Qualitative Methods in Inverse Electromagnetic Scattering Theory

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Introduction

The inverse scattering problem we are considering is to determine the shape and physical properties of an obstacle from a knowledge of the scattered field due to the scattering of an incident time-harmonic electromagnetic wave at fixed frequency.

- A solution is needed in real time.
- The scattering obstacle may be either penetrable, a perfect conductor or partially coated but such information is not known a priori.
- Often only partial information on the scatterer is needed.

Approaches to Inverse Scattering

- Born or weak scattering approximation
  - multiple scattering and polarization effects are ignored
  - a priori information is needed

- Optimization techniques, Newton iteration methods
  - multiple scattering and polarization effects are included
  - a priori information is needed
  - typically it requires the solution of the forward problem

Approaches to Inverse Scattering

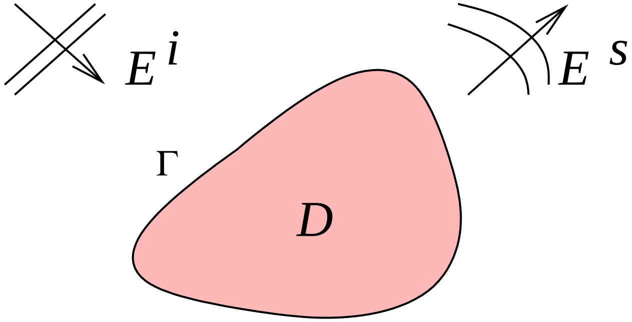
- Qualitative methods
  - multiple scattering and polarization effects are included
  - the problem is linear
  - essentially no a priori information is needed

Linear Sampling Method and related ideas – Arens, Cakoni, Colton, Kirsch, Haddar, Monk, Piana, Potthast, Monk ....

F. Cakoni - D. Colton *"Qualitative Methods in Inverse Scattering"* , Springer, 2006.

Factorization Methods – Kirsch, Grinberg ......
Points Source Methods, One Wave Methods, and others – Ikehata, Potthast, Silvester ....

Scattering from Anisotropic Media



$$
\begin{aligned}
\nabla \times \nabla \times E - k^2 N(x) E &= 0 && \text{in} \quad \mathbb{R}^3 \\
E &= E^s + E^i && \text{in} \quad \mathbb{R}^3 \\
\lim_{|x| \to \infty} (\nabla \times E^s \times x - ik|x|E^s) &= 0
\end{aligned}
$$

The incident field $E^i(x) := \nabla \times \nabla \times p\, e^{ikx\cdot d}$ is a plane wave with polarization $p \in \mathbb{R}^3$ and incident direction $d \in \Omega := \{x : |x| = 1\}$.

The Forward Problem

We assume:

- $N - I$ has compact support $\overline{D}$
- ∂D is piecewise smooth.
- N is a symmetric matrix-valued function with piecewise C^1 entries in $\overline{D}$ with piecewise smooth interfaces.
- $\nu \times E$ and $\nu \times \nabla \times E$ are assumed continuous across the interfaces.
- $\mathcal{R}e\left(\overline{\xi} \cdot N\xi\right) \geq \gamma|\xi|^2$ and $\mathcal{R}e\left(\overline{\xi} \cdot N^{-1}\xi\right) \geq \gamma|\xi|^2$, $\gamma > 0$
- $\mathcal{I}m\left(\overline{\xi} \cdot N\xi\right) \geq 0$.

Inverse Scattering Problem

The scattered field E^s has the asymptotic behavior

$$E^s(x) = \frac{e^{ikr}}{r} E_\infty(\hat{x}, d, p) + O\left(\frac{1}{r^2}\right)$$

as $r \longrightarrow \infty$ where $r = |x|$, $\hat{x} = x/r$ is the observation direction, and $E_\infty(\hat{x}, d, p)$ is the far field pattern of the scattered field E^s.

The inverse scattering problem is to determine D and N from a knowledge of $E_\infty(\hat{x}, d, p)$ for $\hat{x}, -d \in \Omega_0 \subset \Omega$, two linearly independent polarizations p tangent to the unit sphere and a range of frequencies k.

Uniqueness Theorems

Theorem (*Cakoni-Colton, 2003*): Assume that either $\mathcal{R}e\left(\overline{\xi}\cdot N\xi\right)\geq\gamma|\xi|^2$ or $\mathcal{R}e\left(\overline{\xi}\cdot N^{-1}\xi\right)\geq\gamma|\xi|^2$ for some $\gamma>1$. Then D is uniquely determined by $E_\infty(\hat{x},d,p)$ for $\hat{x},d\in\Omega$, two linearly independent polarizations p tangent to Ω and a fixed value of the wave number k.

However $E_\infty(\hat{x},d,p)$ for $\hat{x},d\in\Omega$, two linearly independent polarizations p tangent to Ω does not uniquely determine N even if it is known for an interval of values of k.

Far Field Operator

We define the far field operator $F : L_t^2(\Omega) \to L_t^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))\, ds(d).$$

Given $g \in L_t^2(\Omega)$, Fg is the far field pattern of the scattered field corresponding to the incident field being a Herglotz wave function with kernel g given by

$$E_g(x) := ik \int_\Omega e^{ikx\cdot d} g(d) ds(d).$$

The Far Field Operator

The far field operator F is injective with dense range if and only if the interior transmission problem

$$\begin{aligned} \nabla \times \nabla \times E - k^2 N(x) E &= 0 && \text{in} \quad D \\ \nabla \times \nabla \times E^0 - k^2 E^0 &= 0 && \text{in} \quad D \\ \nu \times E &= \nu \times E^0 && \text{on} \quad \partial D \\ \nu \times \nabla \times E &= \nu \times \nabla \times E^0 && \text{on} \quad \partial D \end{aligned}$$

does not have a solution $E_0, E \in L^2(D)$, $E - E_0 \in H(curl, D)$ and $\nabla \times (E - E_0) \in H(curl, D)$ such that E^0 is a Herglotz wave function with kernel $g \neq 0$.

Transmission Eigenvalues

Definition: If $k > 0$ is such that the interior transmission problem has a nontrivial solution then k is called a transmission eigenvalue.

Theorem: If $\mathcal{I}m\left(\bar{\xi} \cdot N\xi\right) > 0$ in D then k is not a transmission eigenvalue.

Open Problem: Do transmission eigenvalues exist, if $\mathcal{I}m\left(\bar{\xi} \cdot N\xi\right) = 0$ in D? (*It can be shown that they exist in the case of an isotropic spherically stratified medium, Colton-Kress book*).

Theorem (*Cakoni-Haddar, 2007*): If $\mathcal{I}m\left(\bar{\xi} \cdot N\xi\right) = 0$, and $\bar{\xi} \cdot (N - I)\,\xi \geq \alpha|\xi|^2$ or $\bar{\xi} \cdot N(I - N)^{-1}\,\xi \geq \alpha|\xi|^2$ where $\alpha > 0$ is a constant, then the set of transmission eigenvalues is a discrete set.

Electric Dipoles

The radiating solution to Maxwell's equations

$$E_e(x,z,q) := \frac{i}{k}\nabla_x \times \nabla_x \times q\,\Phi(x,z)$$

with

$$\Phi(x,z) := \frac{1}{4\pi}\,\frac{e^{ik|x-z|}}{|x-z|},\; q \in \mathbb{R}^3$$

is called the electric dipole located at z and polarized in the direction $q \in \mathbb{R}^3$.

$E_{e,\infty}(\hat{x},z,q)$ denotes the far field pattern of the corresponding electric field.

The Far Field Equation

Consider the far field equation

$$\boxed{(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q)}$$

It is a **linear** but severely **ill-posed** first kind integral equation.

Assume that k is not a transmission eigenvalue. We can say the following about the solvability of the far field equation.

Solution of the Far Field Equation

For $z \in D$ and an arbitrary $\epsilon > 0$ there exists an approximate solution g_z^ϵ of the far field equation

$$\|Fg_z^\epsilon - E_{e,\infty}(\cdot, z, q)\|_{L_t^2(\Omega)} < \epsilon$$

such that the corresponding Herglotz function $E_{g_z^\epsilon}$ converges to E_z^0 in the $L^2(D)$-norm where $E_{0,z}, E_z \in L^2(D)$ solves the non-homogeneous interior transmission problem

$$\begin{aligned} \nabla \times \nabla \times E_z - k^2 N(x) E_z &= 0 && \text{in} \quad D \\ \nabla \times \nabla \times E_z^0 - k^2 E_z^0 &= 0 && \text{in} \quad D \\ \nu \times E_z &= \nu \times (E_z^0 + E_{e,\infty}(\cdot, z, q)) && \text{on} \quad \partial D \\ \nu \times \nabla \times E_z &= \nu \times \nabla \times (E_z^0 + E_{e,\infty}(\cdot, z, q)) && \text{on} \quad \partial D \end{aligned}$$

Solution of the Far Field Equation

Furthermore, for fixed $\epsilon > 0$ this g_z^ϵ satisfies

$$\lim_{z\to\partial D} \|E_{g_z^\epsilon}\|_{L^2(D)} \to \infty \qquad \text{and} \qquad \lim_{z\to\partial D} \|g_z^\epsilon\|_{L^2(\Omega)} \to \infty$$

For $z \notin D$ every approximate solution g_z^ϵ

$$\|F g_z^\epsilon - E_{e,\infty}(\cdot, z, q)\|_{L_t^2(\Omega)} < \epsilon$$

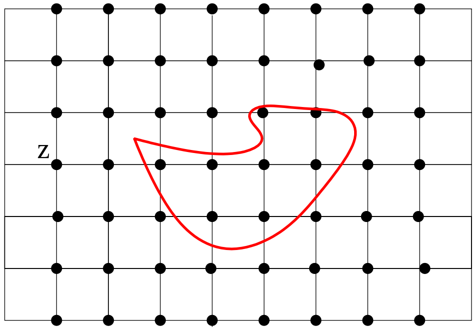
is such that

$$\lim_{\epsilon\to 0} \|E_{g_z^\epsilon}\|_{L^2(D)} \to \infty \qquad \text{and} \qquad \lim_{\epsilon\to 0} \|g_z^\epsilon\|_{L^2(\Omega)} \to \infty$$

for fixed z

Determination of D

The linear sampling method determines D by trying to reconstruct g_z^ϵ.



- Construct a grid $\mathcal{G}$ and for $z_i \in \mathcal{G}$, solve

$$(\alpha I + F^* F) g_{z_i,q} = F^* E_{e,\infty}$$

- For $z_i \in \mathcal{G}$ and three linearly independent $q_1, q_2, q_3 \in \mathbb{R}^3$ evaluate

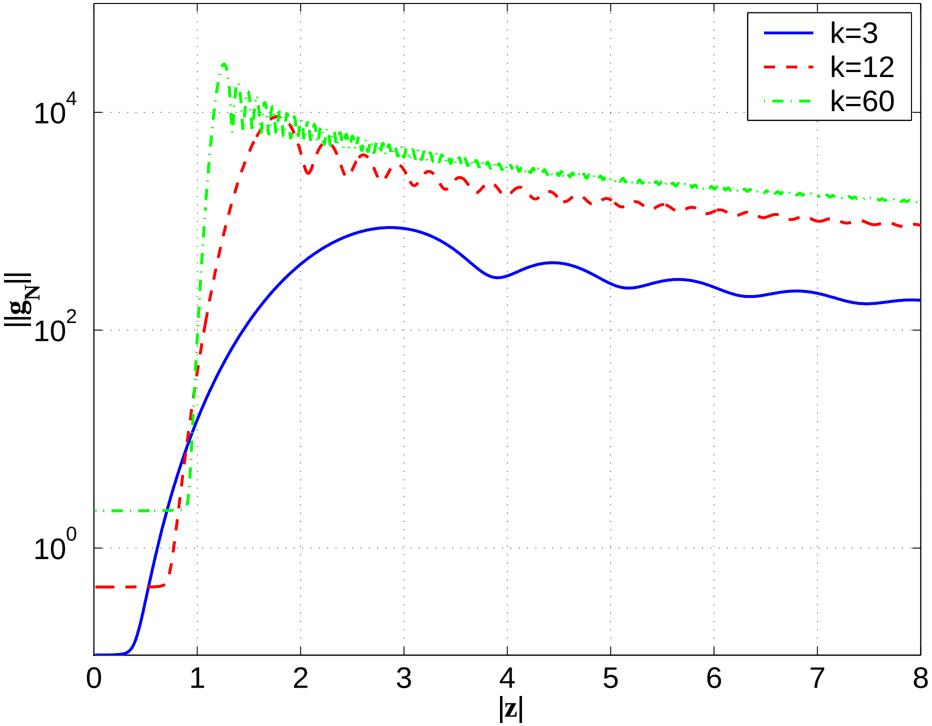
$$G(z_i) = \frac{1}{3}\left(\|g_{z_i,q_1}\|_{\ell^2}^{-1} + \|g_{z_i,q_2}\|_{\ell^2}^{-1} + \|g_{z_i,q_3}\|_{\ell^2}^{-1}\right).$$

- Fix $C > 0$ and visualize the boundary by plotting

$$\mathcal{G}(z) = C \max_{z_i \in \mathcal{G}} G(z_i).$$

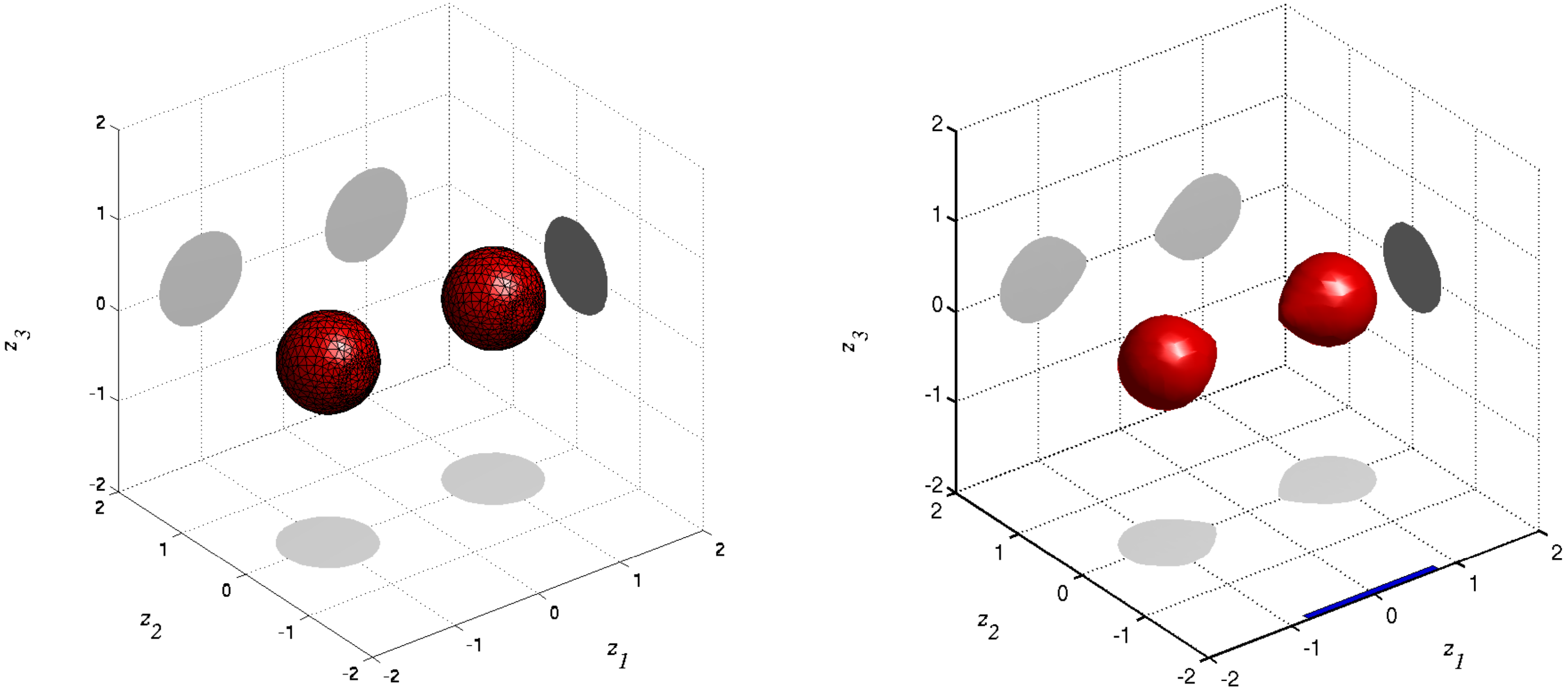
Behavior of $\|g\|$

Collino-Fares-Haddar, 2004



$\|g\|$ with respect to z. Homogeneous penetrable sphere.

Examples of Reconstruction



Exact Geometry

Reconstruction

$N = 3I, k = 4$

Interior Transmission Problem

What, if anything, can be said about N from a knowledge of E_∞?

To this end we return to the interior transmission problem.

Theorem (*Cakoni-Haddar, 2007*) Assume that $\mathcal{I}m\left(\bar{\xi}\cdot N\xi\right)=0$.

- If $\bar{\xi}\cdot(N-I)^{-1}\xi\geq\alpha|\xi|^2$ where $\alpha>0$ is a constant, then the transmission eigenvalues must satisfy $k^2>\alpha\lambda(D)$.
- If $\bar{\xi}\cdot N^{-1}(I-N)^{-1}\xi\geq\alpha|\xi|^2$ where $\alpha>0$ is a constant., then the transmission eigenvalues must satisfy $k^2>\lambda(D)$.

Here $\lambda(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D.

Estimates for N

Since α is the smallest eigenvalue of $(N - I)^{-1}$, the previous theorem implies the following result (*Cakoni-Colton-Haddar, 2007*):

- Assume that $\|N(x)\|_2 \geq \delta > 1$ for all $x \in D$ and some constant δ. Then,

$$\sup_D \|N(x)\|_2 \geq \frac{\lambda(D)}{k^2}$$

- Assume that $0 < \beta \leq \|N(x)\|_2 \leq \delta < 1$ for all $x \in D$ and some constants β and δ. Then,

$$k^2 > \lambda(D)$$

where k is a transmission eigenvalue and $\lambda(D)$ is the first eigenvalue of $-\Delta$ on D.

Computation of Eigenvalues

The linear sampling method can be expected to fail when k is a transmission eigenvalue.

In particular, the norm of the (regularized) solution to

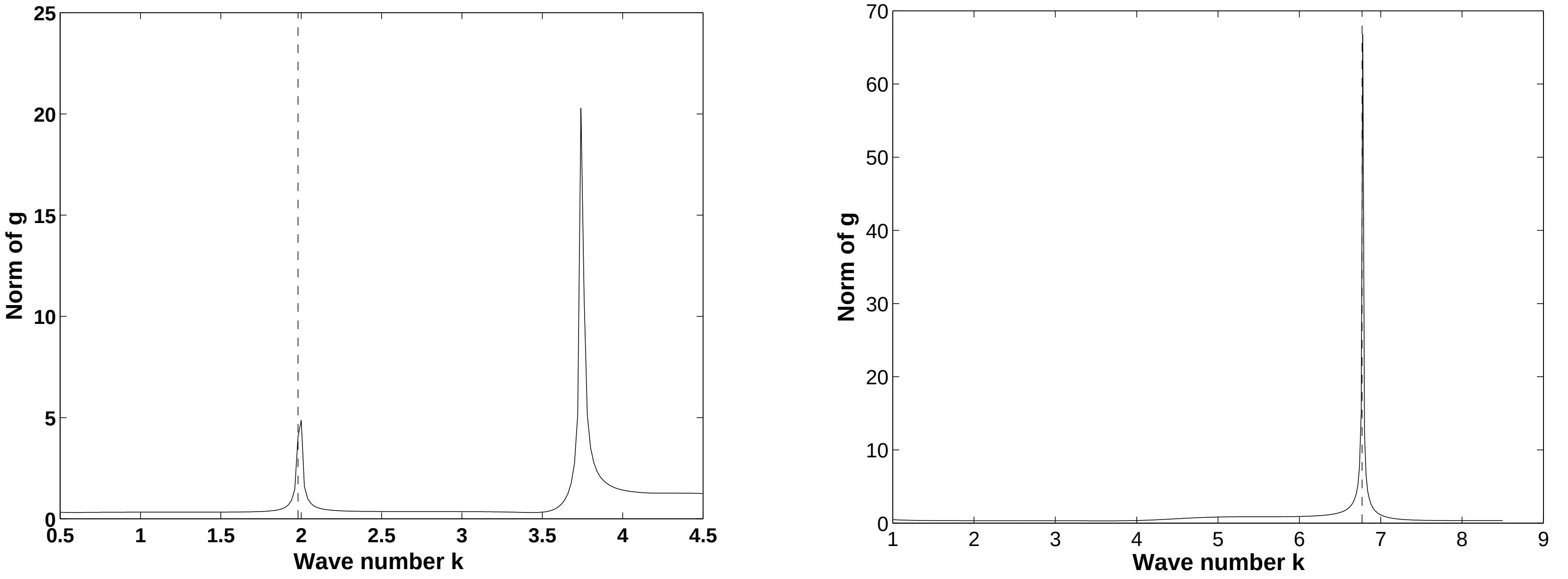
$$(Fg)(\hat{x}) = E_\infty(\hat{x}, z_0, q) \qquad z_0 \in D$$

should be large for such values of k.

Open Problem: Derive an estimate for the norm of N in the case when $0 < \beta \leq \|N(x)\|_2 \leq \delta < 1$

Numerical Examples

D is a disk of diameter 1.



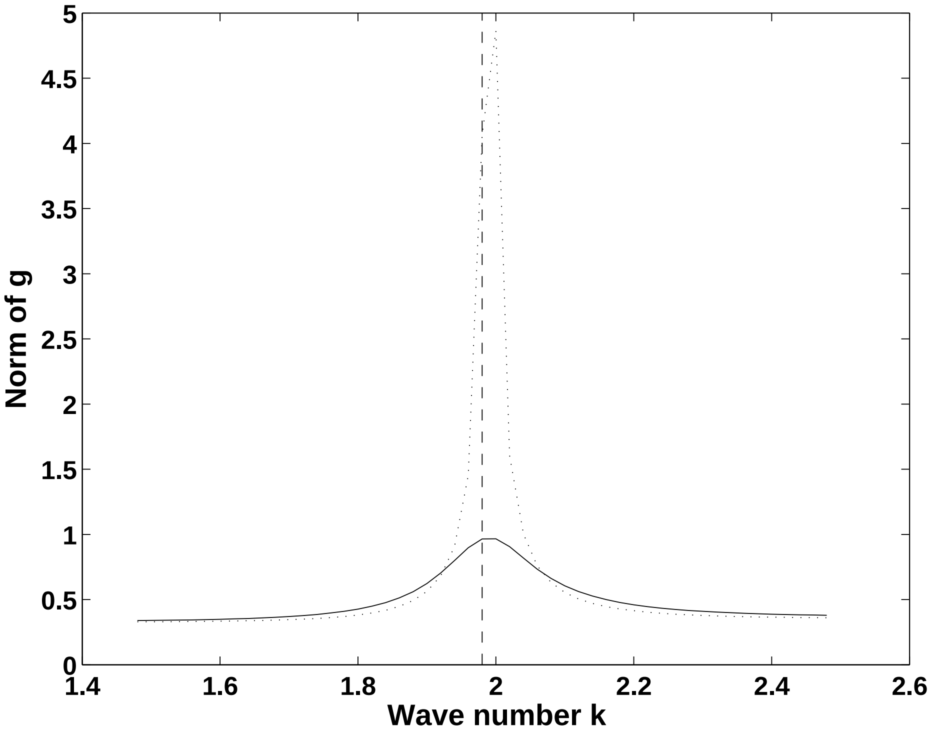
$n = 16$ $n = 4$

$$\Delta u + k^2 n u = 0$$

Cakoni-Colton-Monk, 2006

Numerical Examples

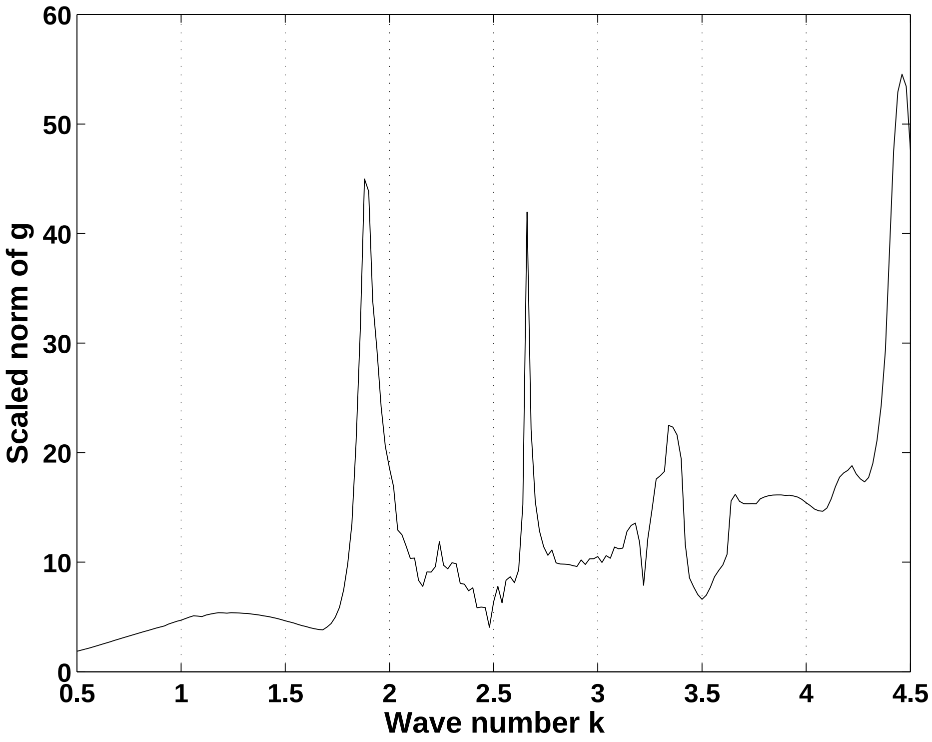
D is a disk of diameter 1.



$$n = 16 + i$$

Numerical Examples

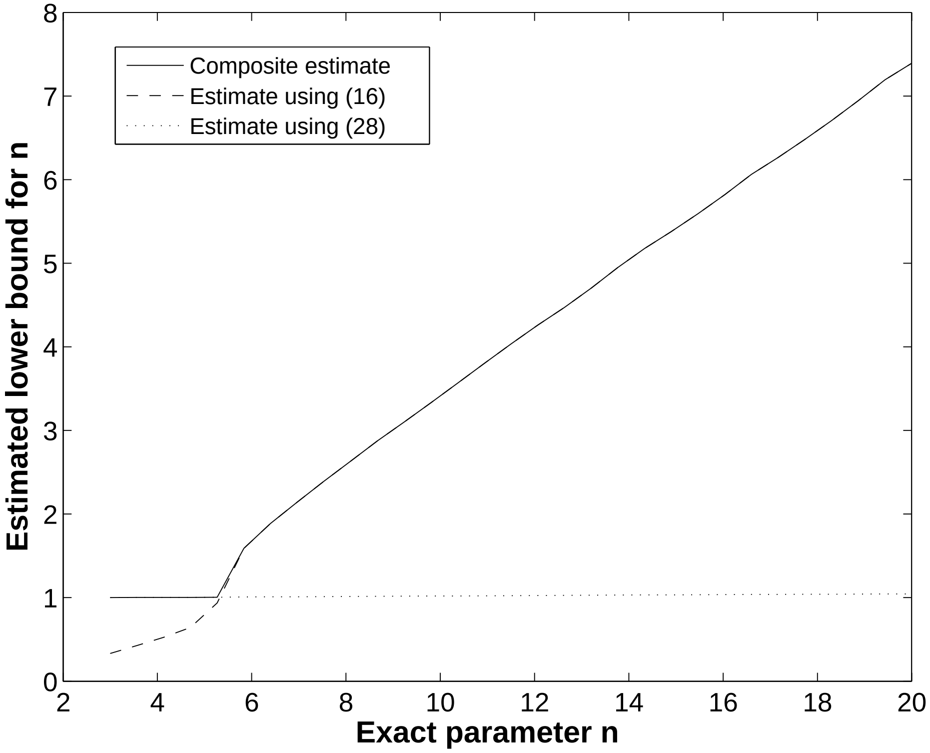
D is the rectangle $[-0.5,\ 0.5] \times [-0.4,\ 0.4]$, $\lambda_0(D) \approx 25.3$.



$n = 16$ and the estimate is $n \geq \dfrac{25.3}{(1.88)^2} \approx 7.1$

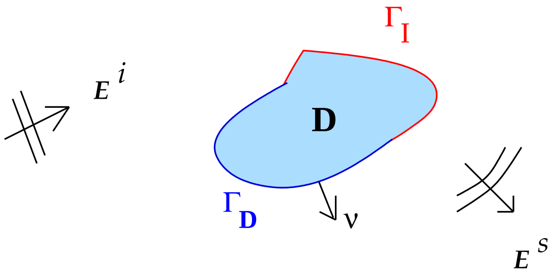
Numerical Examples

D is the rectangle $[-0.5,\ 0.5] \times [-0.4,\ 0.4]$, $\lambda_0(D) \approx 25.3$.



Lower bound for n

Obstacle Scattering



Let $\lambda \in L_\infty(\Gamma_I)$. Under appropriate conditions the electric scattered field E^s satisfies

$$\begin{aligned} \nabla \times \nabla \times E^s - k^2 E^s &= 0 && \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \\ \nu \times E &= 0 && \text{on} \quad \Gamma_D \\ \nu \times (\nabla \times E) - i\lambda(\nu \times E) \times \nu &= 0 && \text{on} \quad \Gamma_I \\ \lim_{|x|\to\infty} (\nabla \times E^s \times x - ik|x|E^s) &= 0 \end{aligned}$$

$$E^i(x) := ik(d \times p) \times d\, e^{ikx\cdot d}, \qquad \partial D = \overline{\Gamma}_D \cup \overline{\Gamma}_I$$

$$E = E^i + E^s$$ the total field

Inverse Scattering Problem

The scattered field E^s again has the asymptotic behavior

$$E^s(x) = \frac{e^{ikr}}{r} E_\infty(\hat{x}, d, p) + O\left(\frac{1}{r^2}\right)$$

where $E_\infty(\hat{x}, d, p)$ is the far field pattern of the scattered field E^s.

The inverse scattering problem is to determine D and λ from a knowledge of $E_\infty(\hat{x}, d, p)$ for $\hat{x}, -d \in \Omega_0 \subset \Omega$, two linearly independent polarizations p tangent to the unit sphere and a fixed frequency k.

D and λ are uniquely determine form the above data.

Determination of D

D can be determined by the linear sampling method.

In particular, the L^2-norm of the approximate (regularized) solution $g \in L^2(\Omega)$ of the far field equation

$$\boxed{(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q)}$$

becomes unbounded as $z \longrightarrow \partial D$ and remains large outside D.

Having determine D, we want next to determine λ. To this end let

$$X(D, \Gamma_I) := \{U \in H(curl, D),\ \nu \times U|_{\Gamma_I} \in L_t^2(\Gamma_I)\}.$$

The determination of λ

For $z \in D$ the Herglotz wave function E_{g_z} where g_z is the solution of the far field equation used to determine D converges to E_z in $X(D, \Gamma_I)$, where E_z is the unique solution of

$$\nabla \times \nabla \times E_z - k^2 E_z = 0 \quad \text{in} \quad D$$

$$\nu \times [E_z + E_e(\cdot, z, q)] = 0 \quad \text{on} \quad \Gamma_D$$

$$\nu \times \nabla \times [E_z + E_e(\cdot, z, q)] - i\lambda\nu \times [E_z + E_e(\cdot, z, q)] \times \nu \quad \text{on} \quad \Gamma_I$$

The Determination of λ

Two important properties of $W_z := E_z + E_e(\cdot, z, q)$.

- The set of functions $f := \nu \times W_z|_{\Gamma_I}$ for z in a ball B_r contained in D is complete in $L^2_t(\Gamma_I)$.
- For any $z_1, z_2 \in D$

$$\begin{aligned} 2\int_{\Gamma_I} (\nu \times W_{z_1}) \cdot \lambda\, (\nu \times \overline{W}_{z_2})\, ds &= -\|q\|^2 A(z_1, z_2, k, q) \\ &+ k(q \cdot E_{z_1}(z_2) + q \cdot \overline{E}_{z_2}(z_1)) \end{aligned}$$

where $A(z_1, z_2, k, q)$ is a computable number.

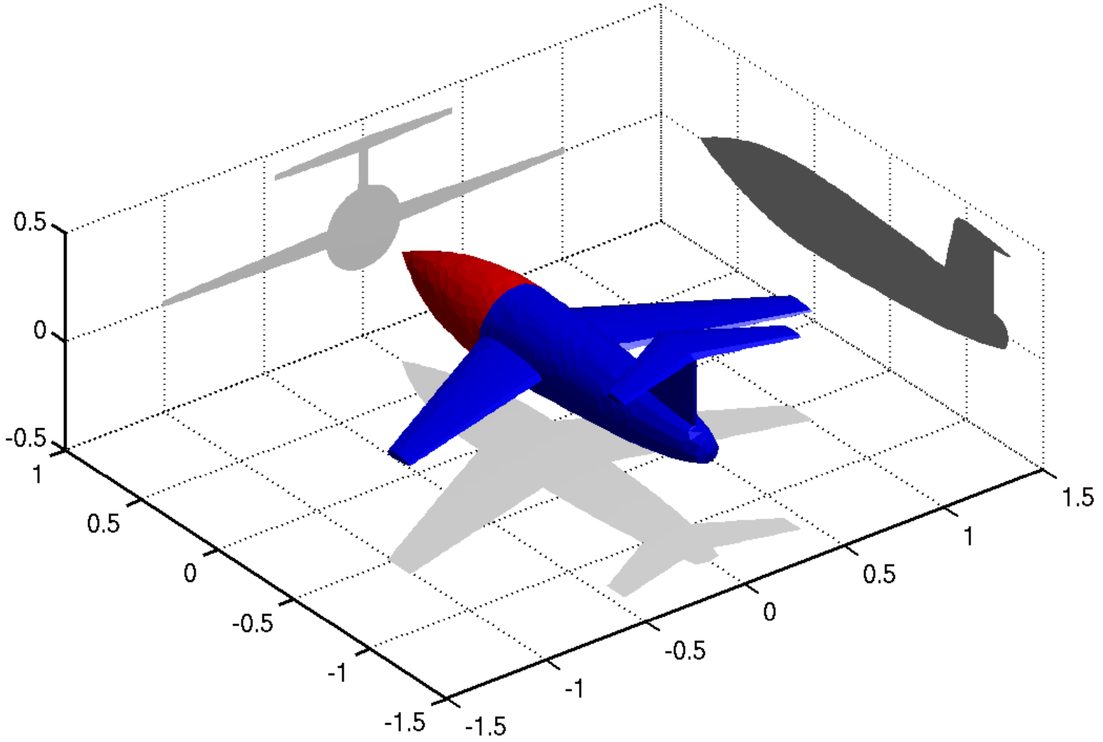
The Determination of λ

- The above properties of W_z provide a method for approximating $\|\lambda\|_{L_\infty(\Gamma_1)}$.
- In particular, if λ is a constant we have

$$\lambda = \frac{-\frac{k^2}{6\pi}\|q\|^2 + k\Re(q \cdot E_{z_0})}{\|\nu \times W_{z_0}\|^2_{L^2_t(\partial D)}} \qquad z_0 \in D.$$

E_z is replaced by E_{g_z} with kernel g_z the (regularized) solution of the far field equation

Examples of Reconstruction

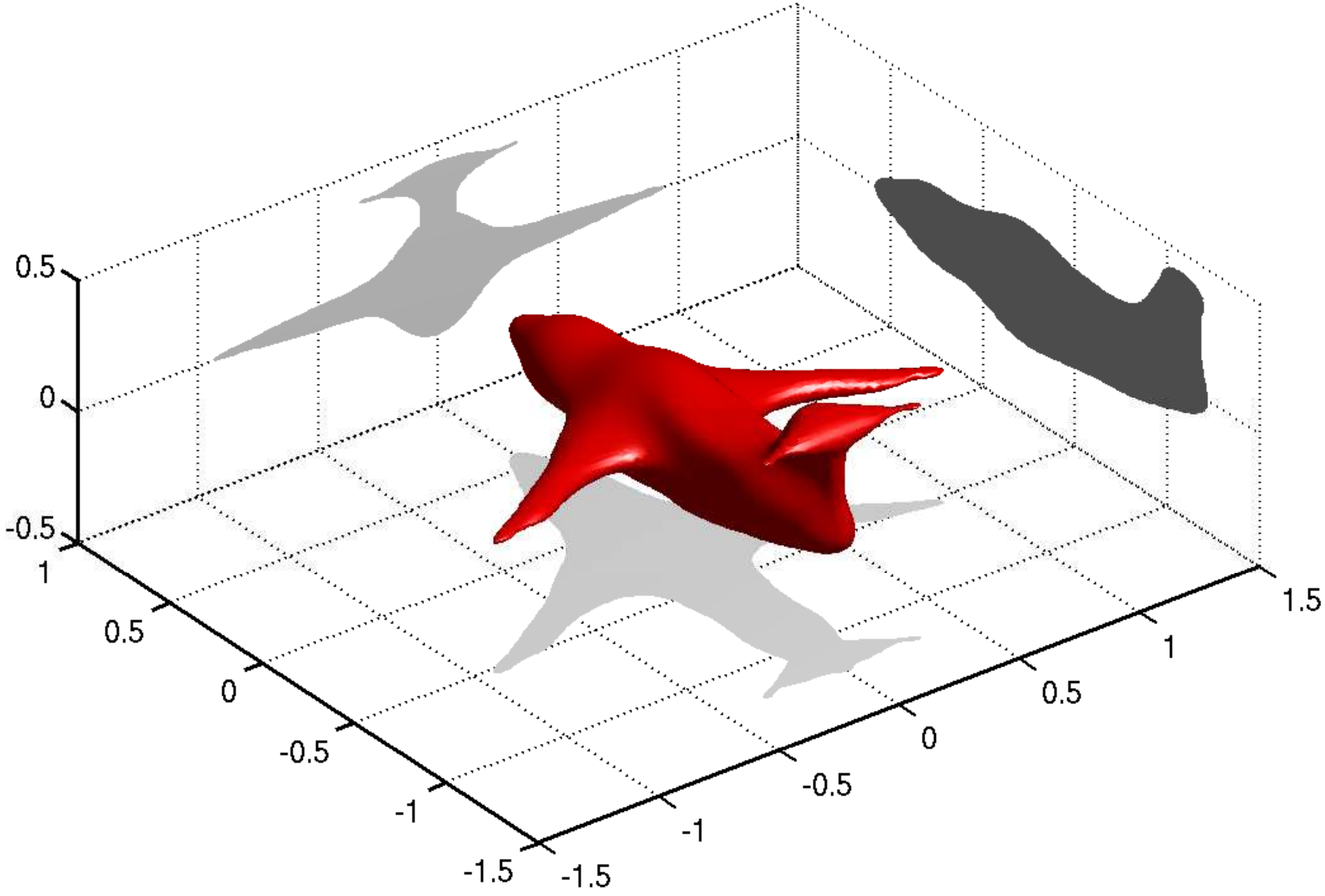


The exact geometry

Impedance boundary condition with $\lambda = 1$ is the red region

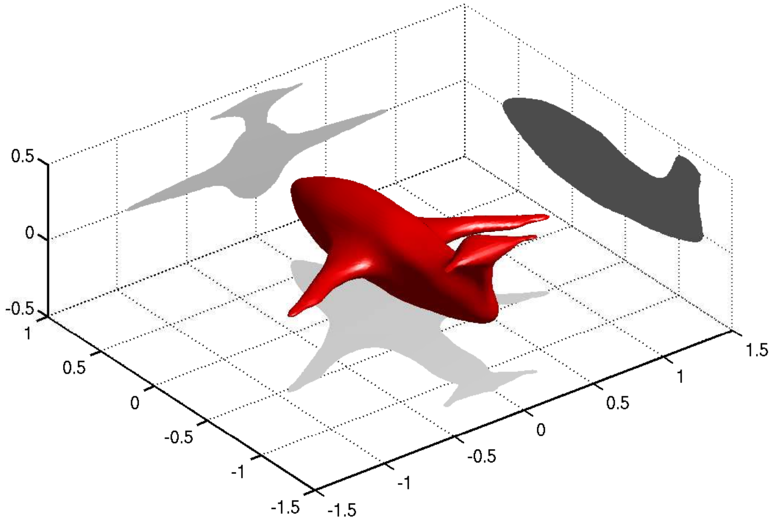
Perfectly conducting boundary condition is the blue region

Examples of Reconstruction

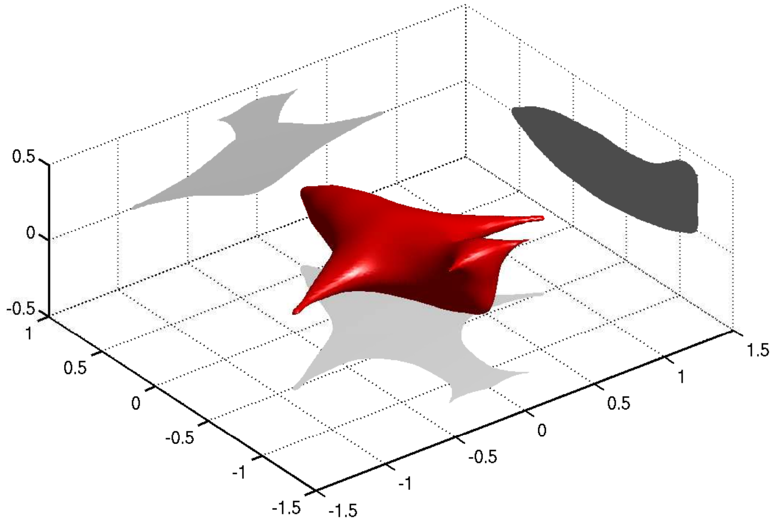


The reconstructed partially coated airplane (wavelength=0.7)

Examples of Reconstruction

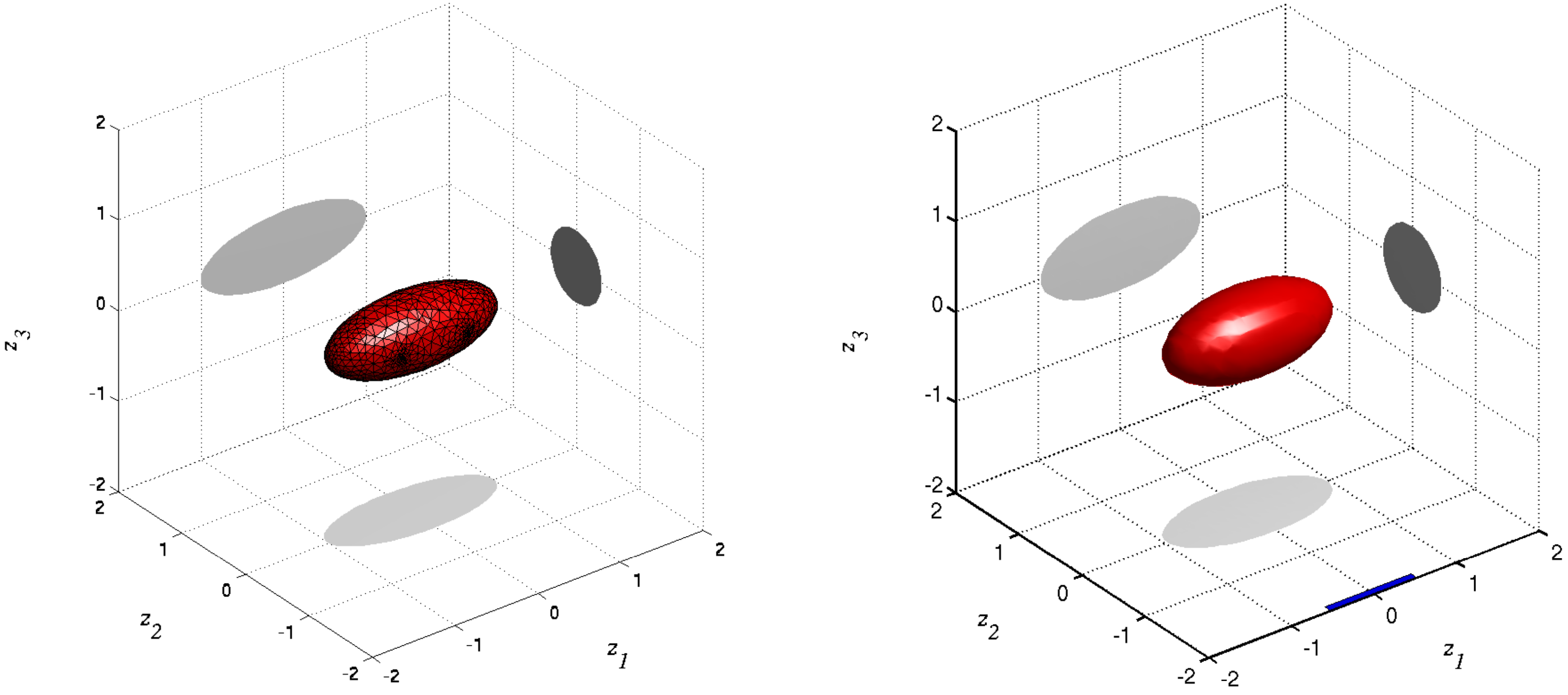


The perfectly conducting airplane



The imperfectly conducting airplane

Examples of Reconstruction



Exact Geometry

Reconstruction

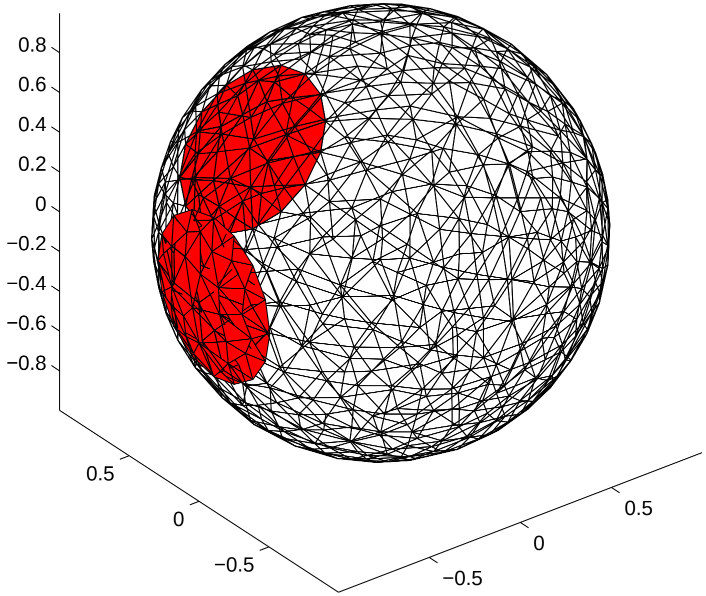
Reconstruction of a fully coated ellipsoid with $\lambda = 1$ and $k = 6$.

Examples of Reconstruction

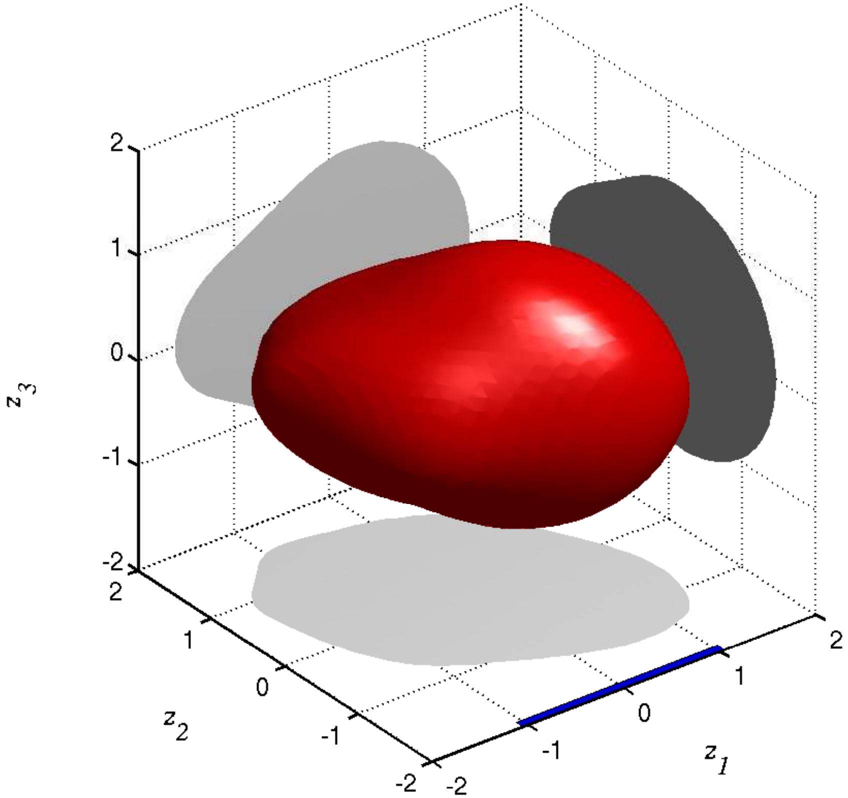
Conducting boundary condition: reconstruction of λ			
Exact	Exact ∂D	LSM	LSM/bound
0.1	0.102	0.099	0.095
1	0.99	1.04	0.80
1.22	1.21	1.25	0.88
2	1.97	1.46	0.89

Reconstruction of λ for the fully coated ellipsoid. Here $k = 6$.

Limited Aperture

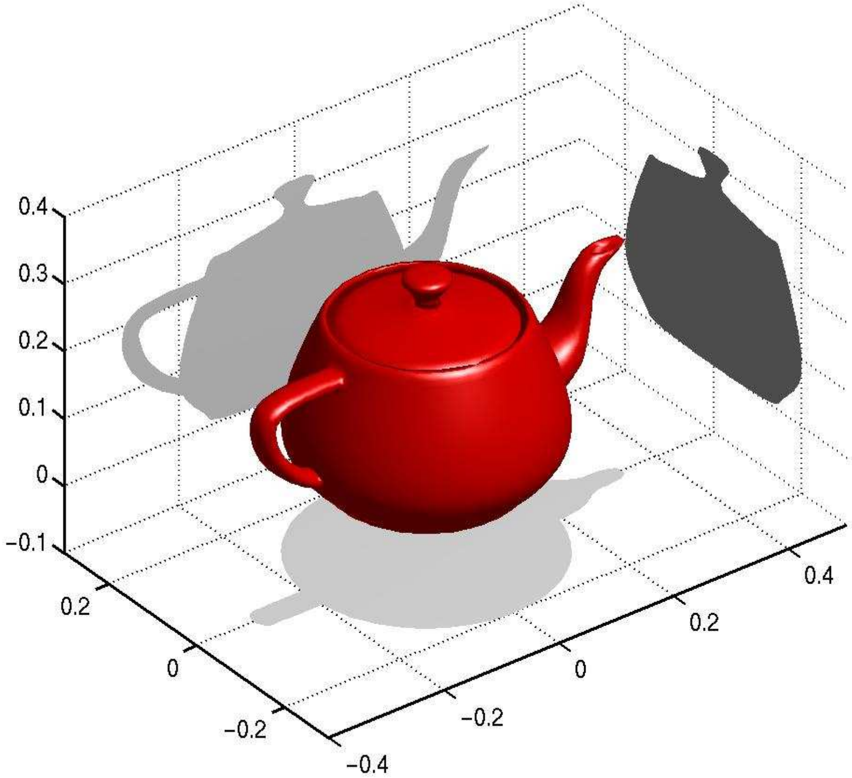


The aperture



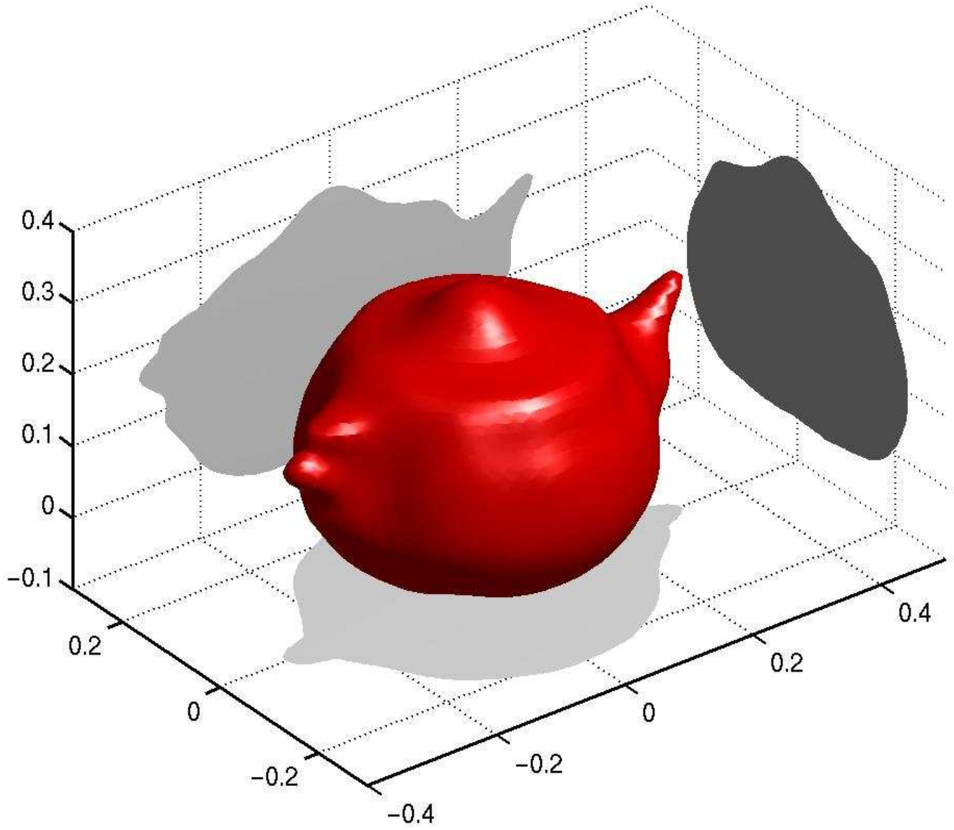
Reconstruction of a coated sphere with $\lambda = 0.1$ with limited aperture data; $k = 3$

Examples of Reconstruction



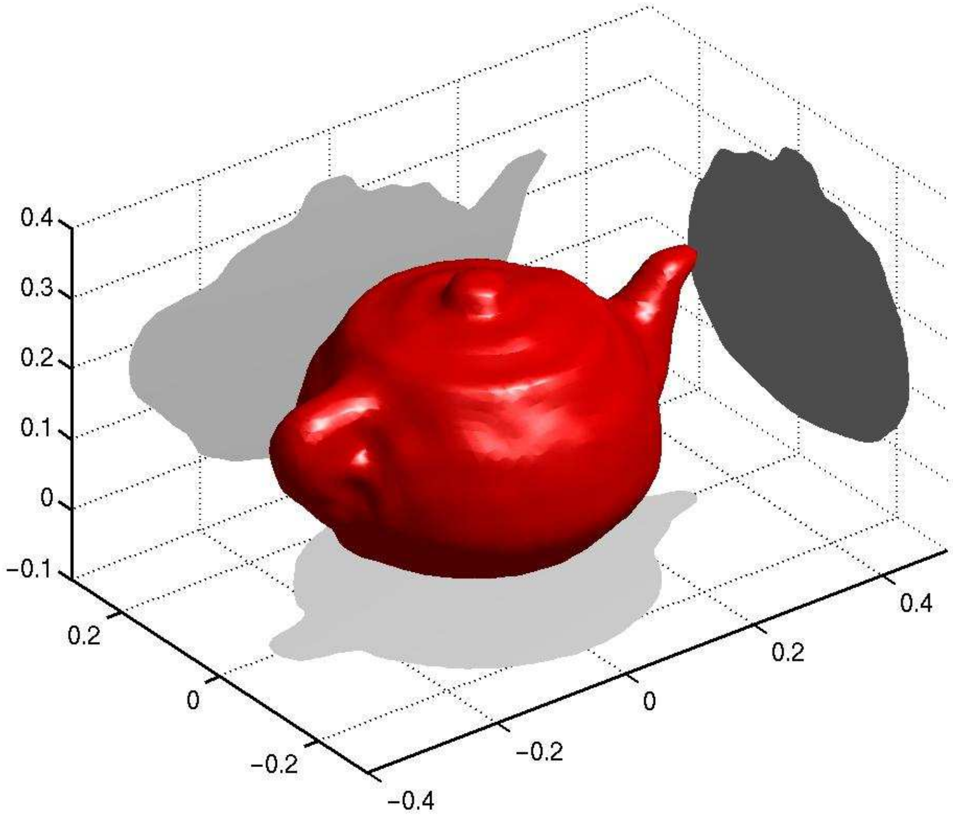
Perfectly conducting teapot, exact geometry

Examples of Reconstruction



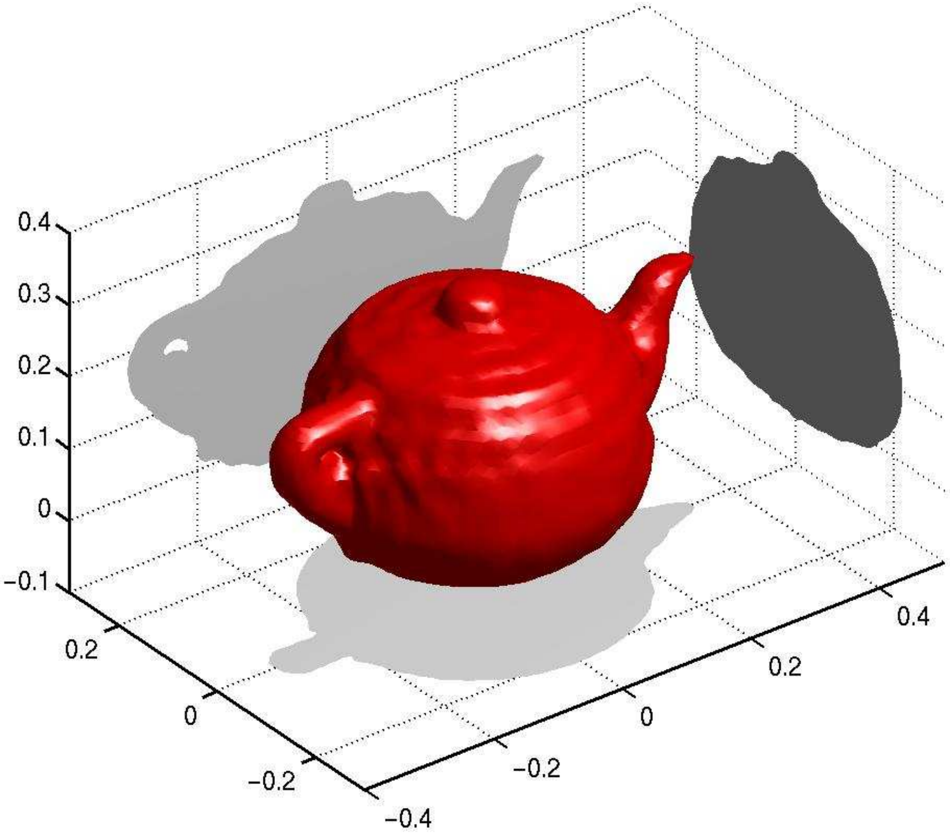
Reconstruction for low frequency

Examples of Reconstruction



Reconstruction for intermediate frequency

Examples of Reconstruction



Reconstruction for high frequency