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- Define probabilities as the long-run frequency of events.
Bayesian vs. Frequentist

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- Compute *point* estimates (e.g. maximum likelihood).
- Define probabilities as the long-run frequency of events.

Bayesian statistics:
- Probabilities are a “state of knowledge” or a “state of belief”.
- Parameters have a probability distribution.
- Prior knowledge is updated in the light of new data.
Example

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- **Bayesian approach:** we treat \( p \) as a *random variable*.
  1. Choose a *prior* distribution for \( p \), say \( P(p) \).
  2. Update the prior distribution using the data via *Bayes’ theorem*:

  \[
P(p|\text{data}) = \frac{P(\text{data}|p)P(p)}{P(\text{data})} \propto P(\text{data}|p)P(p).
  \]
Note: “data\mid p” \sim \text{Binomial}(14, p).
Example (cont.)

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The beta distribution $\text{Beta}(\alpha, \beta)$:

$$P(p; \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} p^{\alpha - 1} (1 - p)^{\beta - 1} \quad (p \in (0, 1)).$$

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Conclusion: $P(p|data) \sim \text{Beta}(10 + \alpha, 4 + \beta)$. 

Example (cont.)

- How should we choose $\alpha, \beta$?

- According to our prior knowledge of $p$.

- Suppose we have no prior knowledge: use a prior: $\alpha = \beta = 1$ (Uniform distribution).

- The resulting posterior distribution is $p \mid \text{data} \sim \text{Beta}(11, 5)$:

- Our knowledge of $p$ has now been updated using the observed data (or evidence).

- Important advantage: Our estimate of $p$ comes with its own uncertainty.
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![Beta distribution graph]

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**Important advantage:** Our estimate of $p$ comes with its own uncertainty.
More generally: suppose we have a model for $X$ that depends on some parameters $\theta$. Then:

1. Choose a prior $P(\theta)$ for $\theta$.
2. Compute the posterior distribution of $\theta$ using $P(\theta | X) \propto P(X | \theta) \cdot P(\theta)$.

Note: Posterior = Prior $\times$ Likelihood.

Advantages:
- Mimics the scientific method: formulate hypothesis, run experiment, update knowledge.
- Can incorporate prior information (e.g. the range of variables).
- Automatically provides uncertainty estimates.

Drawbacks:
- Not always obvious how to choose priors.
- Can be difficult to compute the posterior distribution.
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<table>
<thead>
<tr>
<th>Likelihood</th>
<th>Conjugate prior</th>
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<tbody>
<tr>
<td>Binomial</td>
<td>Beta</td>
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<tr>
<td>Multinomial</td>
<td>Dirichlet</td>
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<tr>
<td>Poisson</td>
<td>Gamma</td>
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<tr>
<td>Normal</td>
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<tr>
<td>$\mu$ unknown, $\sigma^2$ known</td>
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<tr>
<td>$\mu$ known, $\sigma^2$ unknown</td>
<td>Inverse Chi-Square</td>
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One generally then compute some statistics of the sample (e.g. mean, variance, mode, etc.).
A simple way to sample from a distribution:

1. Draw $z \sim h(x)$ and $u \sim \text{Uniform}[0, 1]$.
2. If $u < \frac{f(z)}{c \cdot g(z)}$ accept the draw. Otherwise, discard $z$ and repeat.
A simple way to sample from a distribution:

- We want to sample from a distribution $f(x)$ (complicated).
- We know how to sample from another distribution $g(x)$ (simpler).
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Works well in some cases, but the rejection rate is often large and the resulting algorithm can be very inefficient.
Nicolas Metropolis (1915–1999) was an American physicist. He worked on the first nuclear reactors at the Los Alamos National Laboratory during the second world war. Introduced the algorithm in 1953 in the paper

*Equation of State Calculations by Fast Computing Machines*

with A. Rosenbluth, M. Rosenbluth, A. Teller, and E. Teller

W. K. Hastings (Born 1930) is a Canadian statistician who extended the algorithm to the more general case in 1970.
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We assume

- we can evaluate $f(x)$ at every $x$.
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Some difficulties:
- Choosing an efficient proposal distribution $q(x, y)$.
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- Note: only possible when the conditional distributions are “nice”.

Suppose $X = (X_1, \ldots, X_p)$ and $X^{(i)} = (x_1^{(i)}, \ldots, x_p^{(i)})$ is a given sample. Generate a new sample $X^{(i+1)} = (x_1^{(i+1)}, \ldots, x_p^{(i+1)})$ as follows:

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