Not (To be, or not to be)

The nature of negating

We now revisit something we touched on in the last lecture – what happens when we negate “and” or “or”. We’ll also touch on double negation and what a tautology or contradiction is. To begin, let us turn our title into a statement written in symbolic logic.

We’ll denote “To be” by p.

So “Not to be” is $\neg p$.

In symbolic logic, we use $\vee$ for “or” and $\wedge$ for “and”. So “To be, or not to be” would be $p \vee \neg p$.

Finally, to negate Hamlet as we do in the title, we would write

$$\neg(p \vee \neg p).$$

So now we have the title in symbolic logic. The thing is this is clearly not the simplest way to write this – there’s no way we would have spoken the title as it is written. So let us think a little more about this.

Firstly, we all instinctively understand what the truth value of “p or q” (symbolically $p \vee q$) is and we can easily write down the truth table for this:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

With this understanding now recorded, we can now construct a truth table for $p \vee \neg p$ (“To be or not to be”) and find

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

What is immediately apparent from this table is that $p \vee \neg p$ is always true, regardless of the truth value of p. Such a statement is known as a *tautology* – a statement which is always true, regardless of the truth values of its component parts. Similarly, a *contradiction* is a statement which is always false, regardless of the truth values of its component parts. Hopefully, it shouldn’t surprise you at this point that $\neg(p \vee \neg p)$ is indeed a contradiction (try extending the truth table by adding a column for $\neg(p \vee \neg p)$).

(Note, that as far as symbolic logic is concerned, Hamlet was talking utter nonsense when he said “To be, or not to be: that is the question” for in logical terms he was claiming a tautology was questionable.)

So far we haven’t managed to simplify our symbolic expression, so let us now think a little more deeply about the negation of an “or” statement. (In the last lecture we did a similar process for “and” and you may recall finding this a little taxing.) What is another way of saying

Not (p or q) ?

If you think carefully, you should come up with

not p and not q.

One way to see this is to consider when is $\neg(p \vee q)$ true. You should quickly come to the realization that this is true only when p and q are false – i.e. when not p and not q are true, and this leads to the correct negation of “or”.

Likewise, what is another way of saying

$$\text{not } (p \text{ and } q)?$$

More careful thought should lead you to

$$\text{not } p \text{ or not } q.$$

To see that symbolic logic matches our understanding of language, you should once again construct a truth table. If you do, you should find that

$$\neg(p \vee q) \text{ and } \neg p \wedge \neg q$$

are logically equivalent, as are

$$\neg(p \wedge q) \text{ and } \neg p \vee \neg q$$

Knowing all of this, we can now simplify our original statement from

$$\neg(p \vee \neg p) \text{ to } \neg p \wedge \neg\neg p.$$

There is one final point – what does double negation achieve? The opposite of the opposite of a truth value is... the original truth value. So $\neg\neg p = p$ and we find out title could really be written as $\neg p \wedge p$. Since order of components doesn't matter for an "and" statement, we could even write $p \wedge \neg p$. So in summary,

Not (To be, or not to be)

means the same (to mathematicians) as

To be, and not to be.