Homework 1, Math 245, Fall 2010
Due Wednesday, September 15 in class.

The solution of each exercise should be at most one page long. If you can, try to write your solutions in LaTeX. Each question is worth 2 points.

1. Let $A, B, C$ be sets. Prove that $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.

Proof. We prove this statement by double inclusion.

First, we show that $A \cap (B \setminus C) \subset (A \cap B) \setminus (A \cap C)$.

Let $x \in A \cap (B \setminus C)$. This means $x \in A$ and $x \in B \setminus C$. The last statement implies $x \in B$ and $x \notin C$. Now, as $x \in A$ and $x \in B$, it follows that $x \in A \cap B$. Also, as $x \in A$ and $x \notin C$, we deduce that $x \notin A \cap C$ (otherwise, $x \in A \cap C$ would imply $x \in C$ which is a contradiction with $x \notin C$). Thus, we obtained that $x \in A \cap B$ and $x \notin A \cap C$. Hence, $x \in (A \cap B) \setminus (A \cap C)$ which shows $A \cap (B \setminus C) \subset (A \cap B) \setminus (A \cap C)$.

Now we prove the opposite inclusion $(A \cap B) \setminus (A \cap C) \subset A \cap (B \setminus C)$.

Let $y \in (A \cap B) \setminus (A \cap C)$. This means $y \in A \cap B$ and $y \notin A \cap C$. The fact that $x \in A \cap B$ means that $x \in A$ and $x \in B$. Now as $y \in A$ and $y \notin A \cap C$, we deduce that $y \notin C$ (otherwise, $y \in C$ combined with $y \in A$ would imply $y \in A \cap C$ which is a contradiction with $y \notin A \cap C$). Thus, we have proved that $y \in A, y \in B$ and $y \notin C$. The last two facts imply $y \in B \setminus C$. Combining this with $y \in A$ shows that $y \in A \cap (B \setminus C)$. Hence, we have proved that $(A \cap B) \setminus (A \cap C) \subset A \cap (B \setminus C)$. This finishes the proof of our exercise.

2. Let $A = \{x \in \mathbb{R} : |x + 1| \leq 5\}$. Show that $A = [-6, 4]$.

Proof. 1st proof We can prove this statement by double inclusion.

If $x \in A$, then $|x + 1| \leq 5$. This means $x + 1 \leq |x + 1| \leq 5$ and $-(x + 1) \leq |x + 1| \leq 5$. The first inequality means $x \leq 4$ and the second inequality means $-x - 1 \leq 5$ which is the same as $x \geq -6$. Thus, $-6 \leq x \leq 4$ which means $x \in [-6, 4]$. This shows $A \subset [-6, 4]$.

If $x \in [-6, 4]$, then $-6 \leq x \leq 4$. Adding 1 to these inequalities we get $-5 \leq x + 1 \leq 5$. This implies $-(x + 1) \leq 5$ which means $|x + 1| = \max(x + 1, -(x + 1)) \leq 5$. Thus, $x \in A$. This shows $[-6, 4] \subset A$ and finishes the proof.

2nd proof We can prove this directly as follows.

$x \in A \iff |x + 1| \leq 5$
$\iff \max(x + 1, -(x + 1)) \leq 5$
$\iff (x + 1 \leq 5) \land (-(x + 1) \leq 5)$
$\iff (x \leq 4) \land (x \geq -6)$
$\iff (x \in [-6, 4])$.
In this proof, make sure that every step is actually an equivalence (for example, $|x+1| \leq 5$ is actually equivalent with $\max(x+1,-(x+1)) \leq 5$ etc.).

3. Let $f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{x^2}{1+x^4}$. What is the image of $f$?

Proof. 1st proof using calculus We calculate the derivative of $f(x)$:

$$f'(x) = \left(\frac{x^2}{1+x^4}\right)' = \frac{(x^2)'(1+x^4) - (x^2)(1+x^4)'}{(1+x^4)^2} = \frac{2x(1+x^4) - x^2(4x^3)}{(1+x^4)^2} = \frac{2x - 2x^5}{(1+x^4)^2} = \frac{2x(1-x^4)}{(1+x^4)^2}.$$

This shows that the critical points of $f$ (the points where the derivative is 0) are $-1, 0, 1$. We also find that $f'(x) \leq 0$ when $x \in [-1, 0] \cup [1, +\infty)$ and $f'(x) \geq 0$ when $x \in (-\infty, -1] \cup [0,1]$. Because $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to +\infty} f(x) = 0$, we deduce that $f$ attains a global minimum at $x = 0$ and attains its global maximum at $x = -1$ and at $x = 1$. The values of $f$ at these points are $f(0) = 0$ and $f(-1) = f(1) = \frac{1}{2}$. Because $f$ is continuous, this implies that $f$ will attain all the values between 0 and $\frac{1}{2}$ which means the image of $f$ is the interval $[0, \frac{1}{2}]$.

2nd proof using elementary methods

The arithmetic mean-geometric mean inequality states that $1 + x^4 \geq 2\sqrt{1 \cdot x^4} = 2x^2$ for every $x \in \mathbb{R}$. This implies that $\frac{x^2}{1+x^4} \leq \frac{1}{2}$ for every $x \in \mathbb{R}$ with equality if and only if $x = -1$ or $x = 1$. Stated in terms of $f$, this inequality means $f(x) = \frac{x^2}{1+x^4} \leq f(1) = f(-1) = \frac{1}{2}$. On the other hand, $f(x) = \frac{x^2}{1+x^4} \geq f(0) = 0$ as the numerator $x^2 \geq 0$ and the denominator $1 + x^4 > 0$. This shows that $f(x) \in [0, \frac{1}{2}]$ for every $x \in \mathbb{R}$. This does not quite show that the image of $f$ is $[0, \frac{1}{2}]$. To show this, we can use the fact that $f$ is continuous.

Another way of showing that the image of $f$ is $[0, \frac{1}{2}]$ is to prove that for any $y \in [0, \frac{1}{2}]$, there exists $x \in \mathbb{R}$ such that $f(x) = y$. Let $y \in [0, \frac{1}{2}]$. We are trying to solve for $x$ in the equation $f(x) = y$ which is the same as $\frac{x^2}{1+x^4} = y$. This is equivalent to $yx^4 - x^2 + y = 0$ which is the same as $y(x^2)^2 - x^2 + y = 0$. This is a quadratic equation in $x^2$. A quadratic equation has real solutions if and only if its discriminant $\Delta = b^2 - 4ac$ is greater than or equal to 0. For our equation, $\Delta = b^2 - 4ac = (-1)^2 - 4y \cdot y = 1 - 4y^2$. Because $y \in [0, \frac{1}{2}]$, we get that $y^2 \leq \frac{1}{4}$ which means $1 - 4y^2 \geq 0$ which shows $\Delta \geq 0$. This finishes our proof.

4. Prove that

$$5x^2 - 16xy + 13y^2 + 8x - 12y + 4 \geq 0$$

for any real numbers $x$ and $y$. When does equality hold?
Proof. Complete the square:

\[
5x^2 - 16xy + 13y^2 + 8x - 12y + 4 = 5 \left( x^2 - \frac{16y - 8}{5} x + \frac{13y^2 - 12y + 4}{5} \right)
\]

\[
= 5 \left( x^2 - \frac{2(8y - 4)}{5} x + \frac{13y^2 - 12y + 4}{5} \right)
\]

\[
= 5 \left( x^2 - \frac{2(8y - 4)}{5} x + \left( \frac{8y - 4}{5} \right)^2 - \left( \frac{8y - 4}{5} \right)^2 + \frac{13y^2 - 12y + 4}{5} \right)
\]

\[
= 5 \left[ \left( x - \frac{8y - 4}{5} \right)^2 - \left( \frac{8y - 4}{5} \right)^2 + \frac{13y^2 - 12y + 4}{5} \right]
\]

\[
= 5 \left[ \left( x - \frac{8y - 4}{5} \right)^2 + \frac{-64y^2 + 64y - 16 + 65y^2 - 60y + 20}{25} \right]
\]

\[
= 5 \left[ \left( x - \frac{8y - 4}{5} \right)^2 + \frac{y^2 + 4y + 4}{25} \right]
\]

\[
= 5 \left[ \left( x - \frac{8y - 4}{5} \right)^2 + \left( \frac{y + 2}{5} \right)^2 \right]
\]

As our expression $5x^2 - 16xy + 13y^2 + 8x - 12y + 4$ is the sum of two squares, it must be greater than or equal to zero.

Our expression $5x^2 - 16xy + 13y^2 + 8x - 12y + 4$ equals 0 if and only if $y + 2 = 0$ and $x - \frac{8y - 4}{5} = 0$. This means $y = -2$ and $x = -4$. \(\square\)

5. Prove that

\[ x^4 + y^4 + z^4 + w^4 \geq 4xyzw \]

for any real numbers $x, y, z$ and $w$. Use this inequality to prove that

\[ a^3 + b^3 + c^3 \geq 3abc \]

for any real non-negative numbers $a, b, c$.

(Hint: For the first part, use the Arithmetic Mean-Geometric Mean inequality for two numbers. For the second part, make $w := \sqrt[3]{xyz}$ in the first inequality.)

Proof. For the first inequality, we use the arithmetic mean-geometric mean inequality (or the inequality $u^2 + v^2 \geq 2uv$) for two numbers three times as follows:

\[ x^4 + y^4 \geq 2\sqrt{x^4y^4} = 2x^2y^2, \]

and

\[ z^4 + w^4 \geq 2\sqrt{z^4w^4} = 2z^2w^2 \]
imply that
\[ x^4 + y^4 + z^4 + w^4 \geq 2(x^2y^2 + z^2w^2) \geq 2 \cdot 2 \sqrt{x^2y^2z^2w^2} = 4|xyzw| \geq 4xyzw \]
which finishes the proof.

For the second part, take \( w := \sqrt[3]{xyz} = (xyz)^{\frac{1}{3}} \) in the inequality we just proved:
\[ x^4 + y^4 + z^4 + (xyz)^{\frac{4}{3}} \geq 4(xyz)(xyz)^{\frac{1}{3}} = 4(xyz)^{\frac{4}{3}}. \]
This implies
\[ x^4 + y^4 + z^4 \geq 3(xyz)^{\frac{4}{3}}. \]
Now make \( x = a^{\frac{3}{4}}, y = b^{\frac{3}{4}}, z = c^{\frac{3}{4}} \) and this will yield \( a^3 + b^3 + c^3 \geq 3abc \) and finish our proof.

6. **Bonus Question!** Prove that:
\[ x^2 + y^2 + z^2 \geq xy + yz + zx. \]
for any real numbers \( x, y \) and \( z \).
When does equality hold?

*Proof. 1st proof*

Adding the inequalities \( (x - y)^2 \geq 0, (y - z)^2 \geq 0, (z - x)^2 \geq 0 \) we obtain
\[ (x^2 - 2xy + y^2) + (y^2 - 2yz + z^2) + (z^2 - 2zx + x^2) \geq 0 \]
Simplifying by 2, we get the desired inequality. Equality holds if and only if \( x - y = y - z = z - x = 0 \) which is the same as \( x = y = z \).

*2nd proof* Complete the square:
\[
x^2 + y^2 + z^2 - xy - yz - zx = x^2 - (y + z)x + y^2 + z^2 - yz
\]
\[ = x^2 - 2 \cdot \frac{y + z}{2} x + \left( \frac{y + z}{2} \right)^2 - \left( \frac{y + z}{2} \right)^2 + y^2 + z^2 - yz
\]
\[ = \left( x - \frac{y + z}{2} \right)^2 + \frac{-(y + z)^2 + 4(y^2 + z^2 - yz)}{4}
\]
\[ = \left( x - \frac{y + z}{2} \right)^2 + \frac{-y^2 - z^2 - 2yz + 4y^2 + 4z^2 - 4yz}{4}
\]
\[ = \left( x - \frac{y + z}{2} \right)^2 + \frac{3(y^2 + z^2 - 2yz)}{4}
\]
\[ = \left( x - \frac{y + z}{2} \right)^2 + \frac{3(y - z)^2}{4}. \]
This shows \( x^2 + y^2 + z^2 - xy - yz - zx \geq 0 \) with equality if and only if \( x - \frac{y + z}{2} = 0 \) and \( y = z \). This is the same as \( x = y = z \). \( \square \)