Cospectral Graphs
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Introduction
A graph G is a set of vertices, V(G), with a set of edges, E(G), connecting them. Two vertices are adjacent if they are connected. Two graphs are called isomorphic if there exists an edge-preserving bijection between the set of vertices. In other words, if one can rearrange the vertices of one graph to get another, then the two graphs are isomorphic.

The original graph G and the obtained graph G’ are cospectral, if the two graphs are not isomorphic. Here are a couple of examples of GM switching:

Spectrum of a Graph
For a simple graph G with n vertices {1, 2, ..., n}, the associated adjacency matrix A is an n × n matrix whose (i,j)-th entry is 1 if i and j are adjacent, and 0 otherwise. The set of eigenvalues, including multiplicities, is called the spectrum of A, as well as the spectrum of G.

A good amount of information about a graph is coded in the spectrum. Here, one question is of particular interest: does the spectrum uniquely determine a graph?

If a graph is not determined by its spectrum, there should a non-isomorphic graph that has the same spectrum as the original graph. These two graphs are called cospectral.

There are known examples of DS graphs, as well as graphs that are not cospectral. The set of eigenvalues, including multiplicities, is called the spectrum of A, as well as the spectrum of G. The original graph G and the obtained graph G’ are cospectral, if the two graphs are not isomorphic.

Here, both graphs have the same spectrum: 2,0,0,0,-2, but they are clearly not isomorphic.

Kneser Graphs
A Kneser graph, denoted K(n, k) is a graph whose vertices are all the k-subsets of the set {1,2,3,...,n}. Two vertices are adjacent if and only if their intersection is empty.

On the left is a picture of K(5,3). As n and k get large, the complexity of Kneser graphs become immense, as we can see from K(9,3) on the right, the graph we closely examined.

Godsil-McKay Switching
One of the most efficient methods to generate cospectral graphs is discovered by Godsil and McKay in 1982 [1], thus named Godsil-McKay switching. For a graph G, we can partition the vertex set into a set D and one or more sets C1, C2, ..., Cn, such that:

1. For any i, j from the set {1,2,...,n}, let ci be the number of vertices in Ci. Then each vertex in D is adjacent to exactly 0, ci/2, or ci vertices in Ci; then, the Ci’s are called switching sets, and we take each vertex in D that is adjacent to ci/2 vertices in Ci and switch the adjacencies to the other ci/2 vertices, obtaining graph G.

For \( l = 1,2 \), they found solutions to equation (1), solving the cases. For \( l = 3 \), it is proven that there is no solution. For \( l \geq 4 \), we found by computer that there is no solution for n up to 10000.

It was conjectured by Erdős that for \( l \geq 3 \), there is only finitely many solutions.

Based on the known results, the smallest open case to the problem is \( K(9,3) \).

On the right is a picture of K(9,3), as shown in figure 8 to find switching sets, but it did not work out as nicely as the previously known diagram for K(8,3), in which a switching set can be found easily. For n=9,10, it has been checked by computer that there are no switching sets of size 4 or 6.

We checked by computer that there is no independent switching set of size 8 in K(9,3). Most of the previously found switching sets for Kneser graphs are independent.

We also found by computer that there is no group of 2 or 3 switching sets where one of them has 4 vertices.

My group member, Jason, also tried to use Steiner Triple System and a more generalized triple system to find switching set for K(9,3), but it did not work out.

The results we got seem to indicate that there is no switching set in K(n,3), for \( n \geq 9 \). But because GM switching is not the only way to find cospectral graphs, future work needs to be done about these Kneser graphs are DS.

Are Kneser Graphs Determined by Spectrum?
K(n,k) is DS if k = \( n/2 \), \( k \geq n/4 \), or \( n \geq 2k+1 \). Other than that, very little is known.

However, Haemers and Ramezani [2] used GM switching and solved the problem for some cases. For a Kneser graph K(n,k), if there exists an l such that \( (n-1)l = 2 \binom{n-1}{2} + 1 \), and \( l < k < n/2 \), then a GM switching set can be found. Further, if \( l < k < n/2 \), then the graph obtained by GM switching will not be isomorphic to the original graph. Thus, if we can find an l that satisfies the equation for a certain K(n,k), then we know it is not DS.

For \( l = 4 \), they found solutions to equation (1), solving the cases. For \( l = 5 \), it is proven that there is no solution. For \( l \geq 6 \), we found by computer that there is no solution for n up to 1400.

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Work Cited