On decompositions of complete hypergraphs

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Abstract

We study the minimum number of complete r-partite r-uniform hypergraphs needed to partition the edges of the complete r-uniform hypergraph on n vertices and we improve previous results of Alon.

1 Introduction

Given an r-uniform hypergraph H, let $f_r(H)$ denote the minimum number of complete r-partite r-uniform hypergraphs needed to partition the edge set of H. When $r = 2$, the parameter $f_2(H)$ is also known as the biclique partition number of the graph H and has been well studied (see [2, 9]). When $H = K_n$, this parameter equals $n - 1$ as shown by Graham and Pollak [5] (see also [2, 10, 12, 14] for other proofs).

In this note, we are interested in extending the theorem of Graham and Pollak to complete r-uniform hypergraphs for larger values of r. This seems to be a difficult and interesting extremal problem with connections to other areas such as theoretical computer science (the complexity of computing bilinear forms and symmetric polynomials, see [6, 8, 11]) and linear algebra (tensor rank computation of high dimensional arrays, see [8]).

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To simplify our notation, let $f_r(n) = f_r(K_n^{(r)})$, where $K_n^{(r)}$ denotes the complete r-uniform hypergraph on n vertices. In [1], Alon showed that $f_3(n) = n - 2$ and that for $n \geq 2k \geq 4$,

$$f_{2k}(n) \geq \frac{2\left(\binom{n}{k} - \binom{n}{k-1} - \binom{n}{k-3} - \cdots - \binom{n}{k+1-2\lceil \frac{k}{2} \rceil}\right)}{\binom{2k}{k}}. \tag{1}$$

It was also proved in [1] by a recursive construction that

$$f_r(n) \leq \sum_{i=0}^{r} f_i(\lfloor n/2 \rfloor) f_{r-i}(\lceil n/2 \rceil). \tag{2}$$

Using (2), Alon showed that for fixed $k \geq 2$, $f_{2k}(n) \leq \frac{n^k}{k!}(1 + o(1))$ as $n \to \infty$.

In this note, we use known results on biclique decomposition of the Kneser graphs to improve and simplify (1) and (2). We show that

$$\frac{2\binom{n-1}{k}}{\binom{2k}{k}} \leq f_{2k}(n) \leq \binom{n-k}{k}. \tag{3}$$

2 The proofs of our results

We follow Bollobás [3] for our hypergraph notation. We use $[n]$ to denote the set $\{1, \ldots, n\}$ and $[n]^{(r)}$ to denote the family of all r-subsets of $[n]$. If $X_1, \ldots, X_r$ are pairwise disjoint subsets of $[n]$, we denote by $\prod_{i=1}^{r} X_i$ the set of r-subsets Y of $[n]$ such that $|Y \cap X_i| = 1$ for each $i \in [r]$. The complete r-partite r-uniform hypergraph whose parts are $X_1, \ldots, X_r$ is the r-uniform hypergraph whose edge-set is $\prod_{i=1}^{r} X_i$. When $r = 2$ for example, $\prod_{i=1}^{2} X_i$ is the edge-set of the complete bipartite subgraph (or biclique) of K_n whose colour classes are X_1 and X_2.

In this section, we present the proofs of our main results. We use known results about the biclique partition numbers of the Kneser graphs to prove the lower bound. Recall that the Kneser graph $K_{n:k}$ has vertex set $[n]^{(k)}$ with two k-subsets being adjacent if and only if they are disjoint.

Theorem 2.1. *For $n \geq 2k \geq 2$,*

$$f_{2k}(n) \geq \frac{2\binom{n-1}{k}}{\binom{2k}{k}}.$$

Proof. Let $m = f_{2k}(n)$ and consider a partition of the edge set of $K_n^{(2k)}$ into complete $2k$-partite $2k$-uniform hypergraphs $H_1, \ldots, H_m$. For $i \in [m]$, let $A_1^i, \ldots, A_{2k}^i$ denote the parts of H_i.

For each $i \in [m]$, consider the following $\frac{\binom{2k}{k}}{2}$ bicliques of the Kneser graph $K_{n:k}$. For each partition X, Y of $[2k]$ with $|X| = |Y| = k$, we construct the biclique of $K_{n:k}$ whose colour classes are $\prod_{j \in X} A_j^i$ and $\prod_{l \in Y} A_l^i$.

We claim that these $m \cdot \frac{\binom{2k}{k}}{2}$ bicliques partition the edge set of $K_{n:k}$. Let xy be an edge of $K_{n:k}$, where $x, y \in [n]^{(k)}$. This means $x \cap y = \emptyset$ and thus, $x \cup y$ is a $2k$-subset of $[n]$. Because

$H_1, \dots, H_m$ partition the edge set of $K_n^{(2k)}$, it follows that there is a unique $i \in [m]$ such that $x \cup y \in E(H_i)$. Now there exists a unique partition X, Y of $[2k]$ such that $|X| = |Y| = k$, $x \in \prod_{j \in X} A_j^i$ and $y \in \prod_{l \in Y} A_l^i$. This proves our claim and implies that

$$f_{2k}(n) \geq \frac{2 f_2(K_{n:k})}{\binom{2k}{k}}$$

We show that $f_2(K_{n:k}) = \binom{n-1}{k}$. This was proved by Vander Meulen [13] (see also [7]). For the sake of completeness, we give a short proof here. First, we can partition all the edges of $K_{n:k}$ by stars centered at vertices $x \in [n-1]^{(k)}$. This is possible because $[n-1]^{(k)}$ is a vertex cover of $K_{n:k}$ and implies $f_2(K_{n:k}) \leq \binom{n-1}{k}$. Recall that $f_2(G) \geq h(G)$, where $h(G)$ is the maximum of the number of positive, and of the number of negative eigenvalues of the adjacency matrix of G (see [5, 7] for more details). When $G = K_{n:k}$, it is known (see [4] or [7]) that $h(G) = \binom{n-1}{k}$. This implies $f_2(K_{n:k}) \geq \binom{n-1}{k}$ and finishes the proof of the theorem. □ □

We obtain the upper bound for $f_{2k}(n)$ by a simple direct construction.

Theorem 2.2. *For each $n \geq 2k+1 \geq 3$, we have that*

$$f_{2k}(n-1) \leq f_{2k+1}(n) \leq \binom{n-k-1}{k}.$$

Proof. We prove first that $f_{2k+1}(n) \leq \binom{n-k-1}{k}$. For each k-tuple $1 < i_1 < i_2 < \dots i_k < n$ such that $i_{j+1} - i_j > 1$ for any $1 \leq j \leq k-1$, consider the complete $(2k+1)$-partite $(2k+1)$-uniform hypergraph $H_{i_1,\dots,i_k}$ whose parts are $\{1, \dots, i_1 - 1\}, \{i_1\}, \{i_1 + 1, \dots i_2 - 1\}, \dots, \{i_{k-1} + 1, \dots, i_k - 1\}, \{i_k\}$ and $\{i_k + 1, \dots, n\}$.

Note that the hypergraphs $H_{i_1,\dots,i_k}$ partition the edge set of $K_n^{(2k+1)}$. This is because any edge $j_1 j_2 \dots j_{2k+1}$ with $1 \leq j_1 < j_2 < \dots < j_{2k+1} \leq n$ is contained in precisely one of these hypergraphs, namely $H_{j_2, j_4, \dots, j_{2k}}$.

Because $1 \leq i_1 - 1 < \dots < i_k - k \leq n - k - 1$, it follows that there are $\binom{n-k-1}{k}$ such hypergraphs $H_{i_1,\dots,i_k}$. This implies that $f_{2k+1}(n) \leq \binom{n-k-1}{k}$. The inequality $f_{2k}(n-1) \leq f_{2k+1}(n)$ is obvious and its proof is omitted (see [1] Lemma 2.1). □ □

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References

[1] N. Alon, Decomposition of the complete r-graph into complete r-partite r-graphs, *Graphs and Combinatorics* **2** (1986), 95-100.

[2] L. Babai and P. Frankl, *Linear Algebra Methods in Combinatorics*, Department of Computer Science, The University of Chicago, September 1992.

[3] B. Bollobás, *Combinatorics.* Set systems, hypergraphs, families of vectors and combinatorial probability, Cambridge University Press, 1986.

[4] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer 2001.

[5] R.L. Graham and H.O. Pollak, On the addressing problem for loop switching, *Bell Syst. Tech. J.* **50** (8) (1971), 2495-2519.

[6] V. Grolmusz, Computing elementary symmetric polynomials with a sub-polynomial number of multiplications, *SIAM J. Comput.* **32** (2003), 1475-1487.

[7] D.A. Gregory, B. Heyink and K.N. Vander Meulen, Inertia and Biclique Decomposition of Joins of Graphs, *Journal of Combin. Theory, Series B* **88** (2001), 135-151.

[8] J. Håstad, Tensor rank is NP-complete, *J. of Algorithms* **11** (1990), 644-654.

[9] S.D. Monson, N.J. Pullman and R. Rees, A survey of clique and biclique coverings and factorizations of $(0, 1)$-matrices, *Bull. Inst. Combin. Appl.* **14** (1995), 17-86.

[10] G. Peck, A new proof of a theorem of Graham and Pollak, *Discrete Math.* **49** (1984), 327-328.

[11] J. Radhakrishnan, P. Sen and S. Vishwanathan, Depth-3 arithmetic for $S_n^2(X)$ and extensions of the Graham-Pollack theorem, *FST TCS 2000: Foundations of software technology and theoretical computer science* (New Delhi), 176–187, Lecture Notes in Comput. Sci., Springer, 2000.

[12] H. Tverberg, On the decomposition of K_n into complete bipartite graphs, *J. Graph Theory* **6** (1982), 493-494.

[13] K.N. Vander Meulen, Covers and Decompositions of Graphs by Complete Multipartite Subgraphs, Ph.D. Thesis, Queen's University, Kingston 1995.

[14] S. Vishwanathan, A polynomial space proof of the Graham-Pollak theorem, *J. Combin. Theory Ser. A* **115** (2008), 674-676.