Markov chains

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- A (discrete time) **Markov chain** is a discrete stochastic process \( \{X_n : n = 0, 1, \ldots \} \) such that
  
  For all \( i,j,i_0,\ldots,i_{n-1} \in S \), and all \( n \geq 0 \):
  \[
  P(X_{n+1} = j | X_0 = i_0, \ldots, X_n = i_{n-1}) = P(X_{n+1} = j | X_n = i).
  \]
  
  Interpretation: Given the present \( X_n \), the future \( X_{n+1} \) is independent of the past \( (X_0, \ldots, X_{n-1}) \).
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- The elements of $S$ are called the **states** of the Markov chain.
- When $X_n = j$, we say that the process is in state $j$ at time $n$. 
A Markov chain is **homogeneous** (or **stationary**) if for all \( n \geq 0 \) and all \( i, j \in S \),

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P(X_{n+1} = j \mid X_n = i) = P(X_1 = j \mid X_0 = i) =: p(i, j).
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\forall i, j \in S, \ p(i, j) \geq 0, \quad \text{and} \quad \forall i \in S, \ \sum_{j \in S} p(i, j) = 1.
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- Conversely, every stochastic matrix is the transition matrix of some homogeneous discrete time Markov chain.
Example 1: (Two-state Markov chain)

\[ S = \{0, 1\}, \quad p(0, 1) = a, \quad p(1, 0) = b, \quad a, b \in [0, 1] \]

\[ P = \begin{pmatrix} 1 - a & a \\ b & 1 - b \end{pmatrix}. \]
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We naturally represent \( P \) using a transition (or state) diagram:

Interpretation: machine is either broken (0) or working (1) at start of \( n \)-th day.
Example 2: (Simple random walk) Let $\xi_1, \xi_2, \xi_3, \ldots$ be iid random variables such that $\forall i \geq 1$,

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\xi_i = \begin{cases} 
+1 & P(\xi_i = +1) = p \\
0 & P(\xi_i = 0) = r \\
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\end{cases},
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where $p + r + q = 1$, $p, r, q \geq 0$. 

Let $X_0$ be an integer valued random variable independent of the $\xi_i$'s. Define $\forall n \geq 1$, 

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X_n = X_0 + n \sum_{i=1}^{n} \xi_i.
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**n-step transitions**
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**Moral:** Distributional computations for Markov Chains are just matrix multiplications.
Reducibility:

A state \( j \in S \) is said to be **accessible** from \( i \in S \) (denoted \( i \rightarrow j \)) if a system started in state \( i \) has a non-zero probability of transitioning into state \( j \) at some point.

A Markov chain is said to be **irreducible** if its state space is a single communicating class.
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Transience and periodicity

- **Transience:**
  - A state \( i \in S \) is said to be **transient** if, given that we start in state \( i \), there is a non-zero probability that we will never return to \( i \).

- **Periodicity:**
  - A state \( i \in S \) has period \( k \) if \( k = \gcd \{ n > 0 : P(X_n = i | X_0 = i) > 0 \} \).
  - For example, suppose you start in state \( i \) and can only return to \( i \) at time 6, 8, 10, 12, etc. Then the period of \( i \) is 2.
  - If \( k = 1 \), then the state is said to be **aperiodic**.
  - A Markov chain is **aperiodic** if every state is aperiodic.
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Limiting behavior of Markov chains: What happens to $p^n(i, j)$ as $n \to \infty$?
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Example: (The two-state Markov chain)

If $(a, b) \neq (0, 0)$, we have (exercise):

\[
P^n = \frac{1}{a+b} \begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{(1-a-b)^n}{a+b} \begin{pmatrix} a & -a \\ -b & b \end{pmatrix}.
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$$P^n = \frac{1}{a + b} \begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{(1 - a - b)^n}{a + b} \begin{pmatrix} a & -a \\ -b & b \end{pmatrix}.$$

Thus, if $(a, b) \neq (0, 0)$ and $(a, b) \neq (1, 1)$, then

$$\lim_{n \to \infty} p^n(0, 0) = \lim_{n \to \infty} p^n(1, 0) = \frac{b}{a + b},$$
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Thus, the chain has a limiting distribution. The limiting distribution is independent of the initial state.
Recall: $\mu_{n+1} = \mu_n P$. 

A vector $\pi = (\pi_i : i \in S)$ is said to be a stationary distribution for a Markov chain $\{X_n : n \geq 0\}$ if

1. $0 \leq \pi_i \leq 1 \forall i \in S$.
2. $\sum_{i \in S} \pi_i = 1$.
3. $\pi = \pi P$, where $P$ is the transition probability matrix of the Markov chain.

Remark: In general, a stationary distribution may not exist or be unique.

Theorem: Let $\{X_n : n \geq 0\}$ be an irreducible and aperiodic Markov chain where each state is positive recurrent. Then

1. The chain has a unique stationary distribution $\pi$.
2. For all $i \in S$, $\lim_{n \to \infty} P(X_n = i) = \pi(i)$.
3. $\pi(i) = \mathbb{E}[T_i]$. 

$\pi(i)$ can be interpreted as the average proportion of time spent by the chain in state $i$. 

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