi(r) = \sqrt{r^2 + c^2}$) and Gaussians ($\phi(r) = e^{-(\epsilon r)^2}$), and
- piecewise smooth and parameter-free, such as cubics ($\phi(r) = r^3$) and thin plate splines ($\phi(r) = r^2 \ln r$).

At least in some circumstances, the infinitely smooth class provides spectrally accurate approximations of smooth data [1–3]. Previous authors have observed experimentally that the quality of approximation is influenced by the free parameter, and that the "optimal" parameter value depends strongly on the data [4,5].

Here we prefer to write MQ in the form $\phi(r) = \sqrt{1 + (\epsilon r)^2}$ (different by a constant factor), because the limit $\epsilon \to 0$ is easier to work with than the limit $c \to \infty$. This of course does not change the RBF approximation itself. In the rest of this paper, the limit $\epsilon \to 0$ corresponds to $\phi(r)$ becoming increasingly flat near the origin.

Large values of ϵ lead to well-conditioned linear systems, but the resulting approximations tend to be inaccurate and useless. For example, they resemble uncoupled spikes in the case of Gaussians and a piecewise linear interpolant (in 1-D) with MQs. Intermediate values of ϵ can often be employed successfully and have been extensively explored in the literature. In the limit $\epsilon \to 0$, the condition number of system (1.2) grows rapidly and without bound (equivalently, the expansion coefficients diverge), and this fact has been a barrier to investigation.

Our main point is that, although the coefficient vector $\boldsymbol{\lambda}$ diverges as $\epsilon \to 0$, the RBF interpolant itself (usually) converges to a finite limit. If we rewrite the RBF system (1.2) in slightly different notation as

$$\begin{bmatrix} & \vdots & \\ \cdots & \phi\left(\|\mathbf{x}_i - \mathbf{x}_j\|\right) & \cdots \\ & \vdots & \end{bmatrix} \boldsymbol{\lambda} = \mathbf{f}, \tag{1.3}$$

and the interpolant (1.1) as

$$s(\mathbf{x}, \epsilon) = \left[\cdots \quad \phi\left(\|\mathbf{x} - \mathbf{x}_j\|\right) \quad \cdots\right] \boldsymbol{\lambda}, \tag{1.4}$$

it is perhaps not so surprising that numerical cancellation occurs in s to compensate for the divergence of $\boldsymbol{\lambda}$. One point of view is that transforming $\mathbf{f}$ to $s(\mathbf{x}, \epsilon)$ is a well-conditioned process, but computing $\boldsymbol{\lambda}$ is an ill-conditioned intermediate step in one particular implementation. If this step could be avoided, perhaps a more stable algorithm could be found.

This paper is structured as follows. In Section 2, we provide a rough estimate of how ill-conditioned system (1.2) is for different numbers of spatial dimensions. We then focus on 1-D in Section 3, presenting some analytic results for small N as well as a theorem showing that, subject to some easily stated conditions on $\phi(r)$, as $\epsilon \to 0$ the RBF interpolant converges to the Lagrange interpolating polynomial. In Section 4, we demonstrate that the situation in 2-D is more complicated. The existence of limits evidently depends on the node locations, and the limit itself (again polynomial) depends on the choice of $\phi(r)$. Some concluding remarks are given in Section 5.

2. THE CONDITIONING OF RBF SYSTEMS

It is well known that $\phi(r) = r^n$ is a poor choice for even values of n. For example, $\phi(r) = r^2$ leads to a singular system in 1-D whenever $N > 3$. This is an immediate consequence of the fact that each basis function is then a parabola, and that any linear combination of parabolas is again a parabola. Since each is described by three coefficients, at most three of them can be linearly independent. Even more trivially, if $\phi(r) \equiv 1$, the system becomes singular whenever $N > 1$, regardless of d.

To generalize these observations, suppose

$$\phi(r) = a_0 + a_1 r^2 + a_2 r^4 + \cdots + a_m r^{2m}.$$

In Table 1, we list the maximum possible number of independent translates as a function of m and d. The cases of $m = 0$ and of $m = d = 1$ have already been discussed; other cases can be studied in the same manner.

Table 1. Dependence of the maximum number of independent basis functions on power ($2m$) and dimension (d).

	$d = 1$	2	3	$\cdots$
$m = 0$	1	1	1	$\cdots$
1	3	4	5	$\cdots$
2	5	9	14	$\cdots$
3	7	16	30	$\cdots$
4	9	25	55	$\cdots$
5	11	36	91	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

EXAMPLE 1. Determine the entry in Table 1 for $m = 1$, $d = 2$. Suppose we have five RBF centers, located at (x_k, y_k), and that the corresponding RBF coefficients are λ_k, $k = 1, 2, \ldots, 5$. The RBF approximation becomes

$$\begin{aligned} s(x,y) &= \sum_{k=1}^{5} \lambda_k \left[a_0 + a_1\left((x - x_k)^2 + (y - y_k)^2\right)\right] \\ &= x^2 \left(a_1 \sum_{k=1}^{5} \lambda_k\right) - 2x\left(a_1 \sum_{k=1}^{5} \lambda_k x_k\right) + y^2 \left(a_1 \sum_{k=1}^{5} \lambda_k\right) \\ &\quad - 2y\left(a_1 \sum_{k=1}^{5} \lambda_k y_k\right) + 1\left(\sum_{k=1}^{5} \lambda_k \left(a_0 + a_1\left(x_k^2 + y_k^2\right)\right)\right). \end{aligned}$$

Assuming $a_1 \neq 0$, this is identically zero if and only if

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ x_1^2 + y_1^2 & x_2^2 + y_2^2 & x_3^2 + y_3^2 & x_4^2 + y_4^2 & x_5^2 + y_5^2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since this system has more columns than rows, a nontrivial solution is guaranteed to exist. Thus, there cannot be more than four independent RBFs of the specified type. ∎

When this approach is applied to larger values of m and d, a pattern emerges. In general, we find that at most

$$\frac{2m + d}{m + d}\binom{m + d}{d}$$

translated basis functions are independent. Based on such data, we can get a lower bound on the condition number of RBF matrices in the case where $\phi(r)$ depends on the parameter ϵ; i.e.,

$$\phi(r) = a_0 + a_1(\epsilon r)^2 + a_2(\epsilon r)^4 + \cdots.$$

For example, with $N = 300$ and $d = 2$, we see that going out only as far as the $m = 16$ term would give a singular RBF matrix. So the fact which "saves" us from singularity is the continuation to $a_{17}(\epsilon r)^{34} + a_{18}(\epsilon r)^{36} + \cdots$. Hence, an $O(\epsilon^{34})$ perturbation of the $O(1)$-sized RBF matrix A would certainly suffice to make A singular, and the condition number of A satisfies $\kappa(A) = O(\epsilon^{-34})$. (This bound is not tight—in fact, we computationally observe $\kappa(A) = O(\epsilon^{-46})$ in this case.) Clearly, the RBF coefficient vector $\boldsymbol{\lambda}$ grows very rapidly as $\epsilon \to 0$.

3. SOME EXAMPLES AND A LIMIT RESULT FOR 1-D

For the smallest values of N, the limit $s(x, 0)$ of $s(x, \epsilon)$ as $\epsilon \to 0$ can be found directly.

EXAMPLE 2. Determine the limiting approximations when $N = 2$ and

$$\phi(r) = a_0 + \epsilon^2 a_1 r^2 + \epsilon^4 a_2 r^4 + O\left(\epsilon^6\right). \tag{3.1}$$

Substituting (3.1) into (1.3) and solving for $\boldsymbol{\lambda}$ in terms of $\mathbf{f}$ gives

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \epsilon^{-2}\left(\dfrac{f_2 - f_1}{2(x_1 - x_2)^2 a_1}\right) + \dfrac{1}{4}\left(\dfrac{f_1 + f_2}{a_0} + \dfrac{2(f_1 - f_2)a_2}{a_1^2}\right) + O\left(\epsilon^2\right) \\ \epsilon^{-2}\left(\dfrac{f_1 - f_2}{2(x_1 - x_2)^2 a_1}\right) + \dfrac{1}{4}\left(\dfrac{f_1 + f_2}{a_0} - \dfrac{2(f_1 - f_2)a_2}{a_1^2}\right) + O\left(\epsilon^2\right) \end{bmatrix}.$$

For the interpolant, we get (after many cancellations)

$$s(x, \epsilon) = \left[\,\phi(x - x_1) \quad \phi(x - x_2)\,\right] \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{(x - x_2)\, f_1 + (x - x_1)\, f_2}{x_1 - x_2} + O\left(\epsilon^2\right).$$

The limiting approximation is simply the interpolating straight line.

Some of the cancellations above required assuming that $a_0/a_0 = 1$ and $a_1/a_1 = 1$. These relations are suspect if either (or both) of $a_0 = 0$ or $a_1 = 0$ hold. These special cases can themselves have special subcases of their own, as summarized in Table 2. We see, however, that the limit is always of polynomial form, and in no case do the expansion coefficients $a_0, a_1, a_2, \ldots$ appear explicitly. ∎

Table 2. Different limits in 1-D with $N = 2$ data points.

Coefficients				Limit
a_0	a_1	a_2	a_3	
$\neq 0$	$\neq 0$			$\dfrac{(x - x_2)\, f_1 + (x - x_1)\, f_2}{x_1 - x_2}$
$= 0$	$\neq 0$			$\dfrac{(x - x_2)^2 f_1 + (x - x_1)^2 f_2}{(x_1 - x_2)^2}$
$= 0$	$= 0$	$\neq 0$		$\dfrac{(x - x_2)^4 f_1 + (x - x_1)^4 f_2}{(x_1 - x_2)^4}$
$= 0$	$= 0$	$= 0$	$\neq 0$	$\dfrac{(x - x_2)^6 f_1 + (x - x_1)^6 f_2}{(x_1 - x_2)^6}$
			$\ddots$	
$\neq 0$	$= 0$			$\{(x - x_2)\left(2x^2 - x(3x_1 + x_2) + 2x_1^2 - x_1x_2 + x_2^2\right) f_1$ $- (x - x_1)\left(2x^2 - x(3x_2 + x_1) + 2x_2^2 - x_1x_2 + x_1^2\right) f_2\} / (x_1 - x_2)^3$

EXAMPLE 3. *Determine the limiting approximations when* $N = 3$. To get a definite answer, it is now necessary to extend (3.1) with terms up to and including $a_4(\epsilon r)^8$. The $\boldsymbol{\lambda}$ components are found to grow like $O(\epsilon^{-4})$. After quite extensive algebra, the final answer for $s(x, 0)$ simplifies to

$$\frac{(x - x_2)(x - x_3)\, f_1}{(x_1 - x_2)(x_1 - x_3)} + \frac{(x - x_1)(x - x_3)\, f_2}{(x_2 - x_1)(x_2 - x_3)} + \frac{(x - x_1)(x - x_2)\, f_3}{(x_3 - x_1)(x_3 - x_2)},$$

i.e., again to the interpolating polynomial of lowest degree. The exceptional cases (featuring different limits) arise this time when $a_1 = 0$ or when $6a_0a_2 - a_1^2 = 0$. These situations again have further exceptional cases, which we do not attempt to describe here. ∎

This explicit approach to finding the limits $s(x,0)$ is useful for illustration and inspiration, but the procedure quickly becomes algebraically intractable as N grows. It turns out, however, that the pattern holds in general: $s(x,0)$ is the Lagrange interpolating polynomial, given some easily stated conditions on the expansion of ϕ.

THEOREM 3.1. *Let N distinct data nodes in 1-D be given. Suppose the basis function*

$$\phi(r) = a_0 + \epsilon^2 a_1 r^2 + \epsilon^4 a_2 r^4 + \cdots \tag{3.2}$$

is such that the RBF system (1.2) has a solution for all $\epsilon > 0$. For integer n, define the symmetric matrices G_{2n-1} and G_{2n} by

$$G_{2n-1} = \begin{bmatrix} \binom{0}{0}a_0 & \binom{2}{2}a_1 & \cdots & \binom{2n-2}{2n-2}a_{n-1} \\ \binom{2}{0}a_1 & \binom{4}{2}a_2 & \cdots & \binom{2n}{2n-2}a_n \\ \vdots & \vdots & & \vdots \\ \binom{2n-2}{0}a_{n-1} & \binom{2n}{2}a_n & \cdots & \binom{4n-4}{2n-2}a_{2n-2} \end{bmatrix}_{n\times n}, \tag{3.3}$$

$$G_{2n} = \begin{bmatrix} \binom{2}{1}a_1 & \binom{4}{3}a_2 & \cdots & \binom{2n}{2n-1}a_n \\ \binom{4}{1}a_2 & \binom{6}{3}a_3 & \cdots & \binom{2n+2}{2n-1}a_{n+1} \\ \vdots & \vdots & & \vdots \\ \binom{2n}{1}a_n & \binom{2n+2}{3}a_{n+1} & \cdots & \binom{4n-2}{2n-1}a_{2n-1} \end{bmatrix}_{n\times n}. \tag{3.4}$$

If G_{N-1} and G_N are nonsingular, then the RBF interpolant $s(x,\epsilon)$ defined by (1.1) satisfies

$$\lim_{\epsilon\to 0} s(x,\epsilon) = L_N(x),$$

where $L_N(x)$ is the Lagrange interpolating polynomial for f on the nodes.

The proof is given in the Appendix. Here we make some remarks.

- For each value of N, only two conditions need to be tested. Since $G_1 = a_0$, $G_2 = 2a_1$, and $G_3 = 6a_0a_2 - a_1^2$, we recognize here the exceptional cases we already found for $N = 2$ ($a_0 = 0$ or $a_1 = 0$) and for $N = 3$ ($a_1 = 0$ or $6a_0a_2 - a_1^2 = 0$).
- Changing ϵ effectively changes the RBF expansion coefficients

$$\begin{aligned} a_0 &\to a_0, \\ a_1 &\to a_1\epsilon^2, \\ a_2 &\to a_2\epsilon^4, \\ &\vdots \end{aligned}$$

  Forming the G-matrices based on such altered coefficients does not affect the issue of singularity—their determinants will just end up scaled by a power of ϵ (as can be verified by cofactor expansion, for instance).
- Our numerical tests suggest that all the G-matrices are nonsingular for all standard choices of $\phi(r)$. However, we have not been able to find proofs for our observations, including the following.
  - With $\phi(r) = e^{-r^2}$, we get $a_k = (-1)^k/k!$ and $\det(G_1) = 1$, $\det(G_2) = -2$. Subsequent determinants in the sequence satisfy $\det(G_{k+1}) = ((-2)^k/k!)\det(G_{k-1})$.

- With $\phi(r) = \cos r$, only G_1 and G_2 are nonsingular. This is quite certainly linked to the fact that the RBF matrix is then always singular whenever $N > 2$. Just as no more than three parabolas can be linearly independent, no more than two different translates of the cosine function can be independent.

- Suppose the nodes are equispaced (say, unit spaced) over $[-\infty, \infty]$ and that all (sufficiently large) G-matrices are nonsingular. Since the approximation on a finite interval converges to the interpolating polynomial of minimal degree, and we can consider increasingly wide finite intervals, the RBF limit on the infinite interval becomes the sinc interpolant

$$s(x,0) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin \pi(x-k)}{\pi(x-k)}. \tag{3.5}$$

This can be seen by comparing Lagrange's interpolation formula to

$$\frac{\sin \pi x}{\pi x} = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right).$$

This limit was demonstrated for Gaussians in [6] and for multiquadrics in [7]. With $\phi(r) = 1/(1+(\epsilon r)^2)$, the interpolant is known in closed form for all ϵ [8], and the limit $\epsilon \to 0$ can be directly reduced to (3.5).

If, moreover, the data are periodic, the sinc expansion (3.5) becomes the standard lowest-degree trigonometric interpolant, thanks to

$$\sum_{k=-\infty}^{\infty} \frac{\sin \pi(x-kN)}{\pi(x-kN)} = \begin{cases} \dfrac{2}{N}\left[\dfrac{1}{2} + \cos\dfrac{2\pi x}{N} + \cos\dfrac{4\pi x}{N} + \cdots + \cos\dfrac{(N-2)\pi x}{N} + \dfrac{1}{2}\cos \pi x\right], & N \text{ even}, \\ \dfrac{2}{N}\left[\cos\dfrac{\pi x}{N} + \cos\dfrac{3\pi x}{N} + \cdots + \cos\dfrac{(N-1)\pi x}{N}\right], & N \text{ odd}. \end{cases}$$

4. OBSERVATIONS ABOUT 2-D

Our investigations for 2-D limits are still preliminary. Here we will show a few illustrative examples of different limiting behaviors in some simple cases. In the first four examples below, the diagrams to the left show how the nodes were distributed. The limits in the first three cases were calculated analytically (using Mathematica). The fourth case was carried out numerically in arbitrary-precision floating point arithmetic.

EXAMPLE 4. $f(x,y) = x - 2y + 3xy$. See Figure 1.

EXAMPLE 5. $f(x,y) = x - y - 2xy - 2y^2$. See Figure 2.

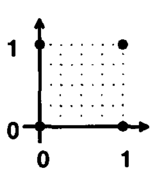

Figure 1. Limit $s(x,y,0) = f(x,y)$ for $\phi(r) = \sqrt{1+(\epsilon r)^2}$ and $\phi(r) = 1/(1+(\epsilon r)^2)$. This is the same as the original function, and is not affected by RBF choice.

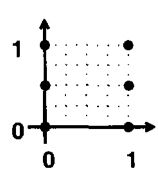

Figure 2. Limit with $\phi(r) = 1/(1+(\epsilon r)^2)$ is $(7/5)x - y - (2/5)x^2 + 2xy - 2y^2$. Limit with $\phi(r) = \sqrt{1+(\epsilon r)^2}$ is different, and very complex algebraically.

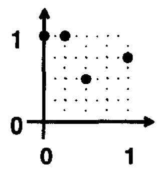

Figure 3. Limit with $\phi(r) = 1/(1+(\epsilon r)^2)$ is $(8485 - 20375x + 4579y + 15228x^2 - 4512xy + 1692y^2)/7378$. Limit with $\phi(r) = \sqrt{1+(\epsilon r)^2}$ is $(8615 - 16345x - 2743y + 11844x^2 - 2256xy + 5076y^2)/5474$. The limits are both quadratics in x and y, but with some differences in their coefficients.

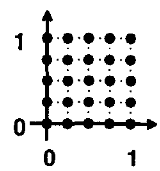

Figure 4. Limit with $\phi(r) = \sqrt{1+(\epsilon r)^2}$ fails to exist—divergence of type $O(\epsilon^{-2})$ as $\epsilon \to 0$. The coefficients of the ϵ^{-2} terms are very small; divergence does not become apparent until ϵ reaches the range of 0.01 to 0.001.

EXAMPLE 6. $f(x,y) = (x+2y)/(3x-y+2)$. See Figure 3.

EXAMPLE 7. $f(x,y)$ arbitrary. See Figure 4.

So far, we have only observed divergence in cases of highly regular grid layouts—never in cases with scattered data points.

5. CONCLUDING REMARKS

In this paper, we have found that the RBF interpolant usually has a well-behaved limit as the basis functions become increasingly flat ($\epsilon \to 0$). In 1-D, conditions which are easily stated and typically satisfied guarantee that the limit is the Lagrange minimal-degree interpolating polynomial. In 2-D, the limit may not exist if the nodes make a tensor-product grid. When a limit does exist, its value clearly depends on $\phi(r)$. All such limits that we have encountered are low-degree polynomials; only the coefficients vary.

The appearance of low-degree polynomials suggests that small values of ϵ will be best when the target function f is well approximated by such a polynomial (for instance, f is so well sampled that just a few Taylor series terms provide a good approximation). This was earlier observed empirically by Carlson and Foley [4].

Since standard finite-difference (FD) methods in 1-D are based on finding the polynomial interpolant and then differentiating it analytically, the $\epsilon \to 0$ limit might be one path to developing FD methods on scattered grids in any dimension. However, there are two serious practical obstacles.

- Tensor-product grids allow a natural refinement process that creates convergence using a fixed FD stencil. This does not seem to be possible on a scattered grid.
- Poor conditioning for small ϵ makes computation of the limit difficult in fixed precision. However, while it has long been clear that computing via the usual path of finding the expansion coefficients is bound to suffer from ill-conditioning, we now also know that the RBF interpolants themselves generally depend smoothly on the input data. This suggests that a more stable algorithm might be feasible.

APPENDIX A
PROOF OF THEOREM 3.1

PROOF. We start with the expansions

$$\phi(r) = a_0 + \epsilon^2 a_1 r^2 + \epsilon^4 a_2 r^4 + \cdots, \tag{A.1}$$

$$\lambda = \epsilon^{-2N+2}\left(\lambda_{-q}\epsilon^{-2q} + \cdots + \lambda_0 + \epsilon^2\lambda_1 + \cdots\right), \tag{A.2}$$

for some integer $q \geq 0$. Equation (A.1) is a definition. (Convergence is assured for small enough ϵ since r is bounded on a fixed node set.) To understand (A.2), recall that $A\boldsymbol{\lambda} = \mathbf{f}$ and that the entries of A can be expanded in even powers of ϵ according to (A.1). It is then clear from Cramer's rule that each entry of $\boldsymbol{\lambda}$ is a rational function of ϵ^2; hence, the expansion (A.2) is possible for a finite q.

Straightforward expansion of (1.1) reveals that

$$s(x,\epsilon) = \epsilon^{-2N+2}\left(\epsilon^{-2q}P_{-q}(x) + \cdots + P_0(x) + \epsilon^2 P_1(x) + \cdots\right), \tag{A.3}$$

where each P_i is a convolution-type polynomial

$$\begin{aligned} P_{-q}(x) &= a_0 \sum_{k=1}^{N} \lambda_{-q,k}, \\ P_{-q+1}(x) &= a_0 \sum_{k=1}^{N} \lambda_{-q+1,k} + a_1 \sum_{k=1}^{N} \lambda_{-q,k}(x-x_k)^2, \\ &\vdots \end{aligned}$$

Polynomial P_{-q+m} has degree at most $2m$. We are about to apply binomial expansion to write out these formulas. To that end, we introduce a notation

$$\sigma_i^{(m)} = \sum_{k=1}^{N} \lambda_{i,k}\, x_k^m.$$

We note that there is a one-to-one correspondence between $\boldsymbol{\lambda}_i$ and the vector

$$\left[\sigma_i^{(0)}\ \sigma_i^{(1)}\ \cdots\ \sigma_i^{(N-1)}\right].$$

In fact, the transformation between the two is just a square Vandermonde matrix for $x_1, \ldots, x_N$.

We now apply the binomial theorem to each of the $(x-x_k)^{2j}$ terms appearing in the polynomials. Separating even and odd powers of x in the result, we find

$$\begin{aligned} P_{-q+m}(x) = \sum_{j=0}^{m} x^{2(m-j)} \sum_{i=0}^{j} a_{m-i} \binom{2(m-i)}{2(j-i)} \sigma_{i-q}^{(2(j-i))} \\ - \sum_{j=0}^{m-1} x^{2(m-j)-1} \sum_{i=0}^{j} a_{m-i} \binom{2(m-i)}{2(j-i)+1} \sigma_{i-q}^{(2(j-i)+1)}. \end{aligned}$$

To make the expression more manageable, we replace the inner sums with inner products. This requires the new definitions

$$\mathbf{b}_{m,j} = \left[\ \binom{2(m-j)}{0} a_{m-j}\quad \binom{2(m-j)+2}{2} a_{m-j+1}\quad \cdots\quad \binom{2m}{2j} a_m\ \right]_{1\times(j+1)}, \tag{A.4}$$

$$\mathbf{v}_j = \left[\ \sigma_{-q+j}^{(0)}\quad \sigma_{-q+j-1}^{(2)}\quad \cdots\quad \sigma_{-q}^{(2j)}\ \right]_{(j+1)\times 1}^{T}, \tag{A.5}$$

$$\mathbf{c}_{m,j} = \left[\ \binom{2(m-j)}{1} a_{m-j}\quad \binom{2(m-j)+2}{3} a_{m-j+1}\quad \cdots\quad \binom{2m}{2j+1} a_m\ \right]_{1\times(j+1)}, \tag{A.6}$$

$$\mathbf{w}_j = \left[\ \sigma_{-q+j}^{(1)}\quad \sigma_{-q+j-1}^{(3)}\quad \cdots\quad \sigma_{-q}^{(2j+1)}\ \right]_{(j+1)\times 1}^{T}. \tag{A.7}$$

We now write

$$P_{-q+m}(x) = \sum_{j=0}^{m} x^{2(m-j)} \mathbf{b}_{m,j}\mathbf{v}_j - \sum_{j=0}^{m-1} x^{2(m-j)-1} \mathbf{c}_{m,j}\mathbf{w}_j.$$

For example,

$$P_{-q}(x) = \mathbf{b}_{0,0}\mathbf{v}_0,$$
$$P_{-q+1}(x) = (\mathbf{b}_{1,0}\mathbf{v}_0)x^2 - (\mathbf{c}_{1,0}\mathbf{w}_0)x + (\mathbf{b}_{1,1}\mathbf{v}_1),$$
$$\vdots$$

If $s(x,\epsilon)$ (as written in (A.3)) is to interpolate f for all ϵ, then

$$P_{-q}, \dots, P_{N-2}, P_N, P_{N+1}, \dots \quad \text{interpolate 0 at } x_1, \dots, x_N; \tag{A.8a}$$
$$P_{N-1} \quad \text{interpolates } f \text{ at } x_1, \dots, x_N. \tag{A.8b}$$

Henceforth, we assume $N = 2n$; the case of odd N differs only slightly. Consider $P_{-q}, \dots, P_{-q+n-1}$. They have maximum degrees $0, 2, \dots, 2(n-1) = N-2$, and each must be zero at N points. Hence, each of these polynomials is identically zero. Looking at the highest-order coefficient of each, we conclude that

$$\begin{bmatrix} \mathbf{b}_{0,0} \\ \mathbf{b}_{1,0} \\ \vdots \\ \mathbf{b}_{n-1,0} \end{bmatrix}_{n \times 1} \mathbf{v}_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

By (A.4), the matrix of this system is precisely the first column of G_{2n-1}, which is guaranteed to be nonsingular by assumption. Therefore, the only solution of this system is

$$\mathbf{v}_0 = 0. \tag{A.9}$$

Now consider the next polynomial, P_{-q+n}. Its leading coefficient is $\mathbf{b}_{n,0}\mathbf{v}_0\, x^{2n}$, which is zero by (A.9). Hence, the degree of P_{-q+n} is no more than $N-1$, and, since it is zero at N points, it is identically zero. If we consider the second-highest terms of $P_{-q+1}, \dots, P_{-q+n}$, we find

$$\begin{bmatrix} \mathbf{c}_{1,0} \\ \mathbf{c}_{2,0} \\ \vdots \\ \mathbf{c}_{n,0} \end{bmatrix}_{n \times 1} \mathbf{w}_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The matrix here is just the first column of G_{2n} (see (A.6)), which is also nonsingular by assumption. So, we conclude

$$\mathbf{w}_0 = 0. \tag{A.10}$$

Collecting the third-highest terms of $P_{-q+1}, \dots, P_{-q+n}$, we see that

$$\begin{bmatrix} \mathbf{b}_{1,1} \\ \mathbf{b}_{2,1} \\ \vdots \\ \mathbf{b}_{n,1} \end{bmatrix}_{n \times 2} \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Here we are using the first *two* columns of G_{2n-1}. They must be independent, so we have

$$\mathbf{v}_1 = 0. \tag{A.11}$$

Equations (A.9)–(A.11) imply that the three highest terms of P_{-q+n+2} must vanish, and thus, it too has degree $\le N-1$, etc. We use this to establish $\mathbf{w}_1 = \mathbf{v}_2 = 0$, which knocks out two more terms of P_{-q+n+3}. This iteration continues up through P_{-q+N-2}, and we can say

$$\begin{aligned} \mathbf{v}_j &= 0, & 0 \le j < n, \\ \mathbf{w}_j &= 0, & 0 \le j < n-1. \end{aligned} \tag{A.12}$$

Now consider P_{-q+N-1}. If $q > 0$, this is also zero at N points by (A.8a), and continuing the above logic leads to

$$\begin{bmatrix} \mathbf{c}_{n,n-1} \\ \mathbf{c}_{n+1,n-1} \\ \vdots \\ \mathbf{c}_{N-1,n-1} \end{bmatrix}_{n\times n} \mathbf{w}_{n-1} = G_{2n}\mathbf{w}_{n-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

so we must conclude $\mathbf{w}_{n-1} = 0$. But then the last entry of $\mathbf{w}_{n-1}$ and all the vectors in (A.12) together imply (refer to (A.5) and (A.7))

$$\sigma_{-q}^{(0)} = \sigma_{-q}^{(1)} = \cdots = \sigma_{-q}^{(N-1)} = 0,$$

which in turn implies $\boldsymbol{\lambda}_{-q} = 0$. In other words, we could have started expansion (A.2) with $q-1$ in place of q. Hence, we are free to assume $q = 0$ in (A.2) without loss of generality.

Thus, $P_{-q+N-1} = P_{N-1}$ must interpolate f at the N nodes, by (A.8b). Since our earlier reasoning implies $\deg(P_{N-1}) \leq N-1$, P_{N-1} must be the Lagrange interpolating polynomial for f. Since $P_m \equiv 0$ for $m < N-1$, expansion (A.3) shows that $s(x,\epsilon) \to P_{N-1}(x) = L_N(x)$ as $\epsilon \to 0$.

REMARK. A side result of the proof is that the condition number of the RBF matrix A must satisfy $\kappa(A) = O(\epsilon^{-2N+2})$. This is clear because there are choices of f such that $\boldsymbol{\lambda}_0 \neq 0$ in (A.3)—in fact, any $f(x)$ for which the Lagrange polynomial has degree exactly $N-1$ will do. This result is in perfect agreement with the data from Table 1, and in this case the bound is tight.

REFERENCES

1. M. Buhmann and N. Dyn, Spectral convergence of multiquadric interpolation, *Proc. Edinburgh Math. Soc.* **36** (2), 319–333, (1993).
2. W.R. Madych and S.A. Nelson, Error bounds for multiquadric interpolation, In *Approximation Theory VI, Volume II*, College Station, TX, 1989, pp. 413–416, Academic Press, Boston, MA, (1989).
3. W.R. Madych and S.A. Nelson, Multivariate interpolation and conditionally positive definite functions. II, *Math. Comp.* **54**, 211–230, (1990).
4. R.E. Carlson and T.A. Foley, The parameter R^2 in multiquadric interpolation, *Computers Math. Applic.* **21** (9), 29–42, (1991).
5. S. Rippa, An algorithm for selecting a good value for the parameter c in radial basis function interpolation, *Adv. Comp. Math.* **11**, 193–210, (1999).
6. S.D. Riemenschneider and N. Sivakumar, Gaussian radial-basis functions: Cardinal interpolation of ℓ^p and power-growth data, *Adv. Comp. Math.* **11**, 229–251, (1999).
7. B.J.C. Baxter, The asymptotic cardinal function of the multiquadratic $\phi(r) = (r^2 + c^2)^{1/2}$ as $c \to \infty$, *Computers Math. Applic.*, Special Issue: "Advances in the Theory and Applications of Radial Basis Functions" **24** (12), 1–6, (1992).
8. B. Fornberg, T.A. Driscoll, G. Wright and R. Charles, Observations on the behavior of radial basis functions near boundaries, *Computers Math. Applic.*, (this issue).