The Universe and Existence

The universal and existential quantifiers

You can pass some of the people all of the time, or
you can pass all of the people some of the time, but
you cannot pass all of the people all of the time.

(Think of this as my teaching philosophy!)

There are two types of statements which often occur in mathematics: one, in effect, states that a certain type of object exists; while the other says that something holds for all objects in a given class. A mathematical example of each is:

Existential quantifier: There exists an integer n whose square can be written as the difference of two squares in at least 3 ways.

Universal quantifier: For all integers n, $n^2 > 0$. (Read as "The square of every integer is positive".)

Note that you can (and often do) nest quantifiers quite naturally:

For all complex numbers a, b, there exists exactly two complex solutions to the equation $x^2 + ax + b = 0$.

Truth values for "There exist" and "For all" are pretty much as we would conclude from a consideration of our understanding of the language.

For a "there exists" statement to be true you need only produce a *single example* satisfying the claim. So you realize the above statement is true when you work out $n = 12$ satisfies the claim (to convince yourself of this, you should play around with Pythagorean triples – that is the sides of right angle triangles where one of the sides has length 12).

For a "for all" statement to be true you need to show the claim holds *for every possibility* outlined. In the example above, the statement is clearly false as $n = 0$ obviously fails the claim. So despite the fact the claim holds for every other integer, the universally quantified example above is false because it does not hold for every possibility.

Let us now return to the "teaching philosophy" outlined above. We want to write this in terms of existential and universal quantifiers.

The symbol for "there exists" is $\exists$ while the symbol for "for all" is $\forall$. Of course, this doesn't quite get us there as we need to think a little bit about just where and when we are using each of the two quantifiers. Lets start with just the first line...

You can pass some of the people all of the time, or...

If you consider carefully what this statement is saying, you'll realize that an equivalent statement is

There exist people who for all exams, they pass, or...

To get a little closer to the true symbolic logic form, we could write

$$\exists \langle \text{people} \rangle \ \forall \ \langle \text{exams they sit} \rangle, \text{ they pass, } \vee \ldots$$

To complete the translation to a completely mathematical form we need some knowledge of set theory (our next topic), so for now we'll leave our translation as it stands. You should now attempt to translate the remaining two lines following similar principles. Hopefully you should end up with the following partial translation:

$$\Big[(\exists\langle\text{people}\rangle\forall\langle\text{exams}\rangle,\text{ they pass})\vee$$
$$(\forall\langle\text{people}\rangle\exists\langle\text{exams}\rangle,\text{ they pass})\Big]\wedge$$
$$\neg(\forall\langle\text{people}\rangle\forall\langle\text{exams}\rangle,\text{ they pass}).$$

The last line brings us to our last point of discussion in logic. What does the negation of a quantifier do to the quantifier? As with negation of "and" and "or" statements, if you think about when the negation of the quantifier is true you should come up with the right answer.

For "not (there exists...)" to be true, the existence claim must be false, which can only happen if the claim is false for every possibility. In other words, the negation of an existential quantifier is a universal quantifier. The one remaining question is what sort of claim is to be made. We want to say "for all" possibilities under scrutiny, the claim is false. In other words, we are claiming *the opposite* of the claim is true. So to negate an existential quantifier, we replace ∃ with ∀ and negate the claim. Using the mathematical example from above: the negation of

<u>There exists</u> an integer n whose square can be written as the difference of two squares in at least 3 ways

we would get

<u>For all</u> integers n, you *cannot* write n^2 as the difference of two squares in at least 3 ways

or possibly

<u>For all</u> integers n, you can write n^2 as the difference of two squares in at most 2 ways.

For "not (for all...)" to be true, the universal claim must be false in at least one instance. So in this case, the negation of a universal quantifier results in an existential quantifier. As with the negation of existential quantifier, you also need to negate the claim. So the negation of

<u>For all</u> integers n, $n^2 > 0$

is the statement

<u>There exists</u> an integer n such that $n^2 \leq 0$.

One must take extra care when negating claims – a common error for negating this last claim is to write $n^2 < 0$ which is not the opposite of $n^2 > 0$. Always think carefully about how you are negating.