Inverse Problems and Imaging: Past, Present and Future

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1 Introduction

Already not recognized as a mathematical discipline until recently, inverse problems are as old as science itself. In particular a working definition of science is the problem of constructing a model of some physical or biological phenomena that, although inexact, is accurate enough to be able to use observations or measurements to obtain information

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about the phenomena under investigation. Such models are typically mathematical in nature and hence the challenge is to "invert" the model to recover useful estimates of the object under investigation. Since the model itself is inexact, such an inversion process typically leads to problems of existence and stability.

Surprisingly enough, given the above description of the scientific method, the mathematical theory of inverse problems was essentially ignored until the middle of the twentieth century. Instead, scientists focused on *direct problems*, i.e. the construction of the model itself rather than the inversion process. In particular, direct problems are based on developing a mathematical model that maps causes into effects and are typically *well-posed*: each cause has a unique effect and causes which are close to one another have effects which are close to each other. The scientific phenomena was then investigated by adjusting the input to the direct problem such that the output fit the measured data. By the beginning of the twentieth century, the idea of direct problems dominated mathematical physics. Indeed, the French mathematician Hadamard held the opinion that an important physical problem *must* be well posed, i.e. the problem must always have a unique solution that depends continuously on the data.

The attitude typified by that of Hadamard persisted well into the middle of the twentieth century. However the advent of quantum mechanics and numerous problems in areas of classical physics such as heat conduction and geophysics soon slowly convinced mathematicians and scientists that well posed direct problems were not the only ones of scientific interest and, pioneered by mathematicians in the Soviet Union led by Tikhonov, the mathematical theory of *inverse problems*. began to be developed. In particular, this theory focused on the problem of determining the parameters and data in the mathematical model of the direct problem from measurements and observations of the data that arise from the physical or biological phenomena taking place. Such problems are almost invariably *ill-posed* in the sense that in general either existence, uniqueness or continuous dependence on data is no longer true. Although the problems of existence and uniqueness in inverse problems can often be ameliorated by generalizing the notion of solution and constructing a generalized solution, the key attribute of stability is often absent in inverse problems unless further a priori information is available. This essential lack of stability usually has dire consequences when numerical methods using measured (and hence inexact) data are applied to inverse problems.

In view of the inherent problem of instability that is characteristic of inverse problems, the mathematical theory of inverse problems focuses on this issue. In particular, the primary problem that needs to be addressed is what type of a priori information is "normally" available and how can this information be brought into the mathematical model?

In this context, the solution space and the space of observations are typically taken to be Hilbert spaces (but not necessarily the same Hilbert space since one desires more of the solution than one demands from the observation). The mathematical model itself is then an operator taking one Hilbert space into another, i.e. the mathematical foundation of inverse problems is the theory of operators in a Hilbert space. Until recently, most of the mathematical theory of inverse problems was focused on linear problems and hence the theory of linear operators in a Hilbert space. However, in recent years, more and more attention has been focused on nonlinear problems where, in addition to stability, uniqueness issues are seen to play a prominent role.

The purpose of this article is to give a brief survey of the far field of inverse problems. However this is by no means an easy task since the field has experienced tremendous growth in the past fifty years covering areas as diverse as computerized tomography, synthetic aperture radar, geophysical prospecting and nondestructive testing. Since the solution of any inverse problem is to "invert" the model to recover useful information about the physical phenomena from the observed image, inverse problems by definition must also deal with the subject of imaging. A comprehensive survey of many areas of inverse problems and imaging can be found in recently published 1600 page handbook [28]. We will make no attempt to survey in twenty pages that which can only be partially done in 1600 pages. Instead we have chosen three "canonical" examples of inverse problems, describe their physical origin and then present mathematical methods which in each case address the basic issues of existence, uniqueness and stability for what are fundamentally ill-posed problems. The mathematical methods that we present are of course also applicable to many other inverse problems which are not discussed here but can be found in the above mentioned handbook. For the mathematically unsophisticated reader we have also presented a brief introduction to theory of Hilbert spaces where we have only supposed a pre-requisite of elementary linear algebra and calculus. We conclude our article by presenting a subjective attempt to see into the future of inverse problems. Here, among many possible choices, we have chosen the area of obtaining inequalities of physical interest in scattering theory from a knowledge of the measured scattered wave. Such techniques provide a rapid method to obtain valuable information from what is basically a complicated multi-dimensional nonlinear inverse scattering problem with many possible future applications in nondestructive testing. Only the future will tell if this direction will in fact bear fruit!

2 Examples

We will now give three examples of inverse problems which will serve as our model problems in what follows. The first example is the backwards heat equation which is perhaps the simplest model of a linear inverse problem and was one of the first ill-posed problems that was systematically studied (c.f. [27]). The second example is again a linear inverse problem, but this time one that is considerably more difficult to analyze. This is the problem of computerized tomography which has revolutionized medical imaging and for which its inventors won the Nobel Prize. Our third and final example is the inverse scattering problem for acoustic waves which is the best known example of a nonlinear inverse problem and, in its electromagnetic version, is thre mathematical basis of synthetic aperture radar (c.f. [7]).

2.1 The backwards heat equation

Before presenting the ill-posed problem of solving the backwards heat equation, we note that in order to investigate a problem that is ill-posed we must answer two basic questions: 1) What do we mean by a solution? and 2) How do we construct this solution? The answers to these questions are by no means trivial. For example, as initially posed a solution may not even exist in the classical sense. In this context it is worthwhile recalling a remark of Lanczos: "A lack of information cannot be remedied by any mathematical trickery". Hence, in order to determine what we mean by a solution it is often necessary to introduce "nonstandard" a priori information gained from a knowledge of the physical situation that one is trying to model. Even after we have resolved the problem of what we mean by a solution, there remains the problem of actually constructing such a solution, and this is often complicated by the fact that the above mentioned nonstandard information has been incorporated into the mathematical model, thus leading to nonstandard problems in analysis.

We now turn our attention to the backwards heat equation. Physically, the problem that we are about to consider is to determine the temperature of a solid in the past from a knowledge of its temperature in the present and the temperature on the boundary of the solid in the past. Mathematically, we can formulate this problem in the following manner (assuming zero boundary data and a homogeneous medium): Find a solution u of

$$\Delta_3 u = u_t \qquad \text{in } D \times [0, T] \tag{1}$$

$$u = 0 \qquad \text{on } \partial D \times [0, T] \tag{2}$$

$$u(x, T) = f(x) \qquad \text{for } x \in D \tag{3}$$

for a prescribed function f, where D denotes the given solid. It can be shown that no solution exists to this problem unless f is an analytic function of its three independent variable. Furthermore, even if f is analytic the solution, if it exists, does not depend continuously on the data f. To see this let φ_n be an (orthonormalized) eigenfunction corresponding to an eigenvalue λ_n of

$$\Delta_3 \varphi + \lambda \varphi = 0 \qquad \text{in } D \tag{4}$$

$$u = 0 \qquad \text{on } \partial D. \tag{5}$$

Then

$$u_n(x,t) = \frac{1}{\lambda_n} \varphi_n(x) e^{-\lambda_n (t-T)}$$

is a solution (1)-(3) for

$$f(x) = f_n(x) = \frac{1}{\lambda_n} \varphi_n(x)$$

and since $\|\varphi\| = 1$ where $\|\cdot\|$ is the L^2-norm over D, we have that $\|f_n\| = 1/\lambda_n$. But for each fixed t, $0 \leq t < T$, we have that

$$\|u_n(x,t)\| = \frac{1}{\lambda_n} e^{-\lambda_n (t-T)}$$

and since $\lambda_n \to \infty$ as $n \to \infty$ [8] we have that $\|f_n\| \to 0$ as $n \to \infty$ whereas $\|u_n(x,t)\| \to \infty$. Thus the solution of (1)-(3) does not depend continuously on the data f.

In Section 4 of this article we will show how the above problems can be avoided if we look for a solution of (1)-(3) in the class of solutions to (1)-(3) that satisfy an a priori bound and define a "solution" to be a function in this class that best approximates the data (3) in $L^2(D)$.

2.2 Computerized tomography

Literally, tomography means *slice imagining.* Today this term is applied to many methods used to reconstruct the internal structure of a solid object from external measurements [17], [26]. We consider here a mathematical model of the measurement process used in transmission computerized tomography. Here, a cross-section of the human body is scanned by a thin x-ray beam whose intensity loss is recorded by a detector and processed by a computer to produce a two-dimensional image of slices of the human body which in turn are displayed on a screen. More specifically, objects of interest in x-ray imaging are described by a real-valued function defined on $\mathbb{R}^3$, called the *attenuation coefficient*, which quantifies the tendency of an object to absorb or scatter x-rays of a given energy.

This function, denoted here by $\mu(x) \geq 0$ for $x \in \mathbb{R}^3$, varies from point-to-point within the object and is usually taken to vanish outside. Our model for the interaction of x-rays with matter is based on three basic assumptions: 1) x-rays travel along straight lines that are not "bent" by the object they pass through, 2) the waves making up the x-ray beam are all of the same frequency, and 3) the intensity I of the x-ray beam satisfies Beer's law

$$\frac{dI}{ds} = -\mu(x) I \tag{6}$$

where s is the arc-length along the straight-line trajectory of the x-ray beam. In a real measurement the total energy, I_i incident on the subject along a given line ℓ is given, and the total energy I_o emerging from the object along ℓ is measured by an x-ray detector. Hence, integrating (6) we obtain

$$\log \frac{I_o}{I_i} = \int_\ell \mu \, ds. \tag{7}$$

By varying the position of the source we can measure the quantity on the left hand side of (7) along a family of lines [17]. In computerized tomography the function $\mu(x)$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, is reconstructed from its two-dimensional slices, i.e. $f_c(x_1, x_2) := \mu(x_1, x_2, c)$ for a given c. Now suppose that the support of $\mu(x)$ is inside the cube $[-a,\, a] \times [-a,\, a] \times [-a,\, a]$. For each fixed c between $\pm a$ and each pair $(t, \omega) \in \mathbb{R} \times S^1$, where S^1 is the unit circle, we measure the line integral of f_c along the line lying on the plane $x_3 = c$

$$\{(x_1, x_2, x_3) : x_3 = c, \text{ and } (x \cdot \omega) = t,\ x = (x_1, x_2)\},$$

that is,

$$\int_{-\infty}^{\infty} f_c(t\omega + s\omega^\perp)\, ds,$$

where $\omega(\theta) := (\cos(\theta), \sin(\theta))$, $\omega^\perp$ is the direction perpendicular to ω, i.e. $\omega^\perp(\theta) := (-\sin(\theta), \cos(\theta))$, and $(\,\cdot\,)$ denotes the $\mathbb{R}^2$ dot product. In this idealized model it is assumed that on the plane $x_3 = c$ the sources and receivers are moved around a circle enclosing the corresponding slice of absorbent material of compact support, i.e $0 \leq \theta < 2\pi$.

The above measurement model in CT scanning, brings us to an essential mathematical problem: *Can a two-variable function be recovered from a knowledge of its line integrals along all lines?* This leads to the definition of the *Radon Transform* which in the following is defined only in $\mathbb{R}^2$ (see [15], [26] for a discussion on the Radon transform in higher dimensions). To this end we identify $\mathbb{R} \times S^1$ with the space of oriented lines $\ell_{t,\omega}$ given by

$$\ell_{t,\omega} := \big\{x \in \mathbb{R}^2 : \langle \omega, x \rangle = t\big\} = \big\{t\omega + s\omega^\perp :\ s \in \mathbb{R}\big\} \tag{8}$$

Definition 2.1 Suppose that f is a function defined in the plane which, for simplicity, we assume is continuous with bounded support. The integral of f along the line $\ell_{t,\omega}$ is denoted by

$$\mathcal{R}f(t,\omega) = \int_{\ell_{t,\omega}} f\,ds = \int_{-\infty}^{\infty} f(t\omega + s\omega^{\perp})ds.$$

The collection of all integrals of f along the line ℓ on the plane defines a function on $\mathbb{R} \times S^1$, called the *Radon transform* of f.

It is not necessary for f to be either continuous or of bounded support. The Radon transform can be defined for a function f whose restriction to each line is locally integrable and

$$\int_{-\infty}^{\infty} |f(t\omega + s\omega^{\perp})|ds < \infty, \quad \text{for all } (t,\omega) \in \mathbb{R} \times S^1. \tag{9}$$

Thus, computerized x-ray tomography becomes the problem of inverting the Radon transform of slices $f_c(x_1, x_2) := \mu(x_1, x_2, c)$, $-a \leq c \leq a$ of the attenuation coefficient μ of the object of interest. We shall address this problem in Section 4 and Section 5, where mathematical questions such as uniqueness, stability and of course reconstructions methods are briefly discussed. Note that many other problems in tomography and imaging can be re-written as the problem of inverting the Radon transform of some function of interest (for more details see [15], [25] and [26]).

In practice the above integrals can be measured only for a finite number of lines using basically two scanning geometries, namely parallel scanning and fan-beam scanning. Thus the real problem in computerized tomography is to reconstruct a slice f_c from a finite number of its line integrals. Sometimes it is not possible nor desirable to scan the whole cross-section. One then has to reconstruct f_c from the integrals corresponding to limited angle aperture, i.e. one speaks of the incomplete data problem. In particular, if a three-dimensional model is adapted in order to increase the efficiency of the procedure, the incomplete data problem is the rule. The line sampling as well as the angle aperture, of course impact the accuracy and the resolution of the image. Finally, the model discussed here is highly idealized and in practice further model corrections are introduced to account for the width of the beam, the energy dependence of the attenuation etc. We refer the reader to [26] for a more detailed discussion on these issues.

2.3 The inverse scattering problem

The propagation of time harmonic acoustic waves of frequency $\omega > 0$ through a homogeneous medium in $\mathbb{R}^3$ with speed of sound c is governed by the *Helmholtz equation*

$$\Delta_3 u + k^2 u = 0 \tag{10}$$

where the wave number $k = \omega/c$. A solution of the Helmholtz equation whose domain of definition contains the exterior of some sphere is called radiating if it satisfies the *Sommerfeld radiation condition*

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{11}$$

where $r = |x|$ and the limit holds uniformly in all directions $\hat{x} = x/|x|$.

We will consider two basic problems in scattering theory, namely scattering by a bounded impenetrable obstacle and scattering by a penetrable inhomogeneous medium of compact support. For a vector $d \in \mathbb{R}^3$ with $|d| = 1$ the function $e^{ikx\cdot d}$ satisfies the Helmholtz equation for all $x \in \mathbb{R}^3$. It is called a *plane wave* since $e^{i(kx\cdot d-\omega t)}$ is constant on the plane $kx \cdot d - \omega t =$ constant. Assume that an incident field is given by the plane wave $u^i(x) = e^{ikx\cdot d}$. Then the simplest obstacle scattering problem is to find the scattered field u^s as a radiating solution to the Helmholtz equation in the exterior of a bounded scatterer D such that the total field

$$u = u^i + u^s \tag{12}$$

satisfies the Dirichlet boundary condition

$$u = 0 \qquad \text{on } \partial D. \tag{13}$$

The simplest scattering problem for an inhomogeneous medium assumes that the speed of sound is constant outside a bounded domain D. Then, if u^i again is given by $u^i(x) = e^{ikx\cdot d}$, the total field $u = u^i + u^s$ satisfies

$$\Delta_3 u + k^2 nu = 0 \qquad \text{in } \mathbb{R}^3 \tag{14}$$

and the scattered field u^s fulfills the Sommerfeld radiation condition (11). Here the wave number k is given by $k = \omega/c_0$ and $n = c_0^2/c^2$ is the *index of refraction* where c_0 is the sound speed in the homogeneous background medium and $c = c(x)$ is the speed of sound in the inhomogeneous medium. We define $n(x) = 1$ for $x \notin D$. An absorbing medium is modeled by adding an absorption term which leads to a refractive index with a positive imaginary part

$$n = \frac{c_0^2}{c^2} + i\frac{\gamma}{k}$$

where $\gamma = \gamma(x)$ is the absorbing coefficient.

It can be shown that radiating solutions u^s to the Helmholtz equation have the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{r} u_\infty(\hat{x}, d) + O\left(\frac{1}{r^2}\right) \tag{15}$$

as $r \to \infty$ uniformly in all directions $\hat{x} = x/|x|$ where the function u_∞ defined on the unit sphere Ω is called the *far field pattern* of the scattered wave u^s. The *direct scattering problem* is, given the physical properties of the scatterer, to find the far field pattern u_∞. The *inverse scattering problem* is to either determine the impenetrable obstacle D or the index of refraction n (and hence D) from a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for all $\hat{x}, d \in \Omega$. The inverse scattering problem is clearly nonlinear and it can further be shown that it is ill-posed [9].

We will consider the uniqueness question for both of these inverse scattering problems in Section 4 of this article and will derive reconstruction methods for determining D or n in Section 5. These reconstruction methods will either be based on decomposing the inverse scattering problem into a linear, ill-posed problem and a nonlinear, well-posed problem or through the use of *qualitative methods* in inverse scattering theory which are based on solution of an ill-posed *linear* integral equation with the far field pattern as its kernel.

Having introduced the above three examples of inverse problems, we now turn to the problem of how to reconstruct a solution to these problems.

3 Hilbert Spaces

The mathematical language used to study inverse problems is functional analysis or, more narrowly, Hilbert space theory. Hence, to enable the reader to understand the sections which follow, we will briefly outline the rudiments of Hilbert spaces. We assume that the reader is familiar with linear algebra, in particular inner product spaces (c.f. Chapter 1 of [1]).

A (separable) Hilbert space H is a finite dimensional or (countably) infinite dimensional inner product space that is *complete* i.e. if $(\cdot, \cdot)$ is the inner product with associated norm

$$\|x\| = \sqrt{(x, x)}$$

then every Cauchy sequence in H converges. In particular, if

$$\lim_{n,m \to \infty} \|x_n - x_m\| = 0$$

for some sequence (x_n) then there exists $x \in H$ such that

$$\|x_n - x\| \to 0.$$

The smallest (in the sense of inclusion) Hilbert space that contains a given inner product space is called the *completion* of the inner product space (Every inner product space has a unique completion). The space, denoted by $L^2[a,\, b]$, of measurable functions on an interval $[a,\, b]$ whose squares are Lebesgue integrable with inner product

$$(f, g) = \int_a^b f(t)\overline{g(t)}\, dt$$

is the prototypical example of a Hilbert space.

Two vectors x, y in a Hilbert space H are called *orthogonal* if $(x, y) = 0$. The *orthogonal complement* of a set S is the closed subspace

$$S^\perp = \{x \in H : \ (x, y) = 0 \text{ for all } y \in S\}\,.$$

If S is a closed subspace of a Hilbert space H then H has the decomposition

$$H = S \oplus S^\perp$$

meaning that each $x \in H$ has a unique representation of the form $x = x_1 + x_2$ where $x_1 \in S$ and $x_2 \in S^\perp$. The element x_1 is called the *projection* of x onto S and we write $x_1 = P_S\, x$. Similarly, $x_2 = P_{S^\perp}\, x$. A set of orthogonal vectors each of which has unit norm is called an *orthonormal set* and is called *complete* if $S^\perp = \{0\}$. Every element $x \in H$ has a convergent expansion in terms of a complete orthonormal set $\{\varphi_n\}_{n=1}^\infty$ of H (if H is finite dimensional so is the set $\{\varphi_n\}$),

$$x = \sum_{n=1}^\infty (x, \varphi_n)\ \varphi_n,$$

and from this expansion we can deduce the *Parseval relation*

$$\|x\|^2 = \sum_{n=1}^\infty |\,(x, \varphi_n)\,|^2.$$

Our real interest is not in Hilbert spaces per se but rather on *linear operators* on a Hilbert space. A *bounded* linear operator from a Hilbert space H_1 into a Hilbert space H_2 is a mapping $T : H_1 \to H_2$ which is linear, $T(\alpha x + \beta y) = \alpha T x + \beta T y$, and for which

$$\|T\| := \sup\{\|Tx\|/\|x\| : \ x \neq 0\}$$

is finite. (Note that we have used the same symbol for the norm in each of the spaces and we will continue to do this in the sequel). An example of a bounded linear operator is the integral operator $T : L^2[a,\, b] \to L^2[c,\, d]$ of the form

$$Tf(t) := \int_a^b k(t, s) f(s) ds, \quad c \leq t \leq d \tag{16}$$

where $k(\cdot, \cdot) \in L^2([a, b] \times [c, d])$ is called the *kernel* of the integral operator. The *adjoint* of a bounded linear operator $T : H_1 \to H_2$ is the bounded linear operator $T^* : H_2 \to H_1$ which satisfies

$$(Tx, y) = (x, T^*y)$$

for all $x \in H_1$ and $y \in H_2$. For example, by interchanging the order of integration, one can see that the adjoint of the integral operator (16) is

$$(T^*g)(s) := \int_c^d \overline{k(t, s)} g(t) \, dt.$$

A bounded linear operator $T : H \to H$ is called *self-adjoint* if $T^* = T$. The *null-space* of a bonded linear operator $T : H_1 \to H_2$ is closed subspace

$$N(T) = \{x \in H_1 : Tx = 0\}.$$

Note that $N(T^*T) = N(T)$ since if $x \in N(T^*T)$ then

$$0 = (T^*Tx, x) = (Tx, Tx) = \|Tx\|^2.$$

There are basic relationships between the null-space and the *range*

$$R(T) := \{Tx : \ x \in H_1\}$$

of a bounded linear operator and its adjoint. For example, $y \in R(T)^\perp$ if and only if

$$0 = (Tx, y) = (x, T^*y)$$

for all $x \in H_1$ and hence $R(T)^\perp = N(T^*)$. In a similar way one can show that $\overline{R(T)} = N(T^*)^\perp$, $R(T^*)^\perp = N(T)$, and $\overline{R(T^*)} = N(T)^\perp$.

We now conclude our short introduction to Hilbert spaces and linear operators on Hilbert spaces by considering *compact operators* on a Hilbert space. A compact operator $T : H_1 \to H_2$ mapping a Hilbert space H_1 into a Hilbert space H_2 is a linear operator that maps bounded sets in H_1 onto relatively compact sets in H_2, i.e. if M is a bounded set in H_1 then $\overline{T(M)}$ is compact in H_2. The canonical example of a compact operator is the integral operator defined in (16). On the other hand, the identity operator $I : H \to H$ on a infinite dimensional Hilbert space H is not compact. To see this let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal set in H. Then $\|\varphi_n\| = 1$ for all n but $\{I\varphi_n\} = \{\varphi_n\}$ has no convergent subsequence since $\|\varphi_n - \varphi_m\| = \sqrt{2}$ for n not equal to m. It can easily be verified that the product of a compact operator and a bounded operator is compact and hence the inverse of a compact operator is *unbounded*. This follows from the fact that if $T : H_1 \to H_2$

is compact and $T^{-1} : H_2 \to H_1$ is bounded then $I = T^{-1}T : H_1 \to H_1$ would be compact, a contradiction to the above example. Since many (if not most) linear inverse problems (such as the backwards heat equation and problems arising in computerized tomography) can be reformulated as the problem of inverting a compact operator, this provides a mathematical explanation of why most (linear) inverse problems are *improperly posed*. In particular, if T is unbounded there exists a bounded sequence $\{x_n\}_{n=1}^{\infty}$ such that $Tx_n \to \infty$, i.e T is not continuous.

In order to solve linear operator equations of the form $Tx = y$ where T is compact, the *singular value decomposition* (SVD) of T plays a central role. The SVD of a compact operator $T : H_1 \to H_2$ is the set of *singular values* $\{\mu_n\}$ and *singular vectors* $\{\varphi_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ where $\varphi_n \in H_1$ and $g_n \in H_2$ such that

$$T\varphi_n = \mu_n g_n, \qquad T^* g_n = \mu_n \varphi_n$$

where $\{\varphi_n\}_{n=1}^{\infty}$ is a complete orthonormal set for $N(T)^{\perp} = \overline{R(T^*)}$ and $\{g_n\}_{n=1}^{\infty}$ is a complete orthonormal set for $N(T^*)^{\perp} = \overline{R(T)}$. Every compact operator T has a *singular system* $\{\varphi_n, \, g_n; \, \mu_n\}$ and in terms of this system we can write

$$Tx = \sum_{n=1}^{\infty} \mu_n (x, \varphi_n) g_n \tag{17}$$

If the range of T is finite dimensional this sum terminates at $n = n_0$ for some iteger n_0; otherwise

$$\mu_n \to 0 \qquad \text{as} \quad n \to \infty.$$

This fact is highly significant in the theory of inverse problems since it says that finite dimensional linear models when pushed too far toward the limiting case of an operator of infinite dimensional range, will inevitably result in instability.

By using a singular system we can provide a criterion, due to Picard [9] that characterizes the existence of solutions of an equation of the *first kind*

$$Tx = y$$

where T is a compact operator that play a role in inverse theory that is analogous to that which the Fredholm alternative (c.f. [24]) plays for integral equations of the *second kind*

$$x - Tx = y.$$

To establish Picard's criterion, let $\{\varphi_n\}_{n=1}^{\infty}$ be a complete orthonormal system for $N(T)^{\perp}$. Then the series

$$\sum_{n=1}^{\infty} |(x, \varphi_n)|^2$$

is convergent (and equals $\|P_{N(T)^\perp}x\|^2$ by Parseval's relation). However, if $y = Tx \in R(T)$ then

$$(x,\, \varphi_n) = \mu_n^{-1}\,(x,\, T^* g_n) = \mu_n^{-1}\,(Tx,\, g_n) = \mu_n^{-1}\,(y, g_n)$$

and hence

$$\sum_{n=1}^{\infty} \mu_n^{-2} |\,(y, g_n)\,|^2 < \infty \tag{18}$$

is a necessary condition for $y \in R(T)$. On the other hand this condition guaranties that the series

$$x = \sum_{n=1}^{\infty} \mu_n^{-1}\,(y, g_n)\,\varphi_n \tag{19}$$

is convergent in $N(T)^\perp = \overline{R(T^*)}$ and the singular value relations show that

$$Tx = \sum_{n=1}^{\infty} (y, g_n)\, g_n$$

is convergent in $N(T*)^\perp = \overline{R(T)}$. Hence we have the following *Picard's criterion*: $y \in R(T)$ if and only if $y \in N(T^*)^\perp$ and (18) is valid. If y satisfies Picard's criterion then $Tx = y$ has a solution x given by (19).

4 Uniqueness and Stability

The first question to ask about an ill-posed problem is the question of uniqueness, i.e. if exact data were available is there at most one solution? Having answered this question, the next question to ask is how can stability be restored and an approximate solution to the problem be constructed? For linear inverse problems, stability can often be restored with a minimum of a priori assumptions whereas for nonlinear problems stability is usually restored by using more ad hoc techniques. In this section we first consider the questions of uniqueness and stability for the backwards hear equation and Radon transform, and then focus only on the issue of uniqueness for the more difficult nonlinear inverse scattering problem.

We begin with the backwards heat equation. Our aim is to show that, in the case of this simple linear problem, uniqueness and stability for the problem (1)-(3) can both be obtained if we require the solution to lie in the set $\mathcal{M}$ defined by

$$\mathcal{M} := \left\{\phi :\ \phi \in C^2(D \times (0,T)) \cap C(\overline{D} \times [0,T]) \ \text{ and } \ \|\phi(x,0)\|^2 \le M\right\}$$

where $\|\cdot\|$ denotes the L_2-norm over D and M is a prescribed positive constant. Physically this means we have an a priori bound on the temperature of the solid at time $t = 0$ (and hence, since $u = 0$ on ∂D, by the maximum principle we have a bound on the temperature for $0 \le t \le T$). We begin by defining the functional $F(t) = F(t; u)$ by

$$F(t) = \int_D [u(x,t)]^2 \, dx = \|u(\cdot, t)\|^2 \tag{20}$$

where u satisfies (1)-(3). We assume that $F(0) \neq 0$ since if this were not true then by the maximum principle $u(x,t) = 0$ for all $(x,t) \in D \times (0, T)$. Under this assumption it is easy to verify that $F(t) \neq 0$ for any $t \in [0, T]$. Now assume further that D is a bounded, simply connected domain with smooth boundary so that we can apply Green's first identity. Then from (20) we have for $0 < t < T$ that

$$F'(t) = 2 \int_D u \frac{\partial u}{\partial t} \, dx = 2 \left(u, \frac{\partial u}{\partial t} \right)$$

and by Green's first identity

$$F'(t) = 2 \int_D u \, \Delta_3 u \, dx = -2 \int_D |\nabla u|^2 dx. \tag{21}$$

From (21) we now have that

$$\begin{aligned} F''(t) &= -4 \int_D \nabla u \cdot \nabla \frac{\partial u}{\partial t} \, dx + 4 \int_D \frac{\partial u}{\partial t} \Delta_3 u \, dx \\ &= 4 \int_D \left(\frac{\partial u}{\partial t} \right)^2 dx = 4 \left\| \frac{\partial u}{\partial t} \right\|^2 \end{aligned} \tag{22}$$

From (21), (22) we now have that

$$F F'' - (F')^2 = 4 \|u\|^2 \left\| \frac{\partial u}{\partial t} \right\|^2 - \left(u, \frac{\partial u}{\partial t} \right)^2$$

and hence by Schwartz's inequality

$$F F'' - (F')^2 \ge 0 \tag{23}$$

i.e.

$$(\log \, F)'' \ge 0$$

which implies that $\log \, F$ is a convex function of t. Hence

$$\log F(t) \le \log F(0) \left(\frac{t - T}{T} \right) + \log F(T) \, \frac{t}{T}$$

i.e.

$$F(t) \le [F(0)]^{(T-t)/T} \, [F(T)]^{t/T} \, . \tag{24}$$

From (24) the uniqueness of a solution to (1)-(3) follows immediately. We furthermore have the following stability result.

Theorem 4.1 *Let $u \in C^2(D \times (0,T)) \cap C(\overline{D} \times [0,T])$ be a solution of (1)-(3) that belongs to the set $\mathcal{M}$. Then*

$$\|u(\cdot,t)\| \leq M^{(1-t/T)} \|f\|^{t/T}$$

for $0 \leq t \leq T$.

In particular, from Theorem 4.1 we can conclude that for $t > 0$ the system (1)-(3) depends Hölder continuously on the data provided we assume a priori that $u \in \mathcal{M}$. We also note that Theorem 4.1 lends itself in a natural manner to the problem of reconstructing an approximate solution to (1)-(3) in the class $\mathcal{M}$. In particular, let $\{u_n\}$ be a complete set of solutions to the heat equation in $D \times [0,\, T]$, for example the *heat polynomials* [8], [29]

$$h_M(x,t) = h_{m_1}(x_1,t) h_{m_2}(x_2,t) h_{m_3}(x_3,t)$$

where $x = (x_1, x_2, x_3)$, $M = (m_1, m_2, m_3)$ and

$$h_m(x,t) = m! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{x^{m-2k} t^k}{(m-2k)!k!} = (-t)^{m/2} H_m \left(\frac{x}{(-2/t)^{1/2}} \right)$$

where H_m denotes the Hermite polynomial of degree m. Then for fixed N we want to minimize the functional

$$\left\| \sum_{n=0}^{N} a_n u_n \right\|^2_{\partial D \times (0,T)} + \left\| \sum_{n=0}^{N} a_n u_n(x,t) - f(x) \right\|^2_D \tag{25}$$

subject to the constraint

$$\left\| \sum_{n=0}^{N} a_n u_n \right\|^2_{\partial D \times (0,T)} + \left\| \sum_{n=0}^{N} a_n u_n(x,0) \right\|^2_D . \tag{26}$$

(25) and (26) can be combined if we fix $\epsilon > 0$ and choose N and a_n, $n = 0, \ldots, N$, such that

$$\begin{aligned} \left\| \sum_{n=0}^{N} a_n u_n \right\|^2_{\partial D \times (0,T)} &+ \left\| \sum_{n=0}^{N} a_n u_n(x,t) - f(x) \right\|^2_D \\ &+ \frac{\epsilon}{M} \left\| \sum_{n=0}^{N} a_n u_n(x,0) \right\|^2_D \leq \epsilon \end{aligned} \tag{27}$$

The minimization problem (27) is essentially *Tikhonov's regulation method* which we will discuss in more detail in Section 5 of this article.

Next we discuss the problem of the inversion of the Radon transform which arises in the computerized x-ray tomography example presented in part b of Section 2. The Radon transform $\mathcal{R}$ defined in Definition 2.1 is a linear transformation mapping an absolutely integrable function f satisfying (9) into $\mathcal{R}f \in \mathbb{R} \times S^1$. We are interested if $\mathcal{R}^{-1}$ exists and if so to study its properties. As previously explained, in medical applications the function $\mathcal{R}f$ is an idealization for what is measured. How sensitive are the measurements to errors? It can be shown that

$$\max_{\omega \in S^1} \int_{-\infty}^{\infty} |\mathcal{R}f(t,\omega)|, dt \le \int_{\mathbb{R}^2} |f(x)|\, dx,$$

i.e. the mapping $f \to \mathcal{R}f$ is continuous. Although this property is important, what we really want to know is how sensitive the *reconstruction method* is to the measurement errors. In other words, we want to understand the continuity properties of $\mathcal{R}^{-1}$. A first attempt to reconstruct (an approximation of) f from $\mathcal{R}f$ is averaging the values of the $\mathcal{R}f$ over the lines passing through a point. For a direction ω, the line in the family $\{\ell_{t,\omega} :\ t \in \mathbb{R}\}$ passing through a point x is given by $t = (x \cdot \omega)$. Hence we can write

$$f(x) \approx \frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}f(x \cdot \omega(\theta), \theta)\, d\theta. \tag{28}$$

(28) is known as the *back-projection formula*, and while is a reasonable guess for f it does not give the correct answer [17]. In order to correctly invert the Radon transform, an indirect approach is used passing through the Fourier transform. For sake of the reader's convenience, we recall here the definition of the n-dimensional Fourier transform. Let $L^1(\mathbb{R}^n)$ be the space of absolutely integrable functions in $\mathbb{R}^n$. Then the *Fourier transform* of an L^1-function g defined on $\mathbb{R}$ is the function $\hat{g}$ defined on $\mathbb{R}^n$ by the integral

$$\mathcal{F}g = \hat{g}(\xi) := \int_{\mathbb{R}^n} g(x) e^{-ix\cdot\xi}\, dx.$$

It is an important property of the Fourier transform that g can be "reconstructed" from its Fourier transform $\hat{g}$. In particular if $g \in L^1(\mathbb{R}^n)$ is such that $\hat{g} \in L^1(\mathbb{R}^n)$, then

$$g(x) = \mathcal{F}^{-1}(\hat{g}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{g}(\xi) e^{ix\cdot\xi}\, d\xi. \tag{29}$$

Although $L^1(\mathbb{R}^n)$ is the natural domain of definition of the Fourier transform, it can be shown that the Fourier transform is an isomorphism in the space of square integrable functions $L^2(\mathbb{R}^n)$ via the above inversion formula [17]. The Fourier transform and Radon transform are connected in a very simple way. Indeed, by definition we have

$$\int_{-\infty}^{\infty} \mathcal{R}f(t,\omega) e^{-itr}\, dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t\omega + s\omega^{\perp}) e^{-itr}\, ds\, dt.$$

Now making the change of variables $x = t\omega + s\omega^\perp$ (which has Jacobian 1) and noting that $t = (x \cdot \omega)$, the preceding integral become

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t\omega + s\omega^\perp)e^{-itr}\,ds\,dt = \int_{\mathbb{R}^2} f(x)e^{-i(x\cdot\omega)r}\,dx.$$

The above calculations prove the following result that in medical imaging is known as the *central slice theorem*:

Theorem 4.2 *Let f be an absolutely integrable function satisfying (9). For any $r \in \mathbb{R}$ and $\omega \in S^1$, we have*

$$\int_{-\infty}^{\infty} \mathcal{R}f(t,\omega)e^{-itr}\,dt = \hat{f}(r\omega). \tag{30}$$

The integral in the left hand side of (30) is in fact the one-dimensional Fourier transform of the Radon transform of f with respect to the affine variable t. To avoid any ambiguity we adapt the notation

$$\tilde{h}(r,\omega) := \int_{-\infty}^{\infty} h(t,\omega)e^{-itr}dt$$

to denote the one-dimesional Forier transform of a function defined in $\mathbb{R}\times S^1$ with respect to the affine variable $t \in \mathbb{R}$. Under this notation (30) reads

$$(\tilde{\mathcal{R}f})(t,\omega) = \hat{f}(r\omega).$$

Now the central slice theorem and the inversion formula for the Fourier transform (29) provide easily an inversion formula for the Radon transform which is stated in the following theorem.

Theorem 4.3 *If f is an absolutely integrable function satisfying (9) and $\hat{f}$ is absolutely integrable, then*

$$f(x) = \frac{1}{(2\pi)^2}\int_0^{\pi}\int_{-\infty}^{\infty} e^{ir(x\cdot\omega)}(\tilde{\mathcal{R}f})(r,\omega)|r|\,dr\,d\omega \tag{31}$$

To see the validity of Theorem 4.3 we compute

$$\begin{aligned} f(x) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{\infty} e^{ir(x\cdot\omega)}\hat{f}(r\omega)r\,dr\,d\omega \\ &= \frac{1}{(2\pi)^2}\int_0^{\pi}\int_{-\infty}^{\infty} e^{ir(x\cdot\omega)}(\tilde{\mathcal{R}f})(r,\omega)|r|\,dr\,d\omega \end{aligned}$$

where the slice theorem and the fact that $(\tilde{\mathcal{R}f})(r,\omega) = (\tilde{\mathcal{R}f})(-r,-\omega)$ are used.

The inversion formula shows that if all line integrals of f are exactly known then f is *uniquely reconstructed*. By straight froward calculation [17], we can also obtain a Parseval formula for the Radon transform

$$\int_{\mathbb{R}}^{2} |f(x)|^2 dx = \frac{1}{(2\pi)^2} \int_0^{\pi} \int_{-\infty}^{\infty} \left| (\tilde{\mathcal{R}f})(r,\omega)|r|^{1/2} \right|^2 dr\, d\omega$$

which can be seen as a stability type estimate. Because $|r|$ varies between zero and infinity, we see that there does *not* exist constants M or M' so that either estimate,

$$\|\mathcal{R}f\|_{L^2(\mathbb{R}\times S^1)} \leq M\|f\|_{L^2(\mathbb{R}^2)} \quad \text{or} \quad \|f\|_{L^2(\mathbb{R}^2)} \leq M'\|\mathcal{R}f\|_{L^2(\mathbb{R}\times S^1)}$$

is valid for f in a dense subset of $L^2(\mathbb{R}^2)$, i.e. one can not obtain uniform estimates. The above Parseval formula implies that in order for a function on the space of lines to be the Radon transform of a square-integrable function, it must have a L^2-half-derivative in the affine parameter, which is a range condition for the Radon transform. Unlike the Fourier transform, the Radon transform is not defined on all of $L^2(\mathbb{R}^2)$. This in practice means that one needs to have control on the half-order L^2-derivative of the measured data (i.e. one needs to control the high-frequecy content of the approximate Radon transform) which due to the noise is not possible and this can be seen as a source of instability (see [25] for a more detailed discussion).

The inversion formula can be understood as a two-step process: 1) The radial integral is interpreted as a *filter* applied to the Radon transform and acting in the affine parameter where the output of the filter is denoted by

$$\mathcal{C}Rf(t,\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{irt} (\tilde{\mathcal{R}f})(r,\omega)|r|\, dr \tag{32}$$

and 2) The angular integral is then interpreted as the back-projection of the filtered Radon transform where the outcome is the function f, i.e.

$$f(x) = \frac{1}{2\pi} \int_0^{\pi} \mathcal{C}Rf((x\cdot\omega), \omega)\, d\omega.$$

Comparing the latter to (28) we observe that if we omit the $|r|$ factor in the inversion formula, the value of f at x is half of the value provided by the back projection formula (28). In the inversion formula the low-frequency components are suppressed by $|r|$ whereas the high-frequeny components are amplified. For this reason the Radon inversion formula is often called the *filtered back-projection formula*.

To understand the mathematical effect of the "filter" $|r|$, we first observe that if there is a function ϕ whose Fourier transform is $\mathcal{F}\phi(r) = |r|$ then (32) can be written as

$$\mathcal{C}Rf(t,\omega) = (\phi \star \mathcal{R}f)(t,\omega). \tag{33}$$

Here the property of the Fourier transform $\mathcal{F}(f \star g) = \mathcal{F}(f)\mathcal{F}(g)$ is used, where the convolution $\star$ is defined by $f \star g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$. Unfortunately, such a function ϕ does not exist. To put a mathematical framework to this issue, we introduce the Hilbert transform

$$\mathcal{H}g = \mathcal{F}^{-1}(\mathrm{sgn}(\hat{g}))$$

where $\mathrm{sgn}(r)$ is the sign function, i.e. 1 for $r > 0$, -1 for $r < 0$ and 0 for $r = 0$. Recalling that

$$\mathcal{F}\left(\frac{df}{dt}\right)(r) = ir\mathcal{F}f(r),$$

it easy to check that

$$\mathcal{H}\left(\frac{\partial}{\partial t}\mathcal{R}f\right)(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} i|r|(\tilde{\mathcal{R}f})(r,\omega)e^{irt}\,dr$$

and putting this into (31) we obtain

$$f(x) = \frac{1}{2\pi i}\int_0^{\pi} \mathcal{H}\left(\frac{\partial}{\partial t}\mathcal{R}f\right)((x\cdot\omega),\omega)\,d\omega. \tag{34}$$

The function f is reconstructed by back-projecting the Hilbert transform of $-i\frac{\partial}{\partial t}\mathcal{R}f$. Mathematical analysis of the Hillbert transform is very well understood and we refer the reader to [17] and the references therein for a more detailed account.

We now turn our attention to the inverse scattering problems discussed in part c of Section 2 of this article. Here we will only consider the question of uniqueness for fixed values of the wave number k, leaving the issues of stability and reconstruction to the next section of our presentation. We begin with the inverse obstacle problem, i.e. given that $u_\infty = u_\infty(\hat{x}, d)$ is the far field pattern corresponding to the direct scattering problem

$$\Delta_3 u + k^2 u = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{D} \tag{35}$$

$$u(x) = e^{ikx\cdot d} + u^s(x) \tag{36}$$

$$u = 0 \qquad \text{on } \partial D \tag{37}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right), \tag{38}$$

determine D from a knowledge of $u_\infty(\hat{x}, d)$ for $\hat{x}$ and d on the unit sphere Ω. To this end we have the following theorem:

Theorem 4.4 *Assume that D_1 and D_2 are two obstacles such that the far field patterns corresponding to (35)-(38) for D_1 and D_2 coincide for $\hat{x}$ and d on the unit sphere Ω. Then $D_1 = D_2$.*

Proof: The proof makes use of *Rellich's lemma* which says that if u^s is a radiating solution to the Helmholtz equation for which the far field pattern vanishes identically then u^s is identically zero in its domain of definition. Now assume that u_1^s and u_2^s are the scattered fields corresponding to D_1 and D_2, respectively. Since u_1^s and u_2^s are analytic functions of their independent variables, we can conclude by the identity theorem and Rellich's lemma that $u_1^s(\cdot, d) = u_2^s(\cdot, d)$ in the unbounded component G of the complement of $\overline{D}_1 \cup \overline{D}_2$ for all $d \in \Omega$. This in turn implies that the scattered fields $u_1^s(x, z)$ and $u_2^s(x, z)$ corresponding to the radiating fundamental solution

$$\Phi(x, z) = \frac{1}{4\pi} \frac{e^{ik|x-z|}}{|x - z|} \tag{39}$$

as incident fields and D_1 and D_2 as the scattering obstacles satisfy $u_1^s(x, z) = u_2^s(x, z)$ for all $x, z \in G$. Now assume that $D_1 \neq D_2$. Then, without loss of generality, there exists $x^* \in \partial G$ such that $x^* \in \partial D_1$ and $x^* \notin \partial \overline{D}_2$. Then setting $z_n := x^* + \frac{1}{n}\nu(x^*)$, where $\nu(x^*)$ is the unit outward normal to ∂D_1 at x^* we have that

$$\lim_{n\to\infty} u_2^s(x^*, z_n)$$

exists but

$$\lim_{n\to\infty} u_1^s(x^*, z_n) = \infty$$

which is a contradiction. Hence $D_1 = D_2$.

We now turn our attention to the inverse medium problem, i.e. given that $u_\infty = u_\infty(\hat{x}, d)$ id the far field pattern corresponding to the direct scattering problem

$$\Delta_3 u + k^2 n u = 0 \qquad \text{in } \mathbb{R}^3 \tag{40}$$

$$u(x) = e^{ikx\cdot d} + u^s(x) \tag{41}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right), \tag{42}$$

determine $n = n(x)$ from a knowledge of $u_\infty(\hat{x}, d)$ for $\hat{x}$ and d on the unit sphere Ω. We have the following analogous to Theorem 4.4:

Theorem 4.5 *The refractive index $n = n(x)$ is uniquely determined by $u_\infty(\hat{x}, d)$ for $\hat{x}$ and D on the unit sphere Ω.*

Proof: Let B be an open ball centered at the origin and containing the support D of $m := 1 - n$. The first step in the proof is to construct a solution of (40) in B of the form

$$w(x) = e^{iz\cdot x}\,(1 + r(x)) \tag{43}$$

where $z \cdot z = 0$, $z \in \mathbb{C}^3$, and

$$\|r\|_{L^2(B)} \leq \frac{C}{|\mathrm{Re}(z)|}$$

for some positive constant C and $|\mathrm{Re}(z)|$ sufficiently large. This can be done using Fourier series [9], [18]. The second step is to show that, given two balls B_1 and B_2 centered at the origin and containing the support of m such that $\overline{B}_1 \subset B_2$, the set of solutions $\{u(\cdot, d) : \; d \in \Omega\}$ satisfying (40)-(42) is complete in

$$H := \left\{ w \in C^2(B_2) : \; \Delta_3 w + k^2 n w = 0 \quad \text{in} \;\; B_2 \right\}$$

with respect to the norm $L^2(B_1)$ [9], [18]. Now assume that n_1 and n_2 are refractive indices such that the corresponding far field patterns satisfy $u_{1,\infty}(\cdot, d) = u_{2,\infty}(\cdot, d)$, $d \in \Omega$, and assume that the support of $1 - n_1$ and $1 - n_2$ are contained in $\overline{B}_1$. Then, using Rellich's lemma and Green's second identity, it can be shown that

$$\int\limits_{B_1} u_1(\cdot, \tilde{d})\, u_2(\cdot, d)\, (n_1 - n_2)\, dx = 0$$

for all $d, \tilde{d} \in \Omega$ and hence

$$\int\limits_{B_1} w_1\, w_2\, (n_1 - n_2)\, dx = 0 \tag{44}$$

for all solutions $w_1, w_2 \in C^2(B_2)$ of $\Delta_3 w_1 + k^2 n_1 w_1 = 0$ and $\Delta_3 w_2 + k^2 n_2 w_2 = 0$ in B_2. Now choose $z_1 := y + \rho a + ib$ and $z_2 := y - \rho a - ib$ such that $\{y, a, b\}$ is an orthogonal basis in $\mathbb{R}^3$ with the properties that $|a| = 1$ and $|b|^2 = |y|^2 + \rho^2$ and substitute these values of z into (43) arriving at functions w_1 and w_2. Substitute these function into (44) and let $\rho \to \infty$ to arrive at

$$\int\limits_{B_1} e^{2iy\cdot x} \left(n_1(x) - n_2(x)\right)\, dx = 0$$

for arbitrary $y \in \mathbb{R}^3$, i.e. $n_1(x) = n_2(x)$ for $x \in B_1$ by the Fourier integral theorem.

5 Reconstruction Methods

We can now consider the problem of reconstructing a stable approximate solution to the ill-posed problems considered above. Methods for doing this are called *regularization methods* and for linear inverse problems such as the backwards heat equation there are numerous methods for doing this [16], [19]. The most popular of such linear regularization methods is *Tikhonov regularization* which we have already briefly encountered in the previous section of this article. We now describe this method for solving the operator equation

$$Tx = y \tag{45}$$

where $T : H_1 \to H_2$ is a compact operator mapping the Hilbert space H_1 into the Hilbert space H_2. As noted previously, this problem is ill-posed in the sense that the inverse operator T^{-1}, if it exists, is not bounded. We note that the problem of solving the backwards heat equations (1)-(3) can be recast as a problem of the form (45) [27]. Furthermore, when we consider methods for solving (nonlinear) inverse scattering problems, linear problems of the form (45) will be part of the reconstruction algorithm.

We begin by considering the problem of finding a vector $x \in H_1$ that minimizes the quadratic functional

$$F(x) = \|Tx - x\|^2.$$

Such a vector is called a *least squares* solution of $Tx = y$. It is well known [16], [19] that the least squares solution x must satisfy the so called normal equation

$$T^*Tx = T^*y$$

which, since T is compact, is ill-posed. The idea of Tikhonov regularization is to replace the normal equation by the second kind integral equation

$$T^*Tx + \alpha x = T^*y \tag{46}$$

where $\alpha > 0$ is a parameter. The key point is that the problem of solving (46) is well-posed. Indeeed,

$$\|x\|^2\|T^*T + \alpha I\| \geq ((T^*T + \alpha I)x, x) = \|Tx\|^2 + \alpha\|x\|^2 \geq \alpha\|x\|^2$$

and hence $(T^*T + \alpha I)^{-1}$ is bounded and

$$\|(T^*T + \alpha I)^{-1}\| \leq \frac{1}{\alpha}.$$

In particular, suppose y^δ is the measured data such that $\|y - y^\delta\| \leq \delta$ and let x_α^δ be the approximation formed by using this approximate data, i.e.

$$x_\alpha^\delta = (T^*T + \alpha I)^{-1}T^*y^\delta.$$

Then, using the singular value decomposition, we have that

$$x_\alpha - x_\alpha^\delta = \sum_{n=1}^{\infty} \frac{\mu_n}{\mu_n^2 + \alpha}(y - y^\delta, g_n)\varphi_n$$

and hence

$$\|x_\alpha - x_\alpha^\delta\|^2 = \sum_{n=1}^{\infty} \frac{\mu_n}{\mu_n^2 + \alpha} \left|(y - y^\delta, g_n)\right|^2 \leq \frac{1}{\alpha} \sum_{n=1}^{\infty} \left|(y - y^\delta, g_n)\right|^2 \leq \frac{\delta^2}{\alpha}.$$

The Tikhonov approximation

$$x_\alpha = (T^*T + \alpha I)^{-1} T^* y \tag{47}$$

also has a useful variational characterization. In particular, (46) is the Euler equation for the functional

$$F_\alpha(x, y) := \|Tx - y\|^2 + \alpha \|x\|^2 \tag{48}$$

and hence the approximation (47) is a global minimizer of (48). This opens the possibility of applying standard optimization techniques for calculating the Tikhonov approximation x_α. In order to do this, we of course need some criteria for selecting the *regularization parameter* α. A popular way of doing this is *Morozov discrepancy principle* which chooses the regularization parameter α in such a way that the error in the residual $\|Tx_\alpha^\delta - y^\delta\|$ is equal to the error level in the data, i.e.

$$\|Tx_\alpha^\delta - y^\delta\| = \delta. \tag{49}$$

Note that the bound for the data error might not be tight. In can be shown that there is a unique computable $\alpha = \alpha(\delta)$ satisfying (49).

We now briefly discuss some reconstruction techniques used in computerized x-ray tomography. Such methods depend on inverting the Radon transform. One method for inverting the Radon transform is the direct implementation of the inversion formula (31) and another method is an algebraic reconstruction technique (ART). When implementing the inversion formula (31), one starts by replacing $|r|$ by a low-pass filter that is the Fourier transform of a band-limited function (i.e. a function whose Fourier transform is nonzero on some finite interval and zero outside that interval). The price of doing this is that the formula (31) is not exact but gives only an approximation for f

$$f(x) \approx \frac{1}{2\pi} \int_0^\pi \left(\mathcal{F}^{-1} A \star \mathcal{R} f\right)(t, \omega)\, d\omega \tag{50}$$

where $t = (x \cdot \omega)$ and A is the low-pass filter. The simplest example of a low-pass filter is the *Ram-Lak filter* defined by $A(r) = |r|$ if $|r| \leq L$ and $A(r) = 0$ if $|r| > L$ where $L > 0$. Other examples include the *Shepp-Logan filter* and the *low-pass cosine filter* [17]. In a parallel beam geometry one has available the discrete version $\mathcal{R}_D f$ of the Radon transform given by

$$\mathcal{R}_D f_{j,k} = \mathcal{R} f(jd, k\pi/N) \tag{51}$$

for $-M \le j \le M$, and $0 \le k \le (N-1)$, where $k\pi/N$ are the angels and jd are the values of t for a chosen distance $d > 0$ where the measurements are taken. The question now become how to compute accurately, efficiently and in a stable manner the discrete version of the convolution in (50) including the computation of the discrete inverse Fourier transform. This issue has been intensively studied in the theory of imaging where techniques such as Nyquist's sampling and the fast Fourier transform are used. At this point the reader can consult a wide literature on this subject (see e.g. [25], [26] and the references therein). In the practical case of incomplete data the structure of Radon's inversion formula gives insight into the nature of the ill-posedeness. Roughly speaking in the "averaging process" in the back projection the biggest contribution comes from the lines which are tangent to discontinuities of f. If integrals over such lines are missing then the approximation of f becomes much worse. Typically an extension of (erroneous inconsistent) data in a consistent way is performed using Tikhonov regularization techniques since this problem is ill-posed [25].

The above Fourier transformation method is used in the algorithms of todays CT scan machines. However, the first CT scanner used an algebraic reconstruction technique (ART) where the approach is grounded in linear algebra and matrix theory to generate an image from the machine readings. This method is conceptually simple and does not involves any Radon inversion formula. Suppose an image is to be constructed in a K by K grid of pixels, a pixel being a small square in the plane. For convenience, we number the pixels from left to right and top to bottom and define the *pixel basis functions* $b_1, b_2 \ldots, b_{K^2}$ such that for $1 \le k \le K^2$, $b_k(x, y) = 1$ if (x, y) is in pixel number k and otherwise is zero. If we assign the color value x_k to the kth pixel, then the resulting image will be represented by

$$f(x, y) = \sum_{k=1}^{K^2} x_k \, b_k(x, y).$$

Applying the discrete Radon transform $\mathcal{R}_D$ to both sides we obtain

$$\mathcal{R}_D f(t_j, \omega_j) = \sum_{k=1}^{K^2} x_k \, \mathcal{R}_D b_k(t_j, \omega_j), \qquad j = 1, \ldots J$$

(the 2×2 discrete data above is recounted as a J-vector). Note that $\mathcal{R}_D b_k(t_j, \omega_j)$ is equal to the length of the intersection of the line $\ell_{t_j,\omega}$ with the pixel k. In principle, these values, which we denote by $r_{j,k}$, $j = 1, \ldots J$, and $k_1, \ldots K^2$ are easy to compute, whereas the left hand side $p_j := \mathcal{R}_D f(t_j, \omega_j)$, $j = 1, \ldots J$ is the measured data. The values x_k, $k = 1, \ldots K^2$ need to be computed and plotted to obtain a image. So the problem is written as solving an non-square algebraic system

$$A\mathbf{x} = \mathbf{p}, \qquad A = (r_{j,k})_{J\times K^2}, \;\; \mathbf{x} = (x_k)_{K^2\times 1}, \;\; \mathbf{p} = (p_j)_{J\times 1}.$$

This system is typically solved by a least squares method that is merely a finite dimensional version of the Tikhonov regularization technique. Alternative ways to solve this large system (in particular K is big in order to obtain a good image) leads to different versions of ART one of which is Kaczmarz's method [26].

We now turn our attention to solving the inverse scattering problem for an obstacle, i.e the problem of determining the scatterer D from the far field pattern $u_\infty(\hat{x}, d)$ associated with the direct scattering problem (35)-(38). We will present two methods for doing this: the *decomposition method* as developed by Kirsch and Kress [21], [22] and the *linear sampling method* due to Colton and Kirsch [11], [13]. We begin with the decomposition method. The main idea of the decomposistion method is to break the inverse obstacle scattering problem into two parts: the first part deals with the ill-posedness by constructing the scattered wave u^s from the far field pattern u_∞ and the second part deals with the nonlinearity by determining the unknown boundary ∂D of the scatterer as the location where the boundary condition for the total field $u = u^i + u^s$ is satisfied in a least square sense. We assume that the unknown scatterer D is bounded and simply connected and enough a priori information on the unknown scatterer is assumed so that one can place a closed surface Γ inside D. Then the scattered field u^s is sought as a single layer potential

$$u^s(x) = \int_\Gamma \phi(y)\, \Phi(x, y)\, ds(y), \qquad x \in \mathbb{R}^2 \setminus \overline{D} \tag{52}$$

where fundamental solution Φ is given by (39) and $\phi \in L^2(\Gamma)$ is a density to be determined. In this case the far field pattern u_∞ has the representation

$$u_\infty(\hat{x}) = \int_\Gamma e^{-ik\hat{x}\cdot y} \phi(y)\, ds(y) \tag{53}$$

and ϕ is now determined by solving the integral equation (53) using Tikhonov regularization (note that the integral operator in (53) is compact). Given an approximation of the scattered wave u^s_∞ obtained by inserting the Tikhonov regularization solution ϕ_α of (53) into (52), the unknown boundary ∂D is then determined by requiring that the boundary condition $u^i + u^s = 0$ on ∂D be satisfied in a least squares sense, i.e. by minimizing

$$\|u^i + u^s_\alpha\|^2_{L^2(\Omega)}$$

over a suitable set of admissible surfaces Λ.

We now direct our attention to a different approach to solving the above inverse scattering problem which is an example of *qualitative methods* in inverse scattering theory [3], [23]. These methods have the advantage over the decomposition method in requiring less a priori information (e.g. it is not necessary to know the topology of the scatterer or the boundary condition satisfied by the total field) and in addition reduces a nonlinear problem to a linear problem. On the other hand, the implementation requires much more data (instead of a singled fixed incident plane wave we now must use plane waves from all directions). We again consider the scattering problem (35)-(38) having far field pattern u_∞ and define the *far filed operator* $F : L^2(\Omega) \to L^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d)\, ds(d), \tag{54}$$

for $\hat{x} \in \Omega$ where Ω is again the unit sphere. By superposition Fg is seen to be the far field pattern corresponding to the *Herglotz wave function*

$$v_g(x) := \int_\Omega e^{ikx\cdot d} g(d)\, ds(d) \tag{55}$$

as incident field. The function $g \in L^2(\Omega)$ is known as the *kernel* of the Herglotz wave function and the far field operator F is compact. It can be shown that F is injective with dense range if and only if there does not exist a Dirichlet eigenfunction for D which is a Herglotz wave function.

The linear sampling method is a qualitative method for solving the inverse scattering problem that was first introduced by Colton and Kirsch [11] and Colton, Piana and Potthast [13]. To describe this method for the case of the inverse scattering scattering problem corresponding to the direct scattering problem (35)-(38), we first assume that there exits a solution $g = g(\cdot, z) \in L^2(\Omega)$ to the *far field equation*

$$Fg = \Phi_\infty(\cdot, z) \tag{56}$$

where Φ_∞ is the far field pattern of radiating fundamental solution (39) given by

$$\Phi_\infty(\hat{x}, z) = \frac{1}{4\pi} e^{-ik\hat{x}\cdot z}.$$

It follows from Rellich's lemma that

$$\int_\Omega u^s(x, d) g(d) ds(d) = \Phi(x, z) \qquad \text{for } x \in \mathbb{R}^3 \setminus D.$$

From the boundary condition $u = 0$ on ∂D we see that

$$v_g(x) + \Phi(x, z) = 0 \tag{57}$$

for $x \in \partial D$ where v_g is the Herglotz wave function with kernel g. We can now conclude from (57) that v_g becomes unbounded as $z \to x \in \partial D$ and hence

$$\lim_{\substack{z \to \partial D \\ z \in D}} \|g(\cdot, z)\|_{L^2(\Omega)} = \infty$$

i.e. ∂D is characterized by points z where the solution of (56) becomes unbounded.

Unfortunately, in general the far field equation (56) does not have a solution nor does the above analysis say anything about what happens when $z \in \mathbb{R}^3 \setminus \overline{D}$. However we have the following result [3]:

Theorem 5.1 *Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta_3$ in $D$and let F be the far field operator corresponding (35)-(38). Then*

1. *For $z \in D$ and a given $\epsilon > 0$ there exists $g_{z,\epsilon} \in L^2(\Omega)$ such that*

$$\|Fg_{z,\epsilon} - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} < \epsilon$$

and the corresponding Herglotz wave function $v_{g_{z,\epsilon}}$ converges to a solution of

$$\begin{aligned} \Delta u + k^2 u = 0 \qquad & \textit{in } D \\ u = -\Phi(\cdot, z) \qquad & \textit{on } \partial D \end{aligned}$$

in the Sobolev space $H^1(D)$ as $\epsilon \to 0$.

2. *For $z \in \mathbb{R}^3 \setminus \overline{D}$ and a given $\epsilon > 0$every $g_{z,\epsilon} \in L^2(\Omega)$ that satisfies*

$$\|Fg_{z,\epsilon} - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} < \epsilon$$

is such that $\lim\limits_{\epsilon \to 0} \|v_{g_{z,\epsilon}}\|_{H^1(D)} = \infty$.

The *linear sampling method* is based on attempting to compute the function $g_{z,\epsilon}$ in the above theorem by using Tikhonov regularization to solve $Fg = \Phi_\infty(\cdot, z)$. Unfortunately it is not clear that doing so will produce the function $g_{z,\epsilon}$. This issue was resolved by Kirsch [20] who proposed replacing the far field equation $Fg = \Phi_\infty(\cdot, z)$ by

$$(F^*F)^{1/4}\, g = \Phi_\infty(\cdot, z) \tag{58}$$

where F^* is the adjoint of F in $L^2(\Omega)$. It was then shown that $\Phi(\cdot, z)$ is in the range of $(F^*F)^{1/4}$ if and only if $z \in D$ and from this one can conclude that if (58) is solved by using Tikhonov regularization and the Morozov discrepancy principle then as the noise level on u_∞ tends to zero the norm of the regularized solution remains bounded if and only if $z \in D$. This method for solving the inverse scattering problem is called the *factorization method*. Using the factorization method one can now establish the following result for the linear sampling method [2].

Theorem 5.2 *Let F be the far field operator associated with the scattering problem (35)-(38) and assume that k^2 is not a Dirichlet eigenvalue of $-\Delta_3$ in D. Then for $z \in D$ let $g_z \in L^2(\Omega)$ be the solution $(F^*F)^{1/4} g_z = \Phi_\infty(\cdot, z)$ and for every $z \in \mathbb{R}^3$ and $\epsilon > 0$ let $g_{z,\epsilon}$ be the solution of $Fg = \Phi_\infty(\cdot, z)$ obtained by Tikhonov regularization, i.e. the unique solution of $\epsilon g + F^*Fg = F^*\Phi$. Then*

1. *Let $v_{g_{z,\epsilon}}$ be the Herglotz wave function with kernel $g_{z,\epsilon}$. Then for every $z \in D$ the limit $\lim_{\epsilon \to 0} v_{g_{z,\epsilon}}(z)$ exists. Furthermore, there exists $c > 0$ depending only on F, such that for every $z \in D$ we have that*

$$c\|g_z\|^2_{L^2(\Omega)} \leq \lim_{\epsilon \to 0} |v_{g_{z,\epsilon}}(z)| \leq \|g_z\|^2_{L^2(\Omega)}$$

2. *For $z \notin D$ we have that $\lim_{\epsilon \to 0} |v_{g_{z,\epsilon}}(z)| = \infty$.*

The linear sampling method and factorization method have been extended to a wide variety of inverse scattering problems and for details we refer the reader to [3], [5] and [23]. In the case of inverse medium problem corresponding to the direct scattering problem (40)-(42) these qualitative methods only provide a reconstruction of the support D of $m := 1 - n$ (However, see the following section of this article). On the other hand the *dual space method* of Colton and Monk [14], [9] is a decomposition method for reconstructing the index of refraction n (and thus also the support D of $m := 1 - n$).

6 Transmission Eigenvalues

The introduction in 1996 of qualitative methods for inverse problems heralded a new era for nonlinear inverse problems [3], [23]. Indeed, instead of trying to reconstruct the solution of an inverse problem in its entirety, attention was now paid to only recovering partial information but in a rapid and simple way. The development of these qualitative methods is still actively being pursued and predicting the future of these methods is fraught with uncertainty. One direction in this new development is to simply obtain upper and lower bounds for a quantity of physical interest. In this closing section we will illustrate this set of ideas by the example of *transmission eigenvalues*.

We again consider the scattering problem for an inhomogeneous medium

$$\Delta_3 u + k^2 n u = 0 \qquad \text{in } \mathbb{R}^3 \tag{59}$$

$$u(x) = e^{ikx\cdot d} + u^s(x) \tag{60}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) \tag{61}$$

where the incindent field u^i is given by $u^i(x) = e^{ikx\cdot d}$, u^s is the scattered field, the Sommerfeld radiation condition (61) is assumed to hold uniformly in $\hat{x} = x/|x|$ and $n = n(x)$ is the index of refraction where we assume that $\mathrm{Im}(n) = 0$. If $u_\infty(\hat{x}, d) = u_\infty(\hat{x}, d; k)$ is the far field pattern with corresponding far field operator F (c.f. equation (54)) then it can be shown that F is injective with dense range if and only if there does not exists a nontrivial solution (in the sense of distributions) $v, w \in L^2(D)$, $v - w \in H_0^2(D)$ of the *interior transmission problem*

$$\Delta_3 w + k^2 n w = 0 \qquad \text{in } D \tag{62}$$

$$\Delta_3 v + k^2 v = 0 \qquad \text{in } D \tag{63}$$

$$v = w \qquad \text{on } \partial D \tag{64}$$

$$\frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} \qquad \text{on } \partial D \tag{65}$$

where ν is the unit outward normal to ∂D and v is a Herglotz wave function (c.f. (55)) [9]. Values of $k \in \mathbb{C}$ such that there exists a nontrivial solution of (62)-(65) are called *transmission eigenvalues*.

Transmission eigenvalues are closely related to non-scattering fields for inhomogeneous media. In particular, if the interrogating frequency is such that the corresponding wave number k is a transmission eigenvalue for the given medium, and v can be extended outside D as a solution to the Helmholtz equation where (v, w) is a nonzero solution to (62)-(65) corresponding yo this eigenvalue, then this v constitutes an incident field that does not generate any scattered field. In general, v cannot be extended outside D to a solution of the Helmholtz equation. However, Herglotz wave functions are dense in the space of solutions to the Helmholtz equation with respect to the $L^2(D)$ norm [9], [10]. Therefore an incident wave of the form of a Herglotz wave function that approximates v arbitrarily closely generates an arbitrarily small scattered field.

Before addressing the question of what physical information is contained in a knowledge of the transmission eigenvalues corresponding to a given scattering problem, we briefly describe how these eigenvalues can be determined from measured scattering data. In particular, we consider the noisy far field equation

$$(F^\delta g)(\hat{x}) = \Phi_\infty(\hat{x}, z) \tag{66}$$

where $z \in D$, Φ_∞ is the far field pattern of the radiating fundamental solution $\Phi(x, z)$ (c.f. (39)) and F^δ is the noisy far field operator (54) with u_∞ replaced by the noisy measured far field data u_∞^δ where u_∞ is the far field pattern corresponding to (59)-(61). We can assume that F has dense range which is essentially always the case unless $n = n(x)$ is spherically stratified. Let $g_{z,\delta}$ be the solution of the far field equation (66) using Tikhonov regularization, i.e. $g_{z,\delta}$ is defined as the unique minimizer of the Tikhonov functional

$$\|F^\delta g - \Phi_\infty(\cdot, z)\|^2_{L^2(\Omega)} + \alpha\|g\|^2_{L^2(\Omega)}$$

where $\alpha = \alpha(\delta)$ is chosen by the Morozov discrepancy principle. Then, if either $n(x) > 1$ or $0 < n(x) < 1$ for $x \in \overline{D}$, for almost every $z \in D$ the Herglotz wave functions $v_{g_{z,\delta}}$ converges in $L^2(D)$ if and only if k is not a transmission eigenvalue [6]. In practice this means that if $g_{z,\delta}$ is the regularized solution of (66), and the noise level δ is sufficiently small then under the above assumption on n if $\|g_{z,\delta}\|_{L^2(\Omega)}$ is plotted against k for a variety of values of z the real transmission eigenvalues will appear as sharp peaks in the graph.

We now turn our attention to what information the transmission eigenvalues contain about the index of refraction $n(x)$. The following theorem [12] shows that if $n(x) > 1$ for $x \in D$ the first transmission eigenvalue contains information about the refractive index provided the support of D is known (obtained for example through the use of the linear sampling or factorization method applied to the scattering problem (59)-(61) – c.f. [3], [23]).

Theorem 6.1 *Suppose that $n(x) > 1$for $x \in \overline{D}$ and let k_1 be the first real transmission eigenvalue for (62)-(65). Then if $\lambda_1(D)$ is the first Dirichlet eigenvalue for $-\Delta_3$ in D we have that*

$$k_1^2 > \frac{\lambda_1(D)}{\sup\limits_{x\in D} n(x)}.$$

Proof: In (62)-(65) we set $u = w - v$. Then

$$\left(\Delta_3 + k^2\right) u = k^2 m w$$

where $m := 1 - n$ and hence

$$\left(\Delta_3 + k^2 n\right) \frac{1}{m} \left(\Delta_3 + k^2\right) u = 0$$

i.e. since $u \in H_0^2(D)$ and using Green's first identity (note that $m < 0$)

$$\begin{aligned} 0 &= -\int_D \left(\Delta_3 u + k^2 n u\right) \frac{1}{m} \left(\Delta_3 \overline{u} + k^2 \overline{u}\right) \\ &= -\int_D \frac{1}{m} \left|\Delta_3 u + k^2 m u\right| \, dx + k^2 \int_D |\nabla u|^2 \, dx - k^4 \int_D n \, |u|^2 \, dx. \end{aligned} \tag{67}$$

But, using the Rayleigh quotient,

$$\inf_{u \in H_0^2(D)} \frac{\int_D |u|^2 \, dx}{\int_D |\nabla u|^2 \, dx} \geq \inf_{u \in H_0^1(D)} \frac{\int_D |\nabla u|^2 \, dx}{\int_D |u|^2 \, dx} = \lambda_1(D)$$

we have that

$$\int_D k^2 \left(|\nabla u|^2 - k^2 n |u|^2\right) dx \geq k^2 \|u\|^2 \left(\lambda_1(D) - k^2 \sup_D n\right)$$

i.e. if $k^2 \leq \dfrac{\lambda_1(D)}{\sup_D \, n}$ then $u = 0$, i.e. k cannot be a transmission eigenvalue (note that if $u \in H_0^2(D)$ satisfies $\Delta_3 u + k^2 n u = 0$ in D then $u = 0$).

Note that the above inequality for k_1 is not *isoperimetric*, i.e. there is no D for which equality is achieved. Isoperimetric inequalities have been obtained by Cakoni, Gintides and Haddar [4]. In particular, they showed that if $n(x) > 1$ for $x \in \overline{D}$ and $n_* = \inf_{x \in D} n(x)$, $n^* = \sup_{x \in D} n(x)$ then

$$0 < k_{1,n^*} \leq k_{1,n(x)} \leq k_{1,n_*}$$

where $k_{1,n}$ is the first transmission eigenvalue corresponding to the index of refraction n (Similar inequalities are also obtained for the case $0 < n(x) < 1$ for $x \in \overline{D}$). In particular, the above inequality implies that if $n(x) = n_0$ is constant then n_0 depends monotonically on k_1. If n is no longer constant, and k_1 is determined from the far field operator as described above, then the n_0 corresponding to k_1 satisfies

$$n^* \leq n_0 \leq n_*,$$

thus giving isoperimetric inequalities for n^* and n_*.

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