Directionless?

Direct and Indirect

Two basic approaches to proving an implication

So far you've seen how to prove "if, then" statements – *assume hypothesis, prove conclusion* – and how an implication is logically equivalent to its contrapositive. This is essentially describing not one but two methods for proving an implication.
If you prove an implication directly by assuming hypothesis, proving conclusion, then this is called a *direct proof.*
If, on the other hand, you prove an implication by proving the contrapositive, then this is a type of *indirect proof.*
You'll no doubt have noticed, perhaps with some trepidation, the word "type" in that last sentence. Yes, it does mean what you suspect (and fear?!).
There is a second type of indirect proof, known as *proof by contradiction* or *reductio ad absurdum.*
Both indirect methods start by assuming something is false.
Suppose we want to prove $p \to q$.
For *proof by contrapositive*, we prove $\neg q \to \neg p$, so we begin by assuming q is false.
For *proof by contradiction*, we begin by assuming $p \to q$ is false. We then aim to derive two statements which contradict one another. This contradiction will follow from some logical (and correct!) argument, so that the only possible error in our argument must come from the original step, our assumption, in which case our assumption must have been incorrect. The contradiction you generate will often involve a statement you already know is true from somewhere else or (as is more often the case) directly from the initial assumption.

Question: What are we really assuming when we assume $p \to q$ is false?

Well, $p \to q$ is false in only one situation – when p is true and q is false. So this is what we assume at the start of every proof by contradiction.

Two Indirect Proofs

We illustrate the two methods by proving the following two statements:

(A) If x is a positive irrational number, then $\sqrt{x}$ is irrational.

(B) If $x \neq 0$ is rational and y is irrational, then xy is irrational.

Proof of (A) by contrapositive (you should write out what the contrapositive is). Suppose $\sqrt{x}$ is rational. Then there exist integers a, b with $b \neq 0$ such that $\sqrt{x} = \frac{a}{b}$. It follows that

$$x = (\sqrt{x})^2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2},$$

so that x is rational, as required. $\square$

Proof of (B) by contradiction. Suppose $\left\{ \begin{array}{c} \text{to the contrary} \\ \text{by way of contradiction} \end{array} \right\}$ that (B) is false. That is, suppose there exists a rational number $x \neq 0$ and irrational number y where xy is rational. Then there exists integers a, b, c, d, with a, b, d all non-zero, such that $x = \frac{a}{b}$ and $xy = \frac{c}{d}$. We have

$$\frac{c}{d} = xy = \frac{a}{b}y,$$

so that $y = \frac{bc}{ad}$ with bc, ad integers and $ad \neq 0$. But this means y is rational, contradicting our assumption that y was irrational. The only possible error in our argument is our original assumption, so that must have been incorrect; that is (B) must be true. $\square$

There are 3 points you should take away from these two proofs.
Firstly, in both cases we relied heavily on definitions after the initial assumption – this is very common.
Secondly, the contradiction generated in our proof of (B) was a contradiction on our original (and as it turned out, false) assumption. This might seem a problem – but its really ok; the point is that in mathematics you cannot have a statement which is simultaneously true and false. The fact we could generate two contradictory statements came directly from us making a false assumption, and so there is really only one false step in our line of reasoning.
Thirdly, anything you prove after the initial assumption in a proof by contradiction is not necessarily true, but it is not necessarily false either. A proof by contradiction only provides you with a proof of the original statement – you cannot trust any statement you derive during the proof as it *may be* dependent on the false assumption you started with.