Eigenvalues and edge-connectivity of regular graphs

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Abstract

In this paper, we show that if the second largest eigenvalue of a d-regular graph is less than $d - \frac{2(k-1)}{d+1}$, then the graph is k-edge-connected. When k is 2 or 3, we prove stronger results. Let $\rho(d)$ denote the largest root of $x^3 - (d-3)x^2 - (3d-2)x - 2 = 0$. We show that if the second largest eigenvalue of a d-regular graph G is less than $\rho(d)$, then G is 2-edge-connected and we prove that if the second largest eigenvalue of G is less than $\frac{d-3+\sqrt{(d+3)^2-16}}{2}$, then G is 3-edge-connected.

1 Introduction

Let $\kappa(G)$ and $\kappa'(G)$ denote the vertex- and edge-connectivity of a connected graph G. If δ is the minimum degree of G, then $1 \leq \kappa(G) \leq \kappa'(G) \leq \delta$. Let $L = D - A$ be the Laplacian matrix of G, where D is the diagonal degree matrix and A is the adjacency matrix of G. We denote by $0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_n$ the eigenvalues of the Laplacian of G. The complement of G is denoted by $\overline{G}$. A graph is called disconnected if it is not connected. The join $G_1 \vee G_2$ of two vertex-disjoint graphs G_1 and G_2 is the graph formed from the union of G_1 and G_2 by joining each vertex of G_1 to each vertex of G_2.

A classical result in spectral graph theory due to Fiedler [7] states that

$$\kappa(G) \geq \mu_2(G) \tag{1}$$

for any non-complete graph G. Fiedler called $\mu_2(G)$ the algebraic connectivity of G and his work stimulated a large amount of research in spectral graph theory over the last forty years (see [1, 11, 13, 15]). In [12], Kirkland, Molitierno, Neumann and Shader characterize the equality case in Fiedler's inequality (1).

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Theorem 1.1 (Kirkland-Molitierno-Neumann-Shader [12]). *Let G be a non-complete connected graph on n vertices. Then $\kappa(G) = \mu_2(G)$ if and only if $G = G_1 \vee G_2$ where G_1 is a disconnected graph on $n - \kappa(G)$ vertices and G_2 is a graph on $\kappa(G)$ vertices with $\mu_2(G_2) \geq 2\kappa(G) - n$.*

Eigenvalue techniques have been also used recently by Brouwer and Koolen [5] to show that the vertex-connectivity of a distance-regular graph equals its degree (see also [2, 4] for related results).

In this paper, we study the relations between the edge-connectivity and the second largest eigenvalue of a d-regular graph. Let $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ denote the eigenvalues of the adjacency matrix of G. If G is d-regular graph, then $\lambda_i(G) = d - \mu_i(G)$ for any $1 \leq i \leq n$. Thus, $\lambda_1(G) = d$ and G is connected if and only if $\lambda_2(G) < d$.

Chandran [6] proved that if G is a d-regular graph of order n and $\lambda_2(G) < d - 1 - \frac{d}{n-d}$, then $\kappa'(G) = d$ and the only disconnecting edge-cuts are trivial, i.e., d edges adjacent to the same vertex. Krivelevich and Sudakov [13] showed that $\lambda_2(G) \leq d - 2$ implies $\kappa'(G) = d$. If G is a d-regular graph on n vertices and $n \leq 2d + 1$, then $\kappa'(G) = d$ regardless of the eigenvalues of G (see also the proof of Lemma 1.3). Both results are based on the following well-known lemma (see [15] for a short proof).

Lemma 1.2. *If $G = (V, E)$ is a connected graph of order n and S is a subset of vertices of G, then*

$$e(S, V \setminus S) \geq \frac{\mu_2 |S|(n - |S|)}{n}$$

where $e(S, V \setminus S)$ denotes the number of edges between S and $V \setminus S$.

We extend and improve the previous results as follows. We prove the following sufficient condition for the k-edge-connectivity of a d-regular graph for any $2 \leq k \leq d$.

Theorem 1.3. *If $d \geq k \geq 2$ are two integers and G is a d-regular graph such that $\lambda_2(G) \leq d - \frac{(k-1)n}{(d+1)(n-d-1)}$, then $\kappa'(G) \geq k$.*

When $k = d$, this result states that if $\lambda_2(G) \leq d - \frac{(d-1)n}{(d+1)(n-d-1)}$, then $\kappa'(G) = d$. We get a small improvement for d even because $\kappa'(G)$ must be even in this case (see Lemma 3.1). When d is even, Theorem 1.3 shows that $\lambda_2(G) \leq d - \frac{(d-2)n}{(d+1)(n-d-1)}$ implies $\kappa'(G) = d$. A simple calculation reveals that these bounds improve the previous result of Chandran.

When $n \geq 2d + 2$, Theorem 1.3 implies that if G is a d-regular graph with $\lambda_2(G) \leq d - 2 + \frac{4}{d+1}$, then $\kappa'(G) = d$. When d is even, the right hand-side of the previous inequality can be replaced by $d - 2 + \frac{6}{d+1}$. These results improve the previous bound of Krivelevich and Sudakov. Note that there are many d-regular graphs G with $d - 2 < \lambda_2(G) \leq d - 2 + \frac{4}{d+1}$. An example is presented in Figure 1.

For $k \in \{2, 3\}$ we further improve these results as shown below. Note that the edge-connectivity of a connected, regular graph of even degree is even (see Lemma 3.1).

Theorem 1.4. *Let $d \geq 3$ be an odd integer and $\rho(d)$ denote the largest root of $x^3 - (d-3)x^2 - (3d-2)x - 2 = 0$. If G is a d-regular graph such that $\lambda_2(G) < \rho(d)$, then $\kappa'(G) \geq 2$.*

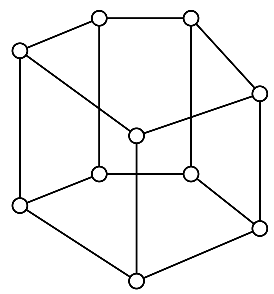

Figure 1: A 3-regular graph with $3 - 2 < \lambda_2 = \frac{1+\sqrt{5}}{2} < 3 - 2 + \frac{4}{3+1}$

The value of $\rho(d)$ is about $d - \frac{2}{d+5}$.

Theorem 1.5. *Let $d \geq 3$ be any integer. If G is a d-regular graph such that*

$$\lambda_2(G) < \frac{d - 3 + \sqrt{(d+3)^2 - 16}}{2}$$

then $\kappa'(G) \geq 3$.

The value of $\frac{d-3+\sqrt{(d+3)^2-16}}{2}$ is about $d - \frac{4}{d+3}$.

We show that Theorem 1.4 and Theorem 1.5 are best possible in the sense that for each odd $d \geq 3$, there exists a d-regular graph X_d such that $\lambda_2(X_d) = \rho(d)$ and $\kappa'(X_d) = 1$. Also, for each $d \geq 3$, there exists a d-regular graph Y_d such that $\lambda_2(Y_d) = \frac{d-3+\sqrt{(d+3)^2-16}}{2}$ and $\kappa'(Y_d) = 2$.

The following result is an immediate consequence of Theorem 1.4 and Theorem 1.5.

Corollary 1.6. *If G is a 3-regular graph and $\lambda_2(G) < \sqrt{5} \approx 2.23$, then $\kappa'(G) = 3$. If G is 4-regular graph and $\lambda_2 < \frac{1+\sqrt{33}}{2} \approx 3.37$, then $\kappa'(G) = 4$.*

Our results imply that if G is a 3-regular graph with $\lambda_2(G) < \sqrt{5}$, then $\kappa(G) = 3$. This is because for 3-regular graphs the vertex- and the edge-connectivity are the same. It is very likely that the eigenvalues results for $\kappa(G)$ and $\kappa'(G)$ can be quite different. We comment on this in the last section of the paper.

2 Proof of Theorem 1.3

In this section, we give a short proof of Theorem 1.3. Recall that a partition $V_1 \cup \cdots \cup V_l = V(G)$ of the vertex set of a graph G is called equitable if for all $1 \leq i, j \leq l$, the number of neighbours in V_j of a vertex v in V_i is a constant b_{ij} independent of v (see Section 9.3 of [8] or [9] for more details on equitable partitions).

Proof of Theorem 1.3. We prove the contrapositive, namely we show that if G is a d-regular graph such that $\kappa'(G) \leq k - 1$, then $\lambda_2(G) > d - \frac{(k-1)n}{(d+1)(n-d-1)}$. Let $V_1 \cup V_2 = V(G)$ be a partition of G such that $r = e(V_1, V_2) \leq k - 1 \leq d - 1$. Note that if $|V_1| \leq d$, then $d - 1 \geq e(V_1, V_2) \geq |V_1|(d - |V_1| + 1) \geq d$ which is a contradiction. Thus, $n_i := |V_i| \geq d + 1$ for each $i = 1, 2$. Since $n_1 + n_2 = n$, this implies that $n_1 n_2 \geq (d+1)(n-d-1)$. The quotient matrix of the partition $V_1 \cup V_2 = V(G)$ (see [8, 9, 10]) is

$$A_2 = \begin{bmatrix} d - \frac{r}{n_1} & \frac{r}{n_1} \\ \frac{r}{n_2} & d - \frac{r}{n_2} \end{bmatrix} \tag{2}$$

The eigenvalues of A_2 are d and $d - \frac{r}{n_1} - \frac{r}{n_2}$. Eigenvalue interlacing (see [8, 9, 10]), $r \leq k - 1$ and $n_1 n_2 \geq (d + 1)(n - d - 1)$ imply that

$$\lambda_2(G) \geq d - \frac{r}{n_1} - \frac{r}{n_2} \geq d - \frac{(k-1)n}{n_1 n_2} \geq d - \frac{(k-1)n}{(d+1)(n-d-1)} \tag{3}$$

We actually have strict inequality here. Otherwise, the partition $V_1 \cup V_2$ would be equitable. This would mean that each vertex of V_1 has the same number of neighbours in V_2 which is impossible since there are vertices in V_1 without a neighbour in V_2. The proof is finished. □

3 Proof of Theorem 1.4

In this section, we present the proof of Theorem 1.4. First, we show that any connected, regular graph of even degree has edge-connectivity larger than two. This follows from the next lemma.

Lemma 3.1. *Let G be a connected d-regular graph. If d is even, then $\kappa'(G)$ is even.*

Proof. Let $V = A \cup B$ be a partition of the vertex set of G such that $e(A, B) = \kappa'(G)$, where $e(A, B)$ denotes the number of edges with one endpoint in A and one endpoint in B. Summing up the degrees of the vertices in A, we obtain that $d|A| = 2e(A) + e(A, B) = 2e(A) + \kappa'(G)$, where $e(A)$ denotes the number of edges with both endpoints in A. Since d is even, this implies $\kappa'(G)$ is even which finishes the proof. □

For the rest of this section, we assume that d is an odd integer with $d \geq 3$. We describe first the d-regular graph X_d having $\kappa'(X_d) = 1$ and $\lambda_2(X_d) = \rho(d)$.

If $H_1, \ldots, H_k$ are pairwise vertex-disjoint graphs, then we define the product $H_1 \vee H_2 \vee \cdots \vee H_k$ recursively as follows. If $k = 1$, the product is H_1. If $k = 2$, then $H_1 \vee H_2$ is the usual join of H_1 and H_2, i.e. the graph formed from the union of H_1 and H_2 by joining each vertex of H_1 to each vertex of H_2. If $k \geq 3$, then $H_1 \vee H_2 \vee \cdots \vee H_k$ is the graph obtained by taking the union of $H_1 \vee H_2 \vee \cdots \vee H_{k-1}$ and H_k and joining each vertex of H_{k-1} to each vertex of H_k.

The extremal graph X_d is defined as:

$$X_d = K_2 \vee \overline{M_{d-1}} \vee K_1 \vee K_1 \vee \overline{M_{d-1}} \vee K_2 \tag{4}$$

where $\overline{M_{d-1}}$ is the unique $(d-3)$-regular graph on $d-1$ vertices (the complement of a perfect matching M_{d-1} on $d - 1$ vertices). Figure 2 shows X_3 and Figure 3 describes X_5.

The graph X_d is d-regular, has $2d+4$ vertices and its edge-connectivity is 1. It has an equitable partition of its vertices into six parts which is described by the following quotient matrix $\tilde{X_d}$. The sizes of the parts in this equitable partition are $2, d-1, 1, 1, d-1, 2$.

$$\tilde{X_d} = \begin{bmatrix} 1 & d-1 & 0 & 0 & 0 & 0 \\ 2 & d-3 & 1 & 0 & 0 & 0 \\ 0 & d-1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & d-1 & 0 \\ 0 & 0 & 0 & 1 & d-3 & 2 \\ 0 & 0 & 0 & 0 & d-1 & 1 \end{bmatrix} \tag{5}$$

Next, we show that the second largest eigenvalue of X_d equals the second largest eigenvalue of $\tilde{X_d}$. This will enable us to get sharp estimates for $\lambda_2(X_d)$ which will be used later in this section.

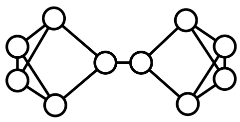

Figure 2: X_3 is 3-regular with $\lambda_2(X_3) = \rho(3) \approx 2.7784$

Lemma 3.2. *For each odd integer $d \geq 3$, we have that $\lambda_2(X_d) = \lambda_2(\tilde{X_d}) = \rho(d)$.*

Proof. Since the partition with quotient matrix $\tilde{X_d}$ is equitable, it follows that the spectrum of X_d contains the spectrum of $\tilde{X_d}$. Using Maple, we determine the characteristic polynomial of $\tilde{X_d}$:

$$P_{\tilde{X_d}}(x) = (x-d)(x-1)(x+2)(x^3 - (d-3)x^2 - (3d-2)x - 2) \tag{6}$$

The roots of $P_{\tilde{X_d}}$ are $d, 1, -2$ and the three roots of $Q(x) = x^3 - (d-3)x^2 - (3d-2)x - 2$. Denote by $\tilde{\lambda}_2 \geq \tilde{\lambda}_3 \geq \tilde{\lambda}_4$ the roots of $Q(x)$. Since these roots sum up to $d-3 \geq 0$ and their product is 2, it follows that $\tilde{\lambda}_2$ is positive and both $\tilde{\lambda}_3$ and $\tilde{\lambda}_4$ are negative. Because $Q(1) = -4d+4 < 0$, we deduce that $\tilde{\lambda}_2 > 1$. Thus, $\lambda_2(\tilde{X_d}) = \tilde{\lambda}_2 = \rho(d)$.

Let $W \subset \mathbb{R}^{2d+4}$ be the subspace of vectors which are constant on each part of the six part equitable partition. The lifted eigenvectors corresponding to the six roots of $P_{\tilde{X_d}}$ form a basis for W. The remaining eigenvectors in a basis of eigenvectors for X_d can be chosen to be perpendicular to the vectors in W. Thus, they may be chosen to be perpendicular to the characteristic vectors of the parts in the six-part equitable partition since these characteristic vectors form a basis for W. This implies that these eigenvectors will correspond to the non-trivial eigenvalues of the graph obtained as a disjoint union of $2K_2$ and $2\overline{M_{d-1}}$. The corresponding eigenvalues will be $(-2)^{(d-3)}, (-1)^{(2)}, 0^{(d-1)}$, where the exponent denotes the multiplicity of each eigenvalue. These $2d-2$ eigenvalues together with the six roots of $P_{\tilde{X_d}}$ form the spectrum of the graph X_d. Thus, $\lambda_2(X_d) = \lambda_2(\tilde{X_d}) = \rho(d)$. $\square$

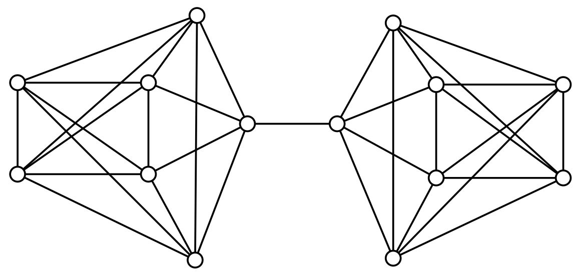

Figure 3: X_5 is 5-regular with $\lambda_2(X_5) = \rho(5) \approx 4.7969$

Using Maple, we find that $\rho(3) \approx 2.7784$ and $\rho(5) \approx 4.7969$. A simple algebraic manipulation shows that $\rho(d)$ satisfies the equation

$$d - \rho(d) = \frac{2d-2}{\rho^2(d) + 3\rho(d) + 2} > \frac{2d-2}{d^2+3d+2}, \tag{7}$$

where the last inequality follows since $\rho(d) < d$. Using (7), we deduce that

$$d - \frac{2}{d+4} < \rho(d) < d - \frac{2}{d+5} \tag{8}$$

for $d \geq 5$. We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. We will prove the contrapositive, namely we will show that if G is a d-regular graph with edge connectivity 1 (containing a bridge), then $\lambda_2(G) \geq \lambda_2(X_d) = \rho(d)$. We also show that equality happens if and only if $G = X_d$.

Consider a connected d-regular graph G that contains a bridge x_1x_2. Deleting the edge x_1x_2 partitions $V(G)$ into two connected components G_1 and G_2 such that $x_i \in V(G_i)$ for $i = 1, 2$. Let $n_i = |V(G_i)|$ for $i = 1, 2$. Without loss of generality, we assume from now on that $n_1 \leq n_2$. The graph G_1 contains $n_1 - 1$ vertices of degree d and one vertex of degree $d - 1$. Because d is odd, this implies that n_1 is odd. By symmetry, n_2 is also odd. Thus, $n_2 \geq n_1 \geq d + 2$. Note that $G = X_d$ if and only if $n_1 = n_2 = d + 2$.

Let A_2 be the quotient matrix of the partition of $V(G)$ into $V(G_1)$ and $V(G_2)$. Then

$$A_2 = \begin{bmatrix} d - \frac{1}{n_1} & \frac{1}{n_1} \\ \frac{1}{n_2} & d - \frac{1}{n_2} \end{bmatrix} \tag{9}$$

The eigenvalues of A_2 are: d and $\lambda_2(A_2) = d - \frac{1}{n_1} - \frac{1}{n_2}$.

Let A_3 be the quotient matrix of the partition of $V(G)$ into $V(G_1), \{x_2\}, V(G_2) \setminus \{x_2\}$. Then

$$A_3 = \begin{bmatrix} d-a & a & 0 \\ 1 & 0 & d-1 \\ 0 & b & d-b \end{bmatrix} \tag{10}$$

where $a = \frac{1}{n_1}$ and $b = \frac{d-1}{n_2-1}$. The eigenvalues of A_3 are d, $\lambda_2(A_3) = \frac{d-a-b+\sqrt{(d-a+b)^2+4(a-b)}}{2}$, $\lambda_3(A_3) = \frac{d-a-b-\sqrt{(d-a+b)^2+4(a-b)}}{2}$. Taking partial derivatives with respect to a and to b respectively, we find that when $d > 1$, the eigenvalue $\lambda_2(A_3)$ is strictly monotone decreasing with respect to both a and b and so is strictly monotone increasing with respect to both n_1 and n_2.

We consider now a partition of $V(G)$ into the following six parts: $V(G_1)\setminus(x_1\cup N(x_1))$, $N(x_1)\setminus\{x_2\}$, $\{x_1\}$, $\{x_2\}$, $N(x_2)\setminus\{x_1\}$ and $V(G_2)\setminus(x_2\cup N(x_2))$. Here $N(u)$ denotes the neighbourhood of vertex u in G. Let e_1 denote the number of edges between $N(x_1)\setminus\{x_2\}$ and $V(G_1)\setminus(x_1\cup N(x_1))$. Let e_2 denote the number of edges between $N(x_2)\setminus\{x_1\}$ and $V(G_2)\setminus(x_2\cup N(x_2))$. Note that $e_i \le (d-1)(n_i-d)$ for $i=1,2$.

Let A_6 be the quotient matrix of the previous partition of $V(G)$ into six parts. Then

$$A_6 = \begin{bmatrix} d-\frac{e_1}{n_1-d} & \frac{e_1}{n_1-d} & 0 & 0 & 0 & 0 \\ \frac{e_1}{d-1} & d-1-\frac{e_1}{d-1} & 1 & 0 & 0 & 0 \\ 0 & d-1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & d-1 & 0 \\ 0 & 0 & 0 & 1 & d-1-\frac{e_2}{d-1} & \frac{e_2}{d-1} \\ 0 & 0 & 0 & 0 & \frac{e_2}{n_2-d} & d-\frac{e_2}{n_2-d} \end{bmatrix} \tag{11}$$

Eigenvalue interlacing (see [8, 9, 10]) implies that

$$\lambda_2(G) \ge \max(\lambda_2(A_2), \lambda_2(A_3), \lambda_2(A_6)). \tag{12}$$

We will use this as well as the inequalities (7) and (8) to show that $\lambda_2(G) \ge \lambda_2(X_d) = \rho(d)$ and that equality happens if and only if $G = X_d$.

Recall that $n_2 \ge n_1 \ge d+2$ and that d, n_1 and n_2 are all odd.

If $n_1 \ge d+6$ and $d \ge 5$, then we use the partition of $V(G)$ into two parts whose quotient matrix A_2 is given in (9). Using inequality (8), we have that

$$\lambda_2(G) \ge \lambda_2(A_2) = d - \frac{1}{n_1} - \frac{1}{n_2} \ge d - \frac{2}{d+6} > \rho(d).$$

If $n_1 \ge d+6$ and $d = 3$, then we use the partition of $V(G)$ into three parts whose quotient matrix A_3 is given in (10). We have that

$$\begin{aligned}\lambda_2(G) \ge \lambda_2(A_3) &\ge \frac{3-\frac{1}{9}-\frac{2}{8}+\sqrt{\left(3-\frac{1}{9}-\frac{2}{8}\right)^2+4\left(\frac{1}{9}-\frac{1}{4}\right)}}{2} \\ &= \frac{95+\sqrt{12049}}{72} > 2.84 > \rho(3).\end{aligned}$$

If $n_2 \ge n_1 = d+4$, then we use the partition of $V(G)$ into three parts whose quotient matrix A_3 is shown in (10). We have that

$$\begin{aligned}\lambda_2(G) \ge \lambda_2(A_3) &\ge \frac{d-\frac{1}{d+4}-\frac{d-1}{d+3}+\sqrt{\left(d-\frac{1}{d+4}+\frac{d-1}{d+3}\right)^2+4\left(\frac{1}{d+4}-\frac{d-1}{d+3}\right)}}{2} \\ &= \frac{d^3+6d^2+8d+1+\sqrt{d^6+16d^5+88d^4+174d^3+8d^2-96d+385}}{2(d^2+7d+12)} \\ &= \frac{d^3+6d^2+8d+1+\sqrt{(d^3+8d^2+12d-9)^2+8d^2+120d+304}}{2(d^2+7d+12)}\end{aligned}$$

If $d=3$, the right hand side of the previous inequality equals $\frac{106+\sqrt{16612}}{84}>2.7962>\rho(3)$. When $d\geq 5$, from the previous inequality and (8) we obtain that

$$\begin{aligned}\lambda_2(G)&\geq \frac{d^3+6d^2+8d+1+\sqrt{(d^3+8d^2+12d-9)^2+8d^2+120d+304}}{2(d^2+7d+12)}\\&>\frac{d^3+6d^2+8d+1+(d^3+8d^2+12d-9)}{2(d^2+7d+12)}=\frac{d^3+7d^2+10d-4}{d^2+7d+12}\\&=d-\frac{2d+4}{d^2+7d+12}>d-\frac{2}{d+5}>\rho(d)\end{aligned}$$

If $n_1=d+2$ and $n_2\geq d+6$, then we use the partition of $V(G)$ into three parts whose quotient matrix is is given in (10). We obtain that

$$\begin{aligned}\lambda_2(G)\geq\lambda_2(A_3)&\geq\frac{d-\frac{1}{d+2}-\frac{d-1}{d+5}+\sqrt{\left(d-\frac{1}{d+2}+\frac{d-1}{d+5}\right)^2+4\left(\frac{1}{d+2}-\frac{d-1}{d+5}\right)}}{2}\\&=\frac{d^3+6d^2+8d-3+\sqrt{d^6+16d^5+80d^4+118d^3-24d^2+56d+329}}{2(d^2+7d+10)}\end{aligned}$$

When $d=3$, the right hand side equals $\frac{102+\sqrt{14564}}{80}>2.7835>\rho(3)$. If $5\leq d\leq 17$, then from the previous identity, we deduce that

$$\begin{aligned}\lambda_2(G)&\geq\frac{d^3+6d^2+8d-3+\sqrt{(d^3+8d^2+8d-5)^2-8d^2+136d+304}}{2(d^2+7d+10)}\\&>\frac{d^3+6d^2+8d-3+(d^3+8d^2+8d-5)}{2(d^2+7d+10)}\\&=\frac{d^3+7d^2+8d-4}{d^2+7d+10}=d-\frac{2}{d+5}>\rho(d).\end{aligned}$$

When $d\geq 19$, from the previous inequality we obtain that

$$\begin{aligned}\lambda_2(G)&\geq\frac{d^3+6d^2+8d-3+\sqrt{(d^3+8d^2+8d-6)^2+2d^3+8d^2+152d+293}}{2(d^2+7d+10)}\\&>\frac{d^3+6d^2+8d-3+(d^3+8d^2+8d-6)}{2(d^2+7d+10)}=\frac{d^3+7d^2+8d-4.5}{d^2+7d+10}\\&=d-\frac{2d+4.5}{d^2+7d+10}>d-\frac{2d-2}{d^2+3d+2}>\rho(d).\end{aligned}$$

The only case which remains to consider is $n_1=d+2$ and $n_2=d+4$. In this case, $e_1=2d-4, n_2-d=4$ and e_2 is an even integer with $4d-12\leq e_2\leq 4d-4$. Thus,

$$A_6=\begin{bmatrix}1 & d-1 & 0 & 0 & 0 & 0\\ 2 & d-3 & 1 & 0 & 0 & 0\\ 0 & d-1 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & d-1 & 0\\ 0 & 0 & 0 & 1 & d-1-\frac{e_2}{d-1} & \frac{e_2}{d-1}\\ 0 & 0 & 0 & 0 & \frac{e_2}{4} & d-\frac{e_2}{4}\end{bmatrix}$$

Let $P_{A_6}(x)$ denote the characteristic polynomial of A_6. Since each row sum of A_6 is d, it follows that d is a root of $P_{A_6}(x)$. Thus, $P_{A_6}(x) = (x-d)P_5(x)$. The second largest root of $P_{A_6}(x)$ is the largest root of $P_5(x)$. Using the results of [3] page 130, we observe that $P_5(x)$ equals the characteristic polynomial of the following matrix:

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & d-1 & d-2 & d-1 & 0 \\ 0 & 0 & 1 & 0 & \frac{e_2}{d-1} \\ 0 & 0 & 0 & 1 & d-\frac{e_2}{d-1}-\frac{e_2}{n_2-d} \end{bmatrix}$$

Using Maple to divide $P_5(x)$ by the polynomial $x^3+(3-d)x^2+(2-3d)x-2$, we obtain the quotient $Q(x) = x^2 - \frac{4d^2-4d-de_2-3e_2}{4(d-1)}x - \frac{e_2}{d-1}$ and the remainder $R(x) = (2d-e_2-2)x + \frac{de_2+e_2+4d-4d^2}{2}$. Thus,

$$P_5(x) = (x^3+(3-d)x^2+(2-3d)x-2)Q(x) + (2d-e_2-2)x + \frac{de_2+4d+e_2-4d^2}{2}.$$

Because $\rho(d)$ satisfies the equation $x^3+(3-d)x^2+(2-3d)x-2=0$, it follows that

$$\begin{aligned} P_5(\rho(d)) &= (2d-e_2-2)\rho(d) + \frac{de_2+4d+e_2-4d^2}{2} \\ &= e_2\left(-\rho(d)+\frac{d+1}{2}\right) + (2d-2)(\rho(d)-d). \end{aligned}$$

Because $d > \rho(d) > d-1 \geq \frac{d+1}{2}$, we deduce that

$$P_5(\rho(d)) < 0. \tag{13}$$

Assume that $\rho(d) \geq \lambda_2(A_6)$. Then $\rho(d)$ is larger than any root of $P_5(x)$. This implies that $P_5(\rho(d)) = \prod_{\theta \text{ root of } P_5}(\rho(d)-\theta) \geq 0$ which is a contradiction with (13). Thus, $\lambda_2(G) \geq \lambda_2(A_6) > \rho(d)$.

Hence, $\lambda_2(G) > \rho(d)$ whenever $n_2 \geq d+4$. The last case to be considered is $n_1 = n_2 = d+2$. This means that $G = X_d$ and $\lambda_2(G) = \rho(d)$ which finishes the proof. □

4 Proof of Theorem 1.5

In this section we present the proof of Theorem 1.5. We describe first the d-regular graph Y_d having $\kappa'(Y_d) = 2$ and $\lambda_2(Y_d) = \frac{d-3+\sqrt{(d+3)^2-16}}{2}$.

Consider the graph $H_d = K_{d-1} \vee \overline{K_2}$. It has $d-1$ vertices of degree d and two vertices of degree $d-1$. We construct Y_d by taking two disjoint copies of $K_{d-1} \vee \overline{K_2}$ and adding two disjoint edges between the vertices of degree $d-1$ in different copies of $K_{d-1} \vee \overline{K_2}$. Figure 4 describes Y_3 and Figure 5 shows Y_4.

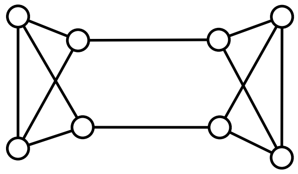

Figure 4: Y_3 is 3-regular with $\lambda_2(Y_3) = \theta(3) = \sqrt{5}$

The graph Y_d is d-regular, has $2d+2$ vertices and its edge-connectivity is 2. It has an equitable partition into four parts of sizes $d-1, 2, 2, d-1$ with the following quotient matrix:

$$\tilde{Y_d} = \begin{bmatrix} d-2 & 2 & 0 & 0 \\ d-1 & 0 & 1 & 0 \\ 0 & 1 & 0 & d-1 \\ 0 & 0 & 2 & d-2 \end{bmatrix} \tag{14}$$

The characteristic polynomial of this matrix equals

$$P_{\tilde{Y_d}}(x) = (x-d)(x+1)(x^2 + (3-d)x + (4-3d))$$

Thus, the eigenvalues of $\tilde{Y_d}$ are $d, 1$ and $\frac{d-3\pm\sqrt{(d+3)^2-16}}{2}$. To simplify our notation, let $\theta(d) = \frac{d-3+\sqrt{(d+3)^2-16}}{2}$. Note that $\theta(d)$ is the largest root of

$$T(x) = x^2 + (3-d)x + (4-3d) \tag{15}$$

and

$$d - \frac{4}{d+2} < \theta(d) < d - \frac{4}{d+3}. \tag{16}$$

Lemma 4.1. *The second largest eigenvalue of Y_d equals $\theta(d)$.*

Proof. Since the previous partition of $V(Y_d)$ into four parts is equitable, it follows that the four eigenvalues of $\tilde{Y_d}$ are also eigenvalues of Y_d. Obviously, the second largest of the eigenvalues of $\tilde{Y_d}$ is $\theta(d)$. By an argument similar to the one of Lemma 3.2, one can show that the other eigenvalues of Y_d are $(-1)^{(2d-2)}$ which implies the desired result. □

We are ready now to prove Theorem 1.5.

Proof of Theorem 1.5. We will prove the contrapositive, namely we will show that among all d-regular graphs with edge-connectivity less than or equal to 2, the graph Y_d has the smallest λ_2.

Let G be a d-regular graph with $\kappa'(G) \le 2$. We will prove that $\lambda_2(G) \ge \theta(d)$ with equality if and only if $G = Y_d$.

If $\kappa'(G) = 1$, then Theorem 1.4, (8) and (16) imply that $\lambda_2(G) \ge \rho(d) \ge d - \frac{2(d-1)}{(d+1)(d+2)} \ge d - \frac{4}{d+3} > \theta(d)$.

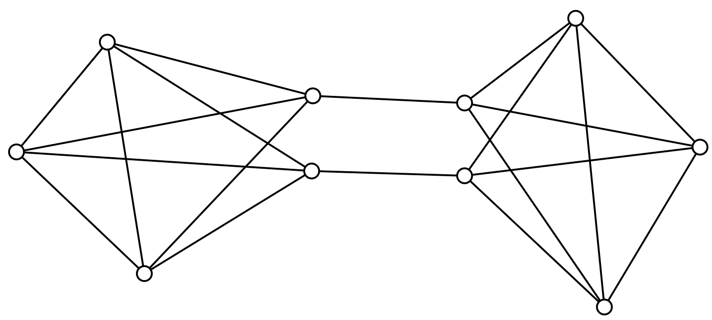

Figure 5: Y_4 is 4-regular with $\lambda_2(Y_4) = \theta(4) = \frac{1+\sqrt{33}}{2}$

If $\kappa'(G) = 2$, then there exists a partition of $V(G)$ into two parts V_1 and V_2 such that $e(V_1, V_2) = 2$. Let $S_1 \subset V_1$ and $S_2 \subset V_2$ denote the endpoints of the two edges between V_1 and V_2. We have that $(|S_1|, |S_2|) \in \{(1, 2), (2, 2), (2, 1)\}$. Let $n_i = |V_i|$ for $i \in \{1, 2\}$. It is easy to see that $n_i \geq d + 1$ for each $i \in \{1, 2\}$. Note that $n_1 = n_2 = d + 1$ is equivalent to $G = Y_d$.

Without loss of generality assume that $n_2 \geq n_1 \geq d + 1$. If $n_1 \geq d + 3$, consider the partition of $V(G)$ into V_1 and V_2. The quotient matrix of this partition is

$$\begin{bmatrix} d - \frac{2}{n_1} & \frac{2}{n_1} \\ \frac{2}{n_2} & d - \frac{2}{n_2} \end{bmatrix}$$

and its eigenvalues are d and $d - \frac{2}{n_1} - \frac{2}{n_2}$. Eigenvalue interlacing and $n_2 \geq n_1 \geq d+3$ imply that $\lambda_2(G) \geq d - \frac{2}{n_1} - \frac{2}{n_2} \geq d - \frac{4}{d+3}$. Using inequality (16), we obtain that $\lambda_2(G) \geq d - \frac{4}{d+3} > \theta(d)$ which finishes the proof of this case.

If $n_1 = d + 2$, then we have a few cases to consider.

If $|S_1| = 2$, then $e(S_1) = 0$ or $e(S_1) = 1$. If $e(S_1) = 0$, consider the partition of $V(G)$ into three parts: $V_1 \setminus S_1, S_1, V_2$. The quotient matrix of this partition is

$$\begin{bmatrix} d - \frac{2d-2}{d} & \frac{2d-2}{d} & 0 \\ d-1 & 0 & 1 \\ 0 & \frac{2}{n_2} & d - \frac{2}{n_2} \end{bmatrix}$$

Its characteristic polynomial equals

$$P_3(x) = (x - d)\left(x^2 - \frac{d^2n_2 + 2n_2 - 2dn_2 - 2d}{dn_2}x - \frac{2(d^2n_2 + n_2 + 2 - 2dn_2 - d)}{dn_2}\right)$$

If $P_2(x) = x^2 - \frac{d^2n_2+2n_2-2dn_2-2d}{dn_2}x - \frac{2(d^2n_2+n_2+2-2dn_2-d)}{dn_2}$, then

$$P_2(x) = T(x) + R(x)$$

where $R(x) = \frac{dn_2+2n_2-2d}{dn_2}\left(\frac{d^2n_2+2d-2n_2-4}{dn_2+2n_2-2d} - x\right)$. The expression $\frac{d^2n_2+2d-2n_2-4}{dn_2+2n_2-2d}$ is decreasing with n_2 and thus it attains its maximum when $n_2 = d+2$. This maximum equals $\frac{d^3+2d^2-8}{d^2+2d+4} =$

$d - \frac{4(d+2)}{d^2+2d+4} < d - \frac{4}{d+2}$. Thus, $P_2(\theta(d)) = R(\theta(d)) < \frac{dn_2+2n_2-2d}{dn_2}\left(d - \frac{4}{d+2} - \theta(d)\right) < 0$ where the last inequality follows from (16). This fact and eigenvalue interlacing imply that $\lambda_2(G) > \theta(d)$.

If $e(S_1) = 1$, then we consider the same partition into three parts: $V_1 \setminus S_1, S_1, V_2$. The quotient matrix is the following

$$\begin{bmatrix} d - \frac{2d-4}{d} & \frac{2d-4}{d} & 0 \\ d-2 & 1 & 1 \\ 0 & \frac{2}{n_2} & d - \frac{2}{n_2} \end{bmatrix}$$

Its characteristic polynomial equals

$$P_3(x) = (x-d)\left(x^2 - \frac{d^2n_2 + 4n_2 - dn_2 - 2d}{dn_2}x - \frac{d^2n_2 + 4n_2 + 8 - 6dn_2}{dn_2}\right).$$

If $P_2(x) = x^2 - \frac{d^2n_2+4n_2-dn_2-2d}{dn_2}x - \frac{d^2n_2+4n_2+8-6dn_2}{dn_2}$ then

$$P_2(x) = T(x) + R(x)$$

where $R(x) = \frac{2(dn_2+2n_2-d)}{dn_2}\left(\frac{d^2n_2+dn_2-2n_2-4}{dn_2+2n_2-d} - x\right)$. The expression $\frac{d^2n_2+dn_2-2n_2-4}{dn_2+2n_2-d}$ decreases with n_2 and thus, its maximum is attained at $n_2 = d+2$. This maximum equals $\frac{d^3+3d^2-8}{d^2+3d+4} = d - \frac{4(d+2)}{d^2+3d+4} < d - \frac{4}{d+2}$. As before, the previous inequality and eigenvalue interlacing imply that $\lambda_2(G) > \theta(d)$.

If $|S_1| = 1$, then d must be even. Indeed, the subgraph induced by V_1 contains $d+1$ vertices of degree d and one vertex of degree $d-2$. This cannot happen when d is odd. Thus, d is even and $d \geq 4$. Consider the partition of G into three parts: $V_1 \setminus S_1, S_1, V_2$. The quotient matrix of this partition is

$$\begin{bmatrix} d - \frac{d-2}{d+1} & \frac{d-2}{d+1} & 0 \\ d-2 & 0 & 2 \\ 0 & \frac{2}{n_2} & d - \frac{2}{n_2} \end{bmatrix}$$

Its characteristic polynomial is

$$P_3(x) = (x-d)\left(x^2 - \frac{d^2n_2 + 2n_2 - 2d - 2}{(d+1)n_2}x - \frac{d^2n_2 + 2d + 4n_2 + 8 - 4dn_2}{(d+1)n_2}\right)$$

If $P_2(x) = x^2 - \frac{d^2n_2+2n_2-2d-2}{(d+1)n_2}x - \frac{d^2n_2+2d+4n_2+8-4dn_2}{(d+1)n_2}$, then

$$P_2(x) = T(x) + R(x)$$

where $R(x) = \frac{2dn_2+5n_2-2d-2}{(d+1)n_2}\left(\frac{2d^2n_2+3dn_2-8n_2-2d-8}{2dn_2+5n_2-2d-2} - x\right)$. Because $d \geq 4$, the expression $\frac{2d^2n_2+3dn_2-8n_2-2d-8}{2dn_2+5n_2-2d-2}$ is decreasing with n_2 and thus, its maximum is attained when $n_2 = d+2$. This maximum equals $\frac{2d^3+7d^2-4d-24}{2d^2+7d+8} = d - \frac{12d+24}{2d^2+7d+8} < d - \frac{4}{d+2}$. This fact and eigenvalue interlacing imply $\lambda_2(G) > \theta(d)$.

Assume now that $n_1 = d+1$. This implies that V_1 induces a subgraph isomorphic to $K_{d-1} \vee \overline{K_2}$ and consequently, $|S_1| = 2$ and $e(S_1) = 0$.

If $n_2 \geq d+3$, then consider the partition of G into three parts: $V_1 \setminus S_1, S_1, V_2$. The quotient matrix of this partition is

$$\begin{bmatrix} d-2 & 2 & 0 \\ d-1 & 0 & 1 \\ 0 & \frac{2}{n_2} & d-\frac{2}{n_2} \end{bmatrix}$$

Its characteristic polynomial is

$$P_3(x) = (x-d)\left(x^2 - \left(d-2-\frac{2}{n_2}\right)x + 2 + \frac{2}{n_2} - 2d\right).$$

If $P_2(x) = x^2 - \left(d-2-\frac{2}{n_2}\right)x + 2 + \frac{2}{n_2} - 2d$, then

$$P_2(x) = T(x) + \frac{dn_2 + 2 - 2n_2 - (n_2-2)x}{n_2}$$

which implies that

$$P_2(\theta(d)) = \frac{dn_2 + 2 - 2n_2 - (n_2-2)\theta(d)}{n_2}$$

The expression $\frac{dn_2+2-2n_2-(n_2-2)\theta(d)}{n_2}$ is decreasing with n_2 and therefore, its maximum is attained when $n_2 = d+3$. Thus,

$$\begin{aligned} P_2(\theta(d)) &\leq \frac{d^2+d-4-(d+1)\theta(d)}{d+3} \\ &= \frac{(d+1)\left(d-\frac{4}{d+1}-\theta(d)\right)}{d+3} < 0. \end{aligned}$$

This inequality and eigenvalue interlacing imply that $\lambda_2(G) > \theta(d)$.

Thus, the remaining case is $n_1 = d+1$ and $n_2 \leq d+2$. If d is odd, then $n_2 \neq d+2$. Otherwise, the sum of the degrees of the graph induced by V_2 would equal $d|V_2| - 2 = d(d+2) - 2$ which is an odd number. This is impossible and thus, $n_2 = d+1$. This implies $G = Y_d$.

If $n_2 = d+2$ and d is even, then we have a few cases to consider. If $|S_2| = 1$, then both vertices in S_1 are adjacent to the vertex a of S_2. Thus, a has exactly $d-2$ neighbours in V_2 which means there are 3 vertices of V_2 which are not adjacent to a. Each of these three vertices has degree d and the only way this can happen is they form a clique and each of them is adjacent to the $d-2$ neighbours of a in V_2. Using a degree argument, it also follows that the $d-2$ neighbours of a in V_2 induce a subgraph isomorphic to the complement of a perfect matching on $d-2$ vertices $\overline{M_{d-2}}$. Using the notation from the previous section, it follows that $G = K_{d-1} \vee \overline{K_2} \vee K_1 \vee \overline{M_{d-2}} \vee K_3$. The graph G has an obvious equitable partition into five parts whose quotient matrix is

$$\begin{bmatrix} d-2 & 2 & 0 & 0 & 0 \\ d-1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & d-2 & 0 \\ 0 & 0 & 1 & d-4 & 3 \\ 0 & 0 & 0 & d-2 & 2 \end{bmatrix}$$

Its characteristic polynomial equals $P_5(x) = (x - d)P_4(x)$ where $P_4(x) = x^4 + (-d+4)x^3 + (-4d+4)x^2 - 8x + 6d - 12$. Dividing $P_4(x)$ by $T(x)$ we get that $P_4(x) = T(x)Q_4(x) + R_4(x)$ where $Q_4(x) = x^2 + x - 3$ and $R_4(x) = -3x - 3d$. It follows that $P_4(\theta(d)) = R_4(\theta(d)) = -3\theta(d) - 3d < 0$. This fact and eigenvalue interlacing imply that $\lambda_2(G) > \theta(d)$.

The case remaining is $|S_2| = 2$. If $e(S_2) = 0$, then each vertex of S_2 has exactly $d-1$ neighbours in V_2. Let $S_2 = \{x_2, y_2\}$. If x_2 and y_2 are adjacent to the same $d-1$ vertices of V_2, then because $|V_2| = n_2 = d+2$, there is exactly one vertex of V_2 outside the vertices of S_2 and their $d-1$ common neighbours. This vertex cannot have degree d. Thus, this case cannot happen and therefore, both x_2 and y_2 have exactly $d-2$ common neighbours in V_2. We call this set U. Let $\{u_2, w_2\} = V_2 \setminus (\{x_2, y_2\} \cup U)$. Because x_2 and y_2 have $d-1$ neighbours in V_2, we may assume that x_2 is adjacent to u_2 and y_2 is adjacent to w_2. A degree argument implies that u_2 and w_2 must be adjacent and both of them are adjacent to each vertex of U. Finally, the subgraph induced by U must be $(d-4)$-regular.

The following is a five-part equitable partition of G: $V_1 \setminus S_1, S_1, S_2, U, \{u_2, w_2\}$. The quotient matrix of this partition is the following:

$$\begin{bmatrix} d-2 & 2 & 0 & 0 & 0 \\ d-1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & d-2 & 1 \\ 0 & 0 & 2 & d-4 & 2 \\ 0 & 0 & 1 & d-2 & 1 \end{bmatrix}$$

Its characteristic polynomial equals $P_5(x) = (x-d)P_4(x)$ where $P_4(x) = x^4 + (5-d)x^3 + (10-5d)x^2 + (-6d+7)x - d$. Dividing $P_4(x)$ by $T(x)$ we get that $P_4(x) = T(x)Q_4(x) + R_4(x)$ where $Q_4(x) = x^2 + 2x$ and $R_4(x) = -x - d$. It follows that $P_4(\theta(d)) = R_4(\theta(d)) = -\theta(d) - d < 0$. This and eigenvalue interlacing show that $\lambda_2(G) > \theta(d)$.

The final case of our proof is $e(S_2) = 1$. Because $|V_2 \setminus S_2| = d$ and both x_2 and y_2 have $d-2$ neighbours in $V_2 \setminus S_2$, it follows that x_2 and y_2 have at least $d-4$ common neighbours in $V_2 \setminus S_2$. If x_2 and y_2 have at least $d-3$ common neighbours in $V_2 \setminus S_2$, we deduce that there exists at least one vertex z of $V_2 \setminus S_2$ that is not adjacent to x_2 nor y_2. The vertex z cannot have degree d since its only possible neighbours are in $V_2 \setminus (S_2 \cup \{z\})$ which has size $d-1$. We conclude that x_2 and y_2 must have precisely $d-4$ common neighbours in $V_2 \setminus S_2$. There are 4 remaining vertices a_2, b_2, c_2, d_2 in $V_2 \setminus S_2$ and without loss of generality, assume that x_2 is adjacent to both a_2 and b_2 and y_2 is adjacent to both c_2 and d_2. Each of the vertices a_2, b_2, c_2, d_2 will be adjacent to every other vertex of $V_2 \setminus S_2$. The degree constraint implies that the remaining $d-4$ vertices of $V_2 \setminus S_2$ induce a $(d-6)$-regular subgraph of G. Obviously, this argument shows this case is only possible when $d \geq 6$.

The following is a five-part equitable partition of G: $V_1 \setminus S_1, S_1, S_2, \{a_2, b_2, c_2, d_2\}, V_2 \setminus (S_2 \cup \{a_2, b_2, c_2, d_\}\})$. Its quotient matrix is the following:

$$\begin{bmatrix} d-2 & 2 & 0 & 0 & 0 \\ d-1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & d-4 \\ 0 & 0 & 1 & 3 & d-4 \\ 0 & 0 & 2 & 4 & d-6 \end{bmatrix}$$

Its characteristic polynomial equals $P_5(x) = (x-d)P_4(x)$ where $P_4(x) = x^4 + (-d+4)x^3 + (-4d+6)x^2 + (-2d-1)x + 3d - 6$. Dividing $P_4(x)$ by $T(x)$ we get that $P_4(x) = T(x)Q_4(x) + R_4(x)$ where $Q_4(x) = x^2 + x - 1$ and $R_4(x) = -2x - 2$. It follows that $P_4(\theta(d)) = R_4(\theta(d)) = -2\theta(d) - 2 < 0$. This and eigenvalue interlacing imply that $\lambda_2(G) > \theta(d)$ which finishes our proof. □

5 Some Remarks

Any strongly regular graph of degree $d \geq 3$ satisfies the condition $\lambda_2 \leq d-2$ and thus, is d-edge-connected. The fact that the edge-connectivity of a strongly regular graph equals its degree, was observed by Plesník in 1975 (cf. [2]). As mentioned in the introduction, much more is true, namely the vertex-connectivity of any distance-regular graph equals its degree (see [5]). It is known that any vertex transitive d-regular graph whose second largest eigenvalue is simple has $\lambda_2(G) \leq d-2$ and consequently, is d-edge-connected. In fact, any vertex transitive d-regular graph is d-edge-connected as shown by Mader in 1971 (see [14] or Chapter 3 of [8]).

I expect that Theorem 1.4 and Theorem 1.5 can be extended to other values of edge-connectivity and vertex-connectivity. For example, it seems that $\lambda_2(G) \leq d - \frac{1}{2}$ implies $\kappa(G) \geq 2$. Note however that in many cases, Fiedler's bound $\kappa(G) \geq d - \lambda_2$ cannot be improved. When $2k-2 \geq d$ and dk is even, consider the graph $2K_{d-k+1} \vee H$ where H is a $(2k-2-d)$-regular graph on k vertices. This graph is d-regular, has vertex connectivity k and its second largest eigenvalue equals $d-k$.

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