Intersecting Family of Triangulations

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Triangulations and Catalan Numbers
Consider the number of ways to divide a convex n-gon into (n-2) triangles using (n-3) non-crossing diagonals. The number of such triangulations is the (n-3)th Catalan number.

<table>
<thead>
<tr>
<th>k Subsets Size</th>
<th>Triangulations</th>
<th>Catalan Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>C_1 = \frac{1}{1}</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>C_2 = \frac{3}{2}</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>C_3 = \frac{11}{6}</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>C_4 = \frac{29}{20}</td>
</tr>
</tbody>
</table>

The first Catalan numbers for n = 0, 1, 2, 3, 4, 5, 6, ... are:
1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, ...

Gil Kalai Conjecture (2013)
Let T_n be the set of all triangulations of an n-gon using (n-3) non-intersecting diagonals. Let \( S \) be a subfamily of those triangulations such that every two triangulations of \( S \) share a common diagonal. Then \( |S| \) is at most the number of triangulations such that every two triangulations of \( S \) share a common diagonal.

Using code in SageMath, we calculated the degree of each vertex to determine the regularity of the graph. The data points are regular for n ≤ 6.

Erdős-Ko-Rado Theorem (1961)
Let \( F \) be an intersecting family of k-subsets from a set S of size n, then \( |F| \leq \binom{n}{k}/2^{k-1} \) and equality implies that \( F \) consists of the k-subsets that contain a fixed element \( i \in S \).

<table>
<thead>
<tr>
<th>k Subsets from [n]</th>
<th>Triangulations of an n-gon</th>
</tr>
</thead>
<tbody>
<tr>
<td>( [i] )</td>
<td>( C_n = \frac{1}{n+1} \binom{2n-1}{n} )</td>
</tr>
<tr>
<td>Upper Bound</td>
<td>( C_n = \frac{1}{n+1} \binom{2n-1}{n} )</td>
</tr>
<tr>
<td>Equality</td>
<td>k-subsets containing a fixed element ( i )</td>
</tr>
</tbody>
</table>

Hoffman’s Ratio Bound Theorem
(1) Let G be a k-regular graph with no loops and \( \lambda_{min} \) the least eigenvalue of its adjacency matrix. For any independent set \( S \) of size \( s \), we have:

\[ s \leq \frac{|S|^2}{\lambda_{min}} \]

Eigenvalue Bound of Independent Sets
(2) Let G be a graph with no loops, and \( \lambda_{min} \) the least eigenvalue of its adjacency matrix. For any independent set \( S \) of size \( s \), we have:

\[ s \leq \frac{|S|^2}{\lambda_{min}} \]

Using the eigenvalue bound for \( n=6 \) we obtain:

\[ |S| \leq 2 \]

Using the eigenvalue bound for \( n=7 \) we obtain:

\[ |S| \leq 2 \]

Since these eigenvalues are larger than the corresponding (n-3)th Catalan number, this indicates algebraically that the maximum size of \( S \) could be greater than \( C_n \) for \( n=5 \) and \( n=7 \). This contradicts the conjecture.

Graph Theory and SageMath
To investigate this problem, we constructed a graph, \( G_n \), using SageMath such that the vertices of \( G_n \) are all the triangulations of an n-gon and the edges represent adjacent vertices. Vertices are adjacent if the triangulations do not contain any common diagonals.

Using the eigenvalue bounds for \( n=6 \) and \( n=7 \), we obtained triangulations into “look-alike” triangulations. In order to construct an equitable partition, in SageMath we found an automorphism of the graph and its orbits.

Equitable Partitions
Definition:
Let \( G = (V,E) \) be a graph and \( \pi \) be a partition of \( V \). Then \( \pi \) is equitable if for any \( U, V \subset X \) there is a constant \( b_x \) such that \( |x \in X, \text{the number of neighbors that x has on } S, \text{the number of neighbors that x has on } X - S | \leq b_x \).

Proposition
Let \( G = (V,E) \) be a graph and \( \pi \) be an equitable partition of \( G \). Also let \( B = [b_{ij}] \) be the quotient matrix of the partition. Then every eigenvalue of \( B \) is an eigenvalue of \( A \), the adjacency matrix of \( G \).

Since there exists an equitable partition, we can investigate the proposition related to the eigenvalues of the matrices.

Quotient Matrix Eigenvalues
We investigate the quotient matrix because it is difficult to calculate the eigenvalues bounds using the adjacency matrix due to its large size. The list of eigenvalues of the adjacency matrix contains all the eigenvalues of the quotient matrix. If the least eigenvalue of the quotient matrix is always the same as the adjacency matrix, then we can obtain the eigenvalue bounds using the quotient matrix.

Let \( A \) be the adjacency matrix of \( G \) and \( B \) be the quotient matrix of the equitable partition.

Future Work
We hope to investigate more about the eigenvalue bounds of independent sets for regular graphs. We intend to partition the vertex set of the graph \( G \) into smaller regular graphs and then apply the eigenvalue bounds of independent sets for regular graphs. We seek to prove the conjecture for any size \( n \).

References