Eigenvalues and edge-connectivity of regular graphs

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Abstract

In this paper, we show that if the second largest eigenvalue of a $d$-regular graph is less than $d - \frac{2(k-1)}{d+1}$, then the graph is $k$-edge-connected. When $k$ is 2 or 3, we prove stronger results. Let $\rho(d)$ denote the largest root of $x^3 - (d - 3)x^2 - (3d - 2)x - 2 = 0$. We show that if the second largest eigenvalue of a $d$-regular graph $G$ is less than $\rho(d)$, then $G$ is 2-edge-connected and we prove that if the second largest eigenvalue of $G$ is less than $\frac{d-3+\sqrt{(d+3)^2-16}}{2}$, then $G$ is 3-edge-connected.

1 Introduction

Let $\kappa(G)$ and $\kappa'(G)$ denote the vertex- and edge-connectivity of a connected graph $G$. If $\delta$ is the minimum degree of $G$, then $1 \leq \kappa(G) \leq \kappa'(G) \leq \delta$. Let $L = D - A$ be the Laplacian matrix of $G$, where $D$ is the diagonal degree matrix and $A$ is the adjacency matrix of $G$. We denote by $0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_n$ the eigenvalues of the Laplacian of $G$. The complement of $G$ is denoted by $\overline{G}$. A graph is called disconnected if it is not connected. The join $G_1 \vee G_2$ of two vertex-disjoint graphs $G_1$ and $G_2$ is the graph formed from the union of $G_1$ and $G_2$ by joining each vertex of $G_1$ to each vertex of $G_2$.

A classical result in spectral graph theory due to Fiedler [7] states that

$$\kappa(G) \geq \mu_2(G)$$

for any non-complete graph $G$. Fiedler called $\mu_2(G)$ the algebraic connectivity of $G$ and his work stimulated a large amount of research in spectral graph theory over the last forty years (see [1, 11, 13, 15]). In [12], Kirkland, Molitierno, Neumann and Shader characterize the equality case in Fiedler’s inequality (1).

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Theorem 1.1 (Kirkland-Molitierno-Neumann-Shader [12]). Let $G$ be a non-complete connected graph on $n$ vertices. Then $\kappa(G) = \mu_2(G)$ if and only if $G = G_1 \vee G_2$ where $G_1$ is a disconnected graph on $n - \kappa(G)$ vertices and $G_2$ is a graph on $\kappa(G)$ vertices with $\mu_2(G_2) \geq 2\kappa(G) - n$.

Eigenvalue techniques have been also used recently by Brouwer and Koolen [5] to show that the vertex-connectivity of a distance-regular graph equals its degree (see also [2, 4] for related results).

In this paper, we study the relations between the edge-connectivity and the second largest eigenvalue of a $d$-regular graph. Let $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ denote the eigenvalues of the adjacency matrix of $G$. If $G$ is $d$-regular, then $\lambda_i(G) = d - \mu_i(G)$ for any $1 \leq i \leq n$. Thus, $\lambda_1(G) = d$ and $G$ is connected if and only if $\lambda_2(G) < d$.

Chandran [6] proved that if $G$ is a $d$-regular graph of order $n$ and $\lambda_2(G) < d - 1 - \frac{d}{n-d}$, then $\kappa'(G) = d$ and the only disconnecting edge-cuts are trivial, i.e., $d$ edges adjacent to the same vertex. Krivelevich and Sudakov [13] showed that $\lambda_2(G) \leq d - 2$ implies $\kappa'(G) = d$. If $G$ is a $d$-regular graph on $n$ vertices and $n \leq 2d + 1$, then $\kappa'(G) = d$ regardless of the eigenvalues of $G$ (see also the proof of Lemma 1.3). Both results are based on the following well-known lemma (see [15] for a short proof).

Lemma 1.2. If $G = (V, E)$ is a connected graph of order $n$ and $S$ is a subset of vertices of $G$, then
\[ e(S, V \setminus S) \geq \frac{\mu_2|S|(n - |S|)}{n} \]
where $e(S, V \setminus S)$ denotes the number of edges between $S$ and $V \setminus S$.

We extend and improve the previous results as follows. We prove the following sufficient condition for the $k$-edge-connectivity of a $d$-regular graph for any $2 \leq k \leq d$.

Theorem 1.3. If $d \geq k \geq 2$ are two integers and $G$ is a $d$-regular graph such that $\lambda_2(G) \leq d - \frac{(k-1)n}{(d+1)(n-d-1)}$, then $\kappa'(G) \geq k$.

When $k = d$, this result states that if $\lambda_2(G) \leq d - \frac{(d-1)n}{(d+1)(n-d-1)}$, then $\kappa'(G) = d$. We get a small improvement for $d$ even because $\kappa'(G)$ must be even in this case (see Lemma 3.1). When $d$ is even, Theorem 1.3 shows that $\lambda_2(G) \leq d - \frac{(d-2)n}{(d+1)(n-d-1)}$ implies $\kappa'(G) = d$. A simple calculation reveals that these bounds improve the previous result of Chandran.

When $n \geq 2d + 2$, Theorem 1.3 implies that if $G$ is a $d$-regular graph with $\lambda_2(G) \leq d - 2 + \frac{4}{d+1}$, then $\kappa'(G) = d$. When $d$ is even, the right hand-side of the previous inequality can be replaced by $d - 2 + \frac{6}{d+1}$. These results improve the previous bound of Krivelevich and Sudakov. Note that there are many $d$-regular graphs $G$ with $d - 2 < \lambda_2(G) \leq d - 2 + \frac{4}{d+1}$. An example is presented in Figure 1.

For $k \in \{2, 3\}$ we further improve these results as shown below. Note that the edge-connectivity of a connected, regular graph of even degree is even (see Lemma 3.1).

Theorem 1.4. Let $d \geq 3$ be an odd integer and $\rho(d)$ denote the largest root of $x^3 - (d-3)x^2 - (3d-2)x - 2 = 0$. If $G$ is a $d$-regular graph such that $\lambda_2(G) < \rho(d)$, then $\kappa'(G) \geq 2$. 

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The value of $\rho(d)$ is about $d - \frac{2}{d+5}$.

**Theorem 1.5.** Let $d \geq 3$ be any integer. If $G$ is a $d$-regular graph such that

$$\lambda_2(G) < \frac{d - 3 + \sqrt{(d + 3)^2 - 16}}{2}$$

then $\kappa'(G) \geq 3$.

The value of $\frac{d - 3 + \sqrt{(d + 3)^2 - 16}}{2}$ is about $d - \frac{4}{d+3}$.

We show that Theorem 1.4 and Theorem 1.5 are best possible in the sense that for each odd $d \geq 3$, there exists a $d$-regular graph $X_d$ such that $\lambda_2(X_d) = \rho(d)$ and $\kappa'(X_d) = 1$. Also, for each $d \geq 3$, there exists a $d$-regular graph $Y_d$ such that $\lambda_2(Y_d) = \frac{d - 3 + \sqrt{(d + 3)^2 - 16}}{2}$ and $\kappa'(Y_d) = 2$.

The following result is an immediate consequence of Theorem 1.4 and Theorem 1.5.

**Corollary 1.6.** If $G$ is a 3-regular graph and $\lambda_2(G) < \sqrt{5} \approx 2.23$, then $\kappa'(G) = 3$.

If $G$ is 4-regular graph and $\lambda_2 < \frac{1 + \sqrt{23}}{2} \approx 3.37$, then $\kappa'(G) = 4$.

Our results imply that if $G$ is a 3-regular graph with $\lambda_2(G) < \sqrt{5}$, then $\kappa(G) = 3$. This is because for 3-regular graphs the vertex- and the edge-connectivity are the same. It is very likely that the eigenvalues results for $\kappa(G)$ and $\kappa'(G)$ can be quite different. We comment on this in the last section of the paper.

## 2 Proof of Theorem 1.3

In this section, we give a short proof of Theorem 1.3. Recall that a partition $V_1 \cup \cdots \cup V_l = V(G)$ of the vertex set of a graph $G$ is called equitable if for all $1 \leq i, j \leq l$, the number of neighbours in $V_j$ of a vertex $v$ in $V_i$ is a constant $b_{ij}$ independent of $v$ (see Section 9.3 of [8] or [9] for more details on equitable partitions).

**Proof of Theorem 1.3.** We prove the contrapositive, namely we show that if $G$ is a $d$-regular graph such that $\kappa'(G) \leq k - 1$, then $\lambda_2(G) > d - \frac{(k-1)n}{(d+1)(n-d-1)}$. Let $V_1 \cup V_2 = V(G)$ be
a partition of \( G \) such that \( r = e(V_1, V_2) \leq k - 1 \leq d - 1 \). Note that if \( |V_1| \leq d \), then \( d - 1 \geq e(V_1, V_2) \geq |V_1|/(d - |V_1| + 1) \geq d \) which is a contradiction. Thus, \( n_i := |V_i| \geq d + 1 \) for each \( i = 1,2 \). Since \( n_1 + n_2 = n \), this implies that \( n_1n_2 \geq (d+1)(n-d-1) \). The quotient matrix of the partition \( V_1 \cup V_2 = V(G) \) (see [8, 9, 10]) is
\[
A_2 = \begin{bmatrix} d - \frac{r}{n_1} & \frac{r}{n_2} \\ \frac{r}{n_1} & d - \frac{r}{n_2} \end{bmatrix}
\] (2)
The eigenvalues of \( A_2 \) are \( d \) and \( d - \frac{r}{n_1} - \frac{r}{n_2} \). Eigenvalue interlacing (see [8, 9, 10]), \( r \leq k - 1 \) and \( n_1n_2 \geq (d+1)(n-d-1) \) imply that
\[
\lambda_2(G) \geq d - \frac{r}{n_1} - \frac{r}{n_2} \geq d - \frac{(k-1)n}{n_1n_2} \geq d - \frac{(k-1)n}{(d+1)(n-d-1)}
\] (3)
We actually have strict inequality here. Otherwise, the partition \( V_1 \cup V_2 \) would be equitable. This would mean that each vertex of \( V_1 \) has the same number of neighbours in \( V_2 \) which is impossible since there are vertices in \( V_1 \) without a neighbour in \( V_2 \). The proof is finished. \( \square \)

3 Proof of Theorem 1.4

In this section, we present the proof of Theorem 1.4. First, we show that any connected, regular graph of even degree has edge-connectivity larger than two. This follows from the next lemma.

Lemma 3.1. Let \( G \) be a connected \( d \)-regular graph. If \( d \) is even, then \( \kappa'(G) \) is even.

Proof. Let \( V = A \cup B \) be a partition of the vertex set of \( G \) such that \( e(A, B) = \kappa'(G) \), where \( e(A, B) \) denotes the number of edges with one endpoint in \( A \) and one endpoint in \( B \). Summing up the degrees of the vertices in \( A \), we obtain that \( d|A| = 2e(A) + e(A, B) = 2e(A) + \kappa'(G) \), where \( e(A) \) denotes the number of edges with both endpoints in \( A \). Since \( d \) is even, this implies \( \kappa'(G) \) is even which finishes the proof. \( \square \)

For the rest of this section, we assume that \( d \) is an odd integer with \( d \geq 3 \). We describe first the \( d \)-regular graph \( X_d \) having \( \kappa'(X_d) = 1 \) and \( \lambda_2(X_d) = \rho(d) \).

If \( H_1, \ldots, H_k \) are pairwise vertex-disjoint graphs, then we define the product \( H_1 \vee H_2 \vee \cdots \vee H_k \) recursively as follows. If \( k = 1 \), the product is \( H_1 \). If \( k = 2 \), then \( H_1 \vee H_2 \) is the usual join of \( H_1 \) and \( H_2 \), i.e. the graph formed from the union of \( H_1 \) and \( H_2 \) by joining each vertex of \( H_1 \) to each vertex of \( H_2 \). If \( k \geq 3 \), then \( H_1 \vee H_2 \vee \cdots \vee H_k \) is the graph obtained by taking the union of \( H_1 \vee H_2 \vee \cdots \vee H_{k-1} \) and \( H_k \) and joining each vertex of \( H_{k-1} \) to each vertex of \( H_k \).

The extremal graph \( X_d \) is defined as:
\[
X_d = K_2 \vee \overline{M_{d-1}} \vee K_1 \vee K_1 \vee \overline{M_{d-1}} \vee K_2
\] (4)
where \( \overline{M_{d-1}} \) is the unique \((d-3)\)-regular graph on \( d-1 \) vertices (the complement of a perfect matching \( M_{d-1} \) on \( d-1 \) vertices). Figure 2 shows \( X_3 \) and Figure 3 describes \( X_5 \).
The graph $X_d$ is $d$-regular, has $2d + 4$ vertices and its edge-connectivity is 1. It has an equitable partition of its vertices into six parts which is described by the following quotient matrix $\tilde{X}_d$. The sizes of the parts in this equitable partition are $2, d - 1, 1, 1, d - 1, 2$.

$$\tilde{X}_d = \begin{bmatrix}
1 & d - 1 & 0 & 0 & 0 & 0 \\
2 & d - 3 & 1 & 0 & 0 & 0 \\
0 & d - 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & d - 1 & 0 \\
0 & 0 & 0 & 1 & d - 3 & 2 \\
0 & 0 & 0 & 0 & d - 1 & 1
\end{bmatrix}$$ (5)

Next, we show that the second largest eigenvalue of $X_d$ equals the second largest eigenvalue of $\tilde{X}_d$. This will enable us to get sharp estimates for $\lambda_2(X_d)$ which will be used later in this section.

![Figure 2: $X_3$ is 3-regular with $\lambda_2(X_3) = \rho(3) \approx 2.7784$](image)

**Lemma 3.2.** For each odd integer $d \geq 3$, we have that $\lambda_2(X_d) = \lambda_2(\tilde{X}_d) = \rho(d)$.

**Proof.** Since the partition with quotient matrix $\tilde{X}_d$ is equitable, it follows that the spectrum of $X_d$ contains the spectrum of $\tilde{X}_d$. Using Maple, we determine the characteristic polynomial of $\tilde{X}_d$:

$$P_{\tilde{X}_d}(x) = (x - d)(x - 1)(x + 2)(x^3 - (d - 3)x^2 - (3d - 2)x - 2)$$ (6)

The roots of $P_{\tilde{X}_d}$ are $d, 1, -2$ and the three roots of $Q(x) = x^3 - (d - 3)x^2 - (3d - 2)x - 2$. Denote by $\tilde{\lambda}_2 \geq \tilde{\lambda}_3 \geq \tilde{\lambda}_4$ the roots of $Q(x)$. Since these roots sum up to $d - 3 \geq 0$ and their product is 2, it follows that $\tilde{\lambda}_2$ is positive and both $\tilde{\lambda}_3$ and $\tilde{\lambda}_4$ are negative. Because $Q(1) = -4d + 4 < 0$, we deduce that $\tilde{\lambda}_2 > 1$. Thus, $\lambda_2(X_d) = \tilde{\lambda}_2 = \rho(d)$.

Let $W \subset \mathbb{R}^{2d+4}$ be the subspace of vectors which are constant on each part of the six part equitable partition. The lifted eigenvectors corresponding to the six roots of $P_{\tilde{X}_d}$ form a basis for $W$. The remaining eigenvectors in a basis of eigenvectors for $X_d$ can be chosen to be perpendicular to the vectors in $W$. Thus, they may be chosen to be perpendicular to the characteristic vectors of the parts in the six-part equitable partition since these characteristic vectors form a basis for $W$. This implies that these eigenvectors will correspond to the non-trivial eigenvalues of the graph obtained as a disjoint union of $2K_2$ and $2M_{d-1}$. The corresponding eigenvalues will be $(-2)^{(d-3)}, (-1)^2, 0^{(d-1)}$, where the exponent denotes the multiplicity of each eigenvalue. These $2d - 2$ eigenvalues together with the six roots of $P_{\tilde{X}_d}$ form the spectrum of the graph $X_d$. Thus, $\lambda_2(X_d) = \lambda_2(\tilde{X}_d) = \rho(d)$.

\[\square\]
Using Maple, we find that \( \rho(3) \approx 2.7784 \) and \( \rho(5) \approx 4.7969 \). A simple algebraic manipulation shows that \( \rho(d) \) satisfies the equation

\[
d - \rho(d) = \frac{2d - 2}{\rho^2(d) + 3\rho(d) + 2} > \frac{2d - 2}{d^2 + 3d + 2},
\]

where the last inequality follows since \( \rho(d) < d \). Using (7), we deduce that

\[
d - \frac{2}{d + 4} < \rho(d) < d - \frac{2}{d + 5}
\]

for \( d \geq 5 \). We are now ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** We will prove the contrapositive, namely we will show that if \( G \) is a \( d \)-regular graph with edge connectivity 1 (containing a bridge), then \( \lambda_2(G) \geq \lambda_2(X_d) = \rho(d) \). We also show that equality happens if and only if \( G = X_d \).

Consider a connected \( d \)-regular graph \( G \) that contains a bridge \( x_1x_2 \). Deleting the edge \( x_1x_2 \) partitions \( V(G) \) into two connected components \( G_1 \) and \( G_2 \) such that \( x_i \in V(G_i) \) for \( i = 1, 2 \). Let \( n_i = |V(G_i)| \) for \( i = 1, 2 \). Without loss of generality, we assume from now on that \( n_1 \leq n_2 \). The graph \( G_1 \) contains \( n_1 - 1 \) vertices of degree \( d \) and one vertex of degree \( d - 1 \). Because \( d \) is odd, this implies that \( n_1 \) is odd. By symmetry, \( n_2 \) is also odd. Thus, \( n_2 \geq n_1 \geq d + 2 \). Note that \( G = X_d \) if and only if \( n_1 = n_2 = d + 2 \).

Let \( A_2 \) be the quotient matrix of the partition of \( V(G) \) into \( V(G_1) \) and \( V(G_2) \). Then

\[
A_2 = \begin{bmatrix}
\frac{1}{n_1} & \frac{1}{n_2} \\
\frac{1}{n_1} & d - \frac{1}{n_2}
\end{bmatrix}
\]

The eigenvalues of \( A_2 \) are: \( d \) and \( \lambda_2(A_2) = d - \frac{1}{n_1} - \frac{1}{n_2} \).

Let \( A_3 \) be the quotient matrix of the partition of \( V(G) \) into \( V(G_1), \{x_2\}, V(G_2) \setminus \{x_2\} \). Then

\[
A_3 = \begin{bmatrix}
d - a & a & 0 \\
1 & 0 & d - 1 \\
0 & b & d - b
\end{bmatrix}
\]

where \( a = \frac{1}{n_1} \) and \( b = \frac{d - 1}{n_2 - 1} \). The eigenvalues of \( A_3 \) are \( d, \lambda_2(A_3) = \frac{d - a - b + \sqrt{(d - a + b)^2 + 4(a - b)}}{2}, \lambda_3(A_3) = \frac{d - a - b - \sqrt{(d - a + b)^2 + 4(a - b)}}{2} \). Taking partial derivatives with respect to \( a \) and to \( b \) respectively, we
find that when \( d > 1 \), the eigenvalue \( \lambda_2(A_3) \) is strictly monotone decreasing with respect to both \( a \) and \( b \) and so is strictly monotone increasing with respect to both \( n_1 \) and \( n_2 \).

We consider now a partition of \( V(G) \) into the following six parts: \( V(G_1) \setminus (x_1 \cup N(x_1)), N(x_1) \setminus \{x_2\}, \{x_1\}, \{x_2\}, N(x_2) \setminus \{x_1\} \) and \( V(G_2) \setminus (x_2 \cup N(x_2)) \). Here \( N(u) \) denotes the neighbourhood of vertex \( u \) in \( G \). Let \( e_1 \) denote the number of edges between \( N(x_1) \setminus \{x_2\} \) and \( V(G_1) \setminus (x_1 \cup N(x_1)) \). Let \( e_2 \) denote the number of edges between \( N(x_2) \setminus \{x_1\} \) and \( V(G_2) \setminus (x_2 \cup N(x_2)) \). Note that \( e_i \leq (d - 1)(n_i - d) \) for \( i = 1, 2 \).

Let \( A_6 \) be the quotient matrix of the previous partition of \( V(G) \) into six parts. Then

\[
A_6 = \begin{bmatrix}
  d - \frac{e_1}{n_1-d} & \frac{e_1}{n_1-d} & 0 & 0 & 0 & 0 \\
  \frac{e_1}{d-1} & d - \frac{e_1}{d-1} & 1 & 0 & 0 & 0 \\
  0 & d - 1 & 0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 & d - 1 & 0 \\
  0 & 0 & 0 & 1 & d - 1 - \frac{e_2}{d-1} & 0 \\
  0 & 0 & 0 & 0 & \frac{e_2}{n_2-d} & d - \frac{e_2}{n_2-d}
\end{bmatrix}
\]  

(11)

Eigenvalue interlacing (see [8, 9, 10]) implies that

\[
\lambda_2(G) \geq \max(\lambda_2(A_2), \lambda_2(A_3), \lambda_2(A_6)).
\]  

(12)

We will use this as well as the inequalities (7) and (8) to show that \( \lambda_2(G) \geq \lambda_2(X_d) = \rho(d) \) and that equality happens if and only if \( G = X_d \).

Recall that \( n_2 \geq n_1 \geq d + 2 \) and that \( d, n_1 \) and \( n_2 \) are all odd.

If \( n_1 \geq d + 6 \) and \( d \geq 5 \), then we use the partition of \( V(G) \) into two parts whose quotient matrix \( A_2 \) is given in (9). Using inequality (8), we have that

\[
\lambda_2(G) \geq \lambda_2(A_2) = d - \frac{1}{n_1} - \frac{1}{n_2} \geq d - \frac{2}{d+6} > \rho(d).
\]

If \( n_1 \geq d + 6 \) and \( d = 3 \), then we use the partition of \( V(G) \) into three parts whose quotient matrix \( A_3 \) is given in (10). We have that

\[
\lambda_2(G) \geq \lambda_2(A_3) \geq \frac{3 - \frac{1}{3} - \frac{2}{3} + \sqrt{(3 - \frac{1}{3} - \frac{2}{3})^2 + 4 \left(\frac{1}{3} - \frac{1}{4}\right)}}{2}
\]

\[
= \frac{95 + \sqrt{12049}}{72} > 2.84 > \rho(3).
\]

If \( n_2 \geq n_1 = d + 4 \), then we use the partition of \( V(G) \) into three parts whose quotient matrix \( A_3 \) is shown in (10). We have that

\[
\lambda_2(G) \geq \lambda_2(A_3) \geq \frac{d - \frac{1}{d+4} - \frac{d-1}{d+3} + \sqrt{(d - \frac{1}{d+4} + \frac{d-1}{d+3})^2 + 4 \left(\frac{1}{d+4} - \frac{d-1}{d+3}\right)}}{2}
\]

\[
= \frac{d^3 + 6d^2 + 8d + 1 + \sqrt{d^6 + 16d^5 + 88d^4 + 174d^3 + 8d^2 - 96d + 385}}{2(d^2 + 7d + 12)}
\]

\[
= \frac{d^3 + 6d^2 + 8d + 1 + \sqrt{(d^3 + 8d^2 + 12d - 9)^2 + 8d^2 + 120d + 304}}{2(d^2 + 7d + 12)}
\]

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If \( d = 3 \), the right hand side of the previous inequality equals \( \frac{106 + \sqrt{10612}}{84} > 2.7962 > \rho(3) \). When \( d \geq 5 \), from the previous inequality and (8) we obtain that

\[
\lambda_2(G) \geq \frac{d^3 + 6d^2 + 8d + 1 + \sqrt{(d^3 + 8d^2 + 12d - 9)^2 + 8d^2 + 120d + 304}}{2(d^2 + 7d + 12)} > \frac{d^3 + 6d^2 + 8d + 1 + (d^3 + 8d^2 + 12d - 9)}{2(d^2 + 7d + 12)} = \frac{d^3 + 7d^2 + 10d - 4}{d^2 + 7d + 12} = d - \frac{2d + 4}{d^2 + 7d + 12} > d - \frac{2}{d + 5} > \rho(d)
\]

If \( n_1 = d + 2 \) and \( n_2 \geq d + 6 \), then we use the partition of \( V(G) \) into three parts whose quotient matrix is is given in (10). We obtain that

\[
\lambda_2(G) \geq \lambda_2(A_3) \geq \frac{d - \frac{1}{d+2} - \frac{d-1}{d+5} + \sqrt{(d - \frac{1}{d+2} + \frac{d-1}{d+5})^2 + 4 \left( \frac{1}{d+2} - \frac{d-1}{d+5} \right)}}{2} = \frac{d^3 + 6d^2 + 8d - 3 + \sqrt{d^6 + 16d^4 + 80d^3 + 118d^2 - 24d^2 + 56d + 329}}{2(d^2 + 7d + 10)}
\]

When \( d = 3 \), the right hand side equals \( \frac{102 + \sqrt{14564}}{80} > 2.7835 > \rho(3) \). If \( 5 \leq d \leq 17 \), then from the previous identity, we deduce that

\[
\lambda_2(G) \geq \frac{d^3 + 6d^2 + 8d - 3 + \sqrt{(d^3 + 8d^2 + 8d - 5)^2 - 8d^2 + 136d + 304}}{2(d^2 + 7d + 10)} > \frac{d^3 + 6d^2 + 8d - 3 + (d^3 + 8d^2 + 8d - 5)}{2(d^2 + 7d + 10)} = \frac{d^3 + 7d^2 + 8d - 4}{d^2 + 7d + 10} = d - \frac{2}{d + 5} > \rho(d).
\]

When \( d \geq 19 \), from the previous inequality we obtain that

\[
\lambda_2(G) \geq \frac{d^3 + 6d^2 + 8d - 3 + \sqrt{(d^3 + 8d^2 + 8d - 6)^2 + 2d^3 + 8d^2 + 152d + 293}}{2(d^2 + 7d + 10)} > \frac{d^3 + 6d^2 + 8d - 3 + (d^3 + 8d^2 + 8d - 6)}{2(d^2 + 7d + 10)} = \frac{d^3 + 7d^2 + 8d - 4.5}{d^2 + 7d + 10} = d - \frac{2d + 4.5}{d^2 + 7d + 10} > d - \frac{2d - 2}{d^2 + 3d + 2} > \rho(d).
\]

The only case which remains to consider is \( n_1 = d + 2 \) and \( n_2 = d + 4 \). In this case, \( e_1 = 2d - 4, n_2 - d = 4 \) and \( e_2 \) is an even integer with \( 4d - 12 \leq e_2 \leq 4d - 4 \). Thus,

\[
A_6 = \begin{bmatrix}
1 & d - 1 & 0 & 0 & 0 & 0 \\
2 & d - 3 & 1 & 0 & 0 & 0 \\
0 & d - 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & d - 1 & 0 \\
0 & 0 & 0 & 1 & d - 1 - \frac{e_2}{d - 1} & \frac{e_2}{d - 1} \\
0 & 0 & 0 & 0 & \frac{e_2}{4} & d - \frac{e_2}{4}
\end{bmatrix}
\]
Let $P_{A_6}(x)$ denote the characteristic polynomial of $A_6$. Since each row sum of $A_6$ is $d$, it follows that $d$ is a root of $P_{A_6}(x)$. Thus, $P_{A_6}(x) = (x - d)P_5(x)$. The second largest root of $P_{A_6}(x)$ is the largest root of $P_5(x)$. Using the results of [3] page 130, we observe that $P_5(x)$ equals the characteristic polynomial of the following matrix:

$$
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 \\
0 & d-1 & d-2 & d-1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & d - \frac{e_2}{d-1} - \frac{e_2}{n_2-d}
\end{bmatrix}
$$

Using Maple to divide $P_5(x)$ by the polynomial $x^3 + (3 - d)x^2 + (2 - 3d)x - 2$, we obtain the quotient $Q(x) = x^2 - \frac{4d^2 - 4d - d^2 - 3e_2}{d(d-1)}x - \frac{e_2}{d-1}$ and the remainder $R(x) = (2d - e_2 - 2)x + \frac{de_2 + 4d + e_2 - 4d^2}{2}$. Thus,

$$
P_5(x) = (x^3 + (3 - d)x^2 + (2 - 3d)x - 2)Q(x) + (2d - e_2 - 2)x + \frac{de_2 + 4d + e_2 - 4d^2}{2}.
$$

Because $\rho(d)$ satisfies the equation $x^3 + (3 - d)x^2 + (2 - 3d)x - 2 = 0$, it follows that

$$
P_5(\rho(d)) = (2d - e_2 - 2)\rho(d) + \frac{de_2 + 4d + e_2 - 4d^2}{2} = e_2(\rho(d) + \frac{d+1}{2}) + (2d - 2)(\rho(d) - d).
$$

Because $d > \rho(d) > d - 1 \geq \frac{d+1}{2}$, we deduce that

$$
P_5(\rho(d)) < 0. \quad (13)
$$

Assume that $\rho(d) \geq \lambda_2(A_6)$. Then $\rho(d)$ is larger than any root of $P_5(x)$. This implies that $P_5(\rho(d)) = \prod_{\theta} \text{root of } P_5(\rho(d) - \theta) \geq 0$ which is a contradiction with (13). Thus, $\lambda_2(G) \geq \lambda_2(A_6) > \rho(d)$.

Hence, $\lambda_2(G) > \rho(d)$ whenever $n_2 \geq d + 4$. The last case to be considered is $n_1 = n_2 = d + 2$. This means that $G = X_d$ and $\lambda_2(G) = \rho(d)$ which finishes the proof. \(\square\)

### 4 Proof of Theorem 1.5

In this section we present the proof of Theorem 1.5. We describe first the $d$-regular graph $Y_d$ having $\kappa'(Y_d) = 2$ and $\lambda_2(Y_d) = \frac{d-3 + \sqrt{(d+3)^2 - 16}}{2}$.

Consider the graph $H_d = K_{d-1} \lor \bar{K}_2$. It has $d - 1$ vertices of degree $d$ and two vertices of degree $d - 1$. We construct $Y_d$ by taking two disjoint copies of $K_{d-1} \lor \bar{K}_2$ and adding two disjoint edges between the vertices of degree $d - 1$ in different copies of $K_{d-1} \lor \bar{K}_2$. Figure 4 describes $Y_3$ and Figure 5 shows $Y_4$. 

9
The graph $Y_d$ is $d$-regular, has $2d + 2$ vertices and its edge-connectivity is 2. It has an equitable partition into four parts of sizes $d - 1, 2, 2, d - 1$ with the following quotient matrix:

$$
\tilde{Y}_d = \begin{bmatrix}
d - 2 & 2 & 0 & 0 \\
d - 1 & 0 & 1 & 0 \\
0 & 1 & 0 & d - 1 \\
0 & 0 & 2 & d - 2
\end{bmatrix}
$$

(14)

The characteristic polynomial of this matrix equals

$$
P_{\tilde{Y}_d}(x) = (x - d)(x + 1)(x^2 + (3 - d)x + (4 - 3d))
$$

Thus, the eigenvalues of $\tilde{Y}_d$ are $d, 1$ and $\frac{d - 3 \pm \sqrt{(d+3)^2 - 16}}{2}$. To simplify our notation, let $\theta(d) = \frac{d - 3 + \sqrt{(d+3)^2 - 16}}{2}$. Note that $\theta(d)$ is the largest root of

$$
T(x) = x^2 + (3 - d)x + (4 - 3d)
$$

(15)

and

$$
d - \frac{4}{d + 2} < \theta(d) < d - \frac{4}{d + 3}.
$$

(16)

Lemma 4.1. The second largest eigenvalue of $Y_d$ equals $\theta(d)$.

Proof. Since the previous partition of $V(Y_d)$ into four parts is equitable, it follows that the four eigenvalues of $\tilde{Y}_d$ are also eigenvalues of $Y_d$. Obviously, the second largest of the eigenvalues of $\tilde{Y}_d$ is $\theta(d)$. By an argument similar to the one of Lemma 3.2, one can show that the other eigenvalues of $\tilde{Y}_d$ are $(-1)^{(2d-2)}$ which implies the desired result.

We are ready now to prove Theorem 1.5.

Proof of Theorem 1.5. We will prove the contrapositive, namely we will show that among all $d$-regular graphs with edge-connectivity less than or equal to 2, the graph $Y_d$ has the smallest $\lambda_2$.

Let $G$ be a $d$-regular graph with $\kappa'(G) \leq 2$. We will prove that $\lambda_2(G) \geq \theta(d)$ with equality if and only if $G = Y_d$.

If $\kappa'(G) = 1$, then Theorem 1.4, (8) and (16) imply that $\lambda_2(G) \geq \rho(d) \geq d - \frac{2(d-1)}{(d+1)(d+2)} \geq d - \frac{4}{d+3} > \theta(d)$.
If \( k'(G) = 2 \), then there exists a partition of \( V(G) \) into two parts \( V_1 \) and \( V_2 \) such that \( e(V_1, V_2) = 2 \). Let \( S_1 \subset V_1 \) and \( S_2 \subset V_2 \) denote the endpoints of the two edges between \( V_1 \) and \( V_2 \). We have that \( (|S_1|, |S_2|) \in \{(1, 2), (2, 2), (2, 1)\} \). Let \( n_i = |V_i| \) for \( i \in \{1, 2\} \). It is easy to see that \( n_i \geq d + 1 \) for each \( i \in \{1, 2\} \). Note that \( n_1 = n_2 = d + 1 \) is equivalent to \( G = Y_d \).

Without loss of generality assume that \( n_2 \geq n_1 \geq d + 1 \). If \( n_1 \geq d + 3 \), consider the partition of \( V(G) \) into \( V_1 \) and \( V_2 \). The quotient matrix of this partition is

\[
\begin{pmatrix}
\frac{d}{n_1} & \frac{2}{n_2} \\
\frac{2}{n_2} & d - \frac{2}{n_2}
\end{pmatrix}
\]

and its eigenvalues are \( d \) and \( d - \frac{2}{n_1} - \frac{2}{n_2} \). Eigenvalue interlacing and \( n_2 \geq n_1 \geq d + 3 \) imply that \( \lambda_2(G) \geq d - \frac{2}{n_1} - \frac{2}{n_2} \geq d - \frac{4}{d + 3} \). Using inequality (16), we obtain that \( \lambda_2(G) \geq d - \frac{4}{d + 3} > \theta(d) \) which finishes the proof of this case.

If \( n_1 = d + 2 \), then we have a few cases to consider. If \( |S_1| = 2 \), then \( e(S_1) = 0 \) or \( e(S_1) = 1 \). If \( e(S_1) = 0 \), consider the partition of \( V(G) \) into three parts: \( V_1 \setminus S_1, S_1, V_2 \). The quotient matrix of this partition is

\[
\begin{pmatrix}
\frac{d - 2d - 2}{d} & \frac{2d - 2}{d} & 0 \\
\frac{2d - 2}{d} & d - 1 & 1 \\
0 & 1 & d - \frac{2}{n_2}
\end{pmatrix}
\]

Its characteristic polynomial equals

\[
P_3(x) = (x - d) \left( x^2 - \frac{d^2 n_2 + 2n_2 - 2dn_2 - 2d}{dn_2} x - \frac{2(d^2 n_2 + n_2 + 2 - 2dn_2 - d)}{dn_2} \right)
\]

If \( P_3(x) = x^2 - \frac{d^2 n_2 + 2n_2 - 2dn_2 - 2d}{dn_2} x - \frac{2(d^2 n_2 + n_2 + 2 - 2dn_2 - d)}{dn_2} \), then

\[
P_2(x) = T(x) + R(x)
\]

where \( R(x) = \frac{dn_2 + 2n_2 - 2d}{dn_2} \left( \frac{d^2 n_2 + 2d - 2n_2 - 4}{dn_2 + 2n_2 - 2d} x - x \right) \). The expression \( \frac{d^2 n_2 + 2d - 2n_2 - 4}{dn_2 + 2n_2 - 2d} \) is decreasing with \( n_2 \) and thus it attains its maximum when \( n_2 = d + 2 \). This maximum equals \( \frac{d^3 + 2d^2 - 8}{d^2 + 2d + 4} = \frac{1+\sqrt{33}}{2} \).

Figure 5: \( Y_4 \) is 4-regular with \( \lambda_2(Y_4) = \theta(4) = \frac{1+\sqrt{33}}{2} \)
\[ d - \frac{4(d+2)}{d^2+3d+4} < d - \frac{4}{d+2}. \] Thus, \( P_3(\theta(d)) = R(\theta(d)) < \frac{dn_2+2n_2-2d}{dn_2} (d - \frac{4}{d+2} - \theta(d)) < 0 \) where the last inequality follows from (16). This fact and eigenvalue interlacing imply that \( \lambda_2(G) > \theta(d) \).

If \( e(S_1) = 1 \), then we consider the same partition into three parts: \( V_1 \setminus S_1, S_1, V_2 \). The quotient matrix is the following

\[
\begin{bmatrix}
  d - \frac{2d-4}{d} & \frac{2d-4}{d} & 0 \\
  d - 2 & 1 & 1 \\
  0 & \frac{2}{n_2} & d - \frac{2}{n_2}
\end{bmatrix}
\]

Its characteristic polynomial equals

\[
P_3(x) = (x - d) \left( x^2 - \frac{d^2n_2 + 4n_2 - dn_2 - 2d}{dn_2} x - \frac{d^2n_2 + 4n_2 + 8 - 6dn_2}{dn_2} \right).
\]

If \( P_2(x) = x^2 - \frac{d^2n_2 + 4n_2 - dn_2 - 2d}{dn_2} x - \frac{d^2n_2 + 4n_2 + 8 - 6dn_2}{dn_2} \) then

\[
P_2(x) = T(x) + R(x)
\]

where \( R(x) = \frac{2(dn_2+2n_2-1)}{dn_2} \left( \frac{d^2n_2+dn_2-2n_2-4}{dn_2+2n_2-d} - x \right) \). The expression \( \frac{d^2n_2+dn_2-2n_2-4}{dn_2+2n_2-d} \) decreases with \( n_2 \) and thus, its maximum is attained at \( n_2 = d + 2 \). This maximum equals \( \frac{d^2+3d^2-8}{d^2+3d+4} = d - \frac{4(d+2)}{d^2+3d+4} < d - \frac{4}{d+2} \). As before, the previous inequality and eigenvalue interlacing imply that \( \lambda_2(G) > \theta(d) \).

If \( |S_1| = 1 \), then \( d \) must be even. Indeed, the subgraph induced by \( V_1 \) contains \( d + 1 \) vertices of degree \( d \) and one vertex of degree \( d - 2 \). This cannot happen when \( d \) is odd. Thus, \( d \) is even and \( d \geq 4 \). Consider the partition of \( G \) into three parts: \( V_1 \setminus S_1, S_1, V_2 \). The quotient matrix of this partition is

\[
\begin{bmatrix}
  d - \frac{d-2}{d+1} & \frac{d-2}{d+1} & 0 \\
  d - 2 & 0 & 2 \\
  0 & \frac{2}{n_2} & d - \frac{2}{n_2}
\end{bmatrix}
\]

Its characteristic polynomial is

\[
P_3(x) = (x - d) \left( x^2 - \frac{d^2n_2 + 2n_2 - 2d - 2}{(d+1)n_2} x - \frac{d^2n_2 + 2d + 4n_2 + 8 - 4dn_2}{(d+1)n_2} \right)
\]

If \( P_2(x) = x^2 - \frac{d^2n_2 + 2n_2 - 2d - 2}{(d+1)n_2} x - \frac{d^2n_2 + 2d + 4n_2 + 8 - 4dn_2}{(d+1)n_2} \), then

\[
P_2(x) = T(x) + R(x)
\]

where \( R(x) = \frac{2dn_2+5n_2-2d-2}{dn_2+5n_2-2d-2} \left( \frac{2d^2n_2+3dn_2-8n_2-2d-8}{2dn_2+5n_2-2d-2} - x \right) \). Because \( d \geq 4 \), the expression \( \frac{2d^2n_2+3dn_2-8n_2-2d-8}{2dn_2+5n_2-2d-2} \) is decreasing with \( n_2 \) and thus, its maximum is attained when \( n_2 = d + 2 \). This maximum equals \( \frac{2d^4+7d^2-4d-24}{2d^2+7d+8} = d - \frac{12d+24}{2d^2+7d+8} < d - \frac{4}{d+2} \). This fact and eigenvalue interlacing imply \( \lambda_2(G) > \theta(d) \).
Assume now that $n_1 = d + 1$. This implies that $V_1$ induces a subgraph isomorphic to $K_{d-1} \lor K_2$ and consequently, $|S_1| = 2$ and $e(S_1) = 0$.

If $n_2 \geq d + 3$, then consider the partition of $G$ into three parts: $V_1 \setminus S_1, S_1, V_2$. The quotient matrix of this partition is

$$
\begin{bmatrix}
    d - 2 & 2 & 0 \\
    d - 1 & 0 & 1 \\
    0 & \frac{2}{n_2} & d - \frac{2}{n_2}
\end{bmatrix}
$$

Its characteristic polynomial is

$$
P_3(x) = (x - d) \left( x^2 - \left( d - 2 - \frac{2}{n_2} \right) x + 2 + \frac{2}{n_2} - 2d \right).
$$

If $P_2(x) = x^2 - \left( d - 2 - \frac{2}{n_2} \right) x + 2 + \frac{2}{n_2} - 2d$, then

$$
P_2(x) = T(x) + \frac{dn_2 + 2 - 2n_2 - (n_2 - 2)x}{n_2}
$$

which implies that

$$
P_2(\theta(d)) = \frac{dn_2 + 2 - 2n_2 - (n_2 - 2)\theta(d)}{n_2}
$$

The expression $\frac{dn_2 + 2 - 2n_2 - (n_2 - 2)\theta(d)}{n_2}$ is decreasing with $n_2$ and therefore, its maximum is attained when $n_2 = d + 3$. Thus,

$$
P_2(\theta(d)) \leq \frac{d^2 + d - 4 - (d + 1)\theta(d)}{d + 3} = \frac{(d + 1) \left( d - \frac{4}{d + 1} - \theta(d) \right)}{d + 3} < 0.
$$

This inequality and eigenvalue interlacing imply that $\lambda_2(G) > \theta(d)$.

Thus, the remaining case is $n_1 = d + 1$ and $n_2 \leq d + 2$. If $d$ is odd, then $n_2 \neq d + 2$. Otherwise, the sum of the degrees of the graph induced by $V_2$ would equal $d|V_2| - 2 = d(d + 2) - 2$ which is an odd number. This is impossible and thus, $n_2 = d + 1$. This implies $G = Y_d$.

If $n_2 = d + 2$ and $d$ is even, then we have a few cases to consider. If $|S_2| = 1$, then both vertices in $S_1$ are adjacent to the vertex $a$ of $S_2$. Thus, $a$ has exactly $d - 2$ neighbours in $V_2$ which means there are $3$ vertices of $V_2$ which are not adjacent to $a$. Each of these three vertices has degree $d$ and the only way this can happen is they form a clique and each of them is adjacent to the $d - 2$ neighbours of $a$ in $V_2$. Using a degree argument, it also follows that the $d - 2$ neighbours of $a$ in $V_2$ induce a subgraph isomorphic to the complement of a perfect matching on $d - 2$ vertices $\overline{M_{d-2}}$. Using the notation from the previous section, it follows that $G = K_{d-1} \lor K_2 \lor K_1 \lor \overline{M_{d-2}} \lor K_3$. The graph $G$ has an obvious equitable partition into five parts whose quotient matrix is

$$
\begin{bmatrix}
    d - 2 & 2 & 0 & 0 & 0 \\
    d - 1 & 0 & 1 & 0 & 0 \\
    0 & 2 & 0 & d - 2 & 0 \\
    0 & 0 & 1 & d - 4 & 3 \\
    0 & 0 & 0 & d - 2 & 2
\end{bmatrix}
$$
implies that the remaining
Obviously, this argument shows this case is only possible when
vertices
There are 4 remaining vertices
−
cannot have degree
in
there exists at least one vertex
z
d −
quotient matrix of this partition is the following:

P
U
vertex of
Q
argument implies that
u
neighbours in
V
case cannot happen and therefore, both
x
and
y
have at least
2
common neighbours. This vertex cannot have degree
d. Thus, this case cannot happen and therefore, both
x
and
y
have exactly
2
common neighbours in
V. We call this set
U. Let \{u_2, w_2\} = V \setminus (\{x_2, y_2\} \cup U). Because
x
and
y
have
2
neighbours in
V, we may assume that
x
is adjacent to
u_2
and
y
is adjacent to
w_2.
A degree argument implies that
u_2
and
w_2
must be adjacent and both of them are adjacent to each vertex of
U. Finally, the subgraph induced by
U
must be
(d − 4)-regular.
The following is a five-part equitable partition of
G:
V \setminus S_1, S_1, S_2, U, \{u_2, w_2\}. The quotient matrix of this partition is the following:

\[
\begin{bmatrix}
 d - 2 & 2 & 0 & 0 & 0 \\
 d - 1 & 0 & 1 & 0 & 0 \\
 0 & 1 & 0 & d - 2 & 1 \\
 0 & 0 & 2 & d - 4 & 2 \\
 0 & 0 & 1 & d - 2 & 1 \\
\end{bmatrix}
\]

Its characteristic polynomial equals
P_5(x) = (x - d)P_4(x)
where
P_4(x) = x^4 + (-d + 4)x^3 + (-4d + 5)x^2 + (-6d + 7)x - d.
Dividing
P_4(x)
by
T(x)
we get that
P_4(x) = T(x)Q_4(x) + R_4(x)
where
Q_4(x) = x^2 + x - 3
and
R_4(x) = -3x - 3d.
It follows that
P_4(\theta(d)) = R_4(\theta(d)) = -3\theta(d) - 3d < 0.
This fact and eigenvalue interlacing imply that
\lambda_2(G) > \theta(d).
The case remaining is \(|S_2| = 2\). If \(e(S_2) = 0\), then each vertex of
S_2
has exactly
d − 1
neighbours in
V_2. Let \(S_2 = \{x_2, y_2\}\). If
x_2
and
y_2
are adjacent to the same
d − 1
vertices of
V_2,
then because \(|V_2| = n_2 = d + 2\), there is exactly one vertex of
V_2
outside the vertices of
S_2
and their
d − 1
common neighbours. This vertex cannot have degree
d.
Thus, this case cannot happen and therefore, both
x_2
and
y_2
have exactly
2
common neighbours in
V_2.
We call this set
U. Let \(\{u_2, w_2\} = V_2 \setminus (\{x_2, y_2\} \cup U)\). Because
x_2
and
y_2
have
2
neighbours in
V_2,
we may assume that
x_2
is adjacent to
u_2
and
y_2
is adjacent to
w_2.
A degree argument implies that
u_2
and
w_2
must be adjacent and both of them are adjacent to each vertex of
U. Finally, the subgraph induced by
U
must be
(d − 4)-regular.
The following is a five-part equitable partition of
G:
V \setminus S_1, S_1, S_2, U, \{a_2, b_2, c_2, d_2\}. The quotient matrix of this partition is the following:

\[
\begin{bmatrix}
 d - 2 & 2 & 0 & 0 & 0 \\
 d - 1 & 0 & 1 & 0 & 0 \\
 0 & 1 & 1 & 2 & d - 4 \\
 0 & 0 & 1 & 3 & d - 4 \\
 0 & 0 & 2 & 4 & d - 6 \\
\end{bmatrix}
\]
Its characteristic polynomial equals \( P_5(x) = (x - d)P_4(x) \) where \( P_4(x) = x^4 + (-d + 4)x^3 + (-4d + 6)x^2 + (-2d^3 - 1)x + 3d - 6 \). Dividing \( P_4(x) \) by \( T(x) \) we get that \( P_4(x) = T(x)Q_4(x) + R_4(x) \) where \( Q_4(x) = x^2 + x - 1 \) and \( R_4(x) = -2x - 2 \). It follows that \( P_4(\theta(d)) = R_4(\theta(d)) = -2\theta(d) - 2 < 0 \). This and eigenvalue interlacing imply that \( \lambda_2(G) > \theta(d) \) which finishes our proof.

\[ \square \]

### 5 Some Remarks

Any strongly regular graph of degree \( d \geq 3 \) satisfies the condition \( \lambda_2 \leq d - 2 \) and thus, is \( d \)-edge-connected. The fact that the edge-connectivity of a strongly regular graph equals its degree, was observed by Plesník in 1975 (cf. [2]). As mentioned in the introduction, much more is true, namely the vertex-connectivity of any distance-regular graph equals its degree (see [5]). It is known that any vertex transitive \( d \)-regular graph whose second largest eigenvalue is simple has \( \lambda_2(G) \leq d - 2 \) and consequently, is \( d \)-edge-connected. In fact, any vertex transitive \( d \)-regular graph is \( d \)-edge-connected as shown by Mader in 1971 (see [14] or Chapter 3 of [8]).

I expect that Theorem 1.4 and Theorem 1.5 can be extended to other values of edge-connectivity and vertex-connectivity. For example, it seems that \( \lambda_2(G) \leq d - \frac{1}{2} \) implies \( \kappa(G) \geq 2 \). Note however that in many cases, Fiedler’s bound \( \kappa(G) \geq d - \lambda_2 \) cannot be improved. When \( 2k - 2 \geq d \) and \( dk \) is even, consider the graph \( 2K_{d-k+1} \lor H \) where \( H \) is a \((2k - 2 - d)\)-regular graph on \( k \) vertices. This graph is \( d \)-regular, has vertex connectivity \( k \) and its second largest eigenvalue equals \( d - k \).

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### References


