Mixed Boundary Value Problems in Inverse Electromagnetic Scattering

Fioralba Cakoni

cakoni@math.udel.edu

Department of Mathematical Sciences

University of Delaware

Collaborators

David Colton and Peter Monk

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Introduction

The inverse scattering problem we are considering is to determine the shape and physical properties of an obstacle from a knowledge of the scattered field due to the scattering of an incident time-harmonic electromagnetic wave at fixed frequency.

- A solution is needed in real time.
- The scattering obstacle may be either penetrable, a perfect conductor or partially coated but such information is not known a priori.
- Often only partial information on the scatterer is needed.

Mixed Boundary Value Problems

Mixed boundary value problems in electromagnetic scattering theory arise when the scattering object is a composite material such that parts of the scatterer have different electrical properties.

Such scattering objects can be:

- Partially coated perfect conductors.
- Thin objects with one side a perfect conductor and the other side an imperfect conductor or dielectric.
- Partially coated dielectrics.

Mixed Boundary Value Problems

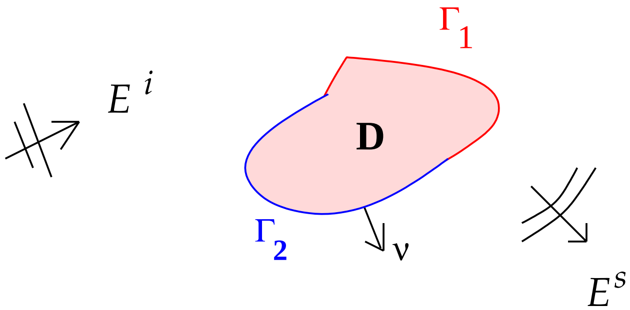
The direct scattering problem

- The mathematical analysis of mixed boundary value problems is difficult due to the non-standard solution space.
- No matter how smooth the boundary data is, the change of boundary conditions causes the scattered field to be singular at the interface. This gives rise to numerical difficulties.

The inverse scattering problem

- Since the physical structure of the composite medium is not known a priori, the use of weak scattering approximations and/or nonlinear optimization techniques are problematic.

Scattering by a Coated Dielectric



The scattered field E^s, H^s and the interior field E^{int}, H^{int} satisfy the equations

$$\begin{cases} \nabla \times E^s - ikH^s = 0 \\ \nabla \times H^s + ikE^s = 0 \end{cases} \quad \textit{in} \ \ D_e := \mathbb{R}^3 \setminus \overline{D}$$

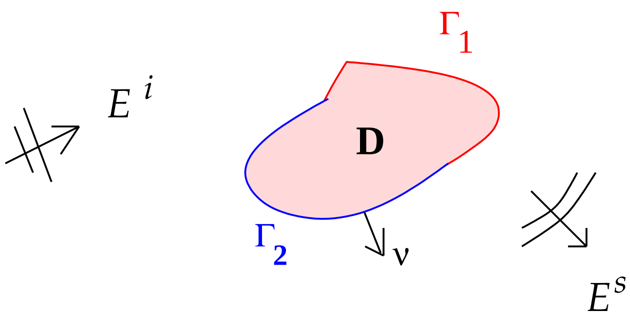
$$\begin{cases} \nabla \times E^{int} - ikH^{int} = 0 \\ \nabla \times H^{int} + ikN(x)E^{int} = 0 \end{cases} \quad \textit{in} \ \ D$$

$$\lim_{|x|\to\infty} (H^s \times x - |x|E^s) = 0,$$

N is a symmetric matrix, $\quad \overline{\xi} \cdot \Re(N)\xi \geq \gamma|\xi|^2, \gamma > 0$

and $\quad \overline{\xi} \cdot \Im(N)\xi \geq 0.$

Scattering by a Coated Dielectric



Let $\eta \in L^\infty(\Gamma_2)$, $\eta > 0$, be the surface conductivity and Γ a Lipshitz boundary.

Then

$$
\begin{aligned}
\nu \times E^s - \nu \times E^{int} &= -\nu \times E^i \quad \text{on} \quad \Gamma = \Gamma_1 \cup \Gamma_2 \\
\nu \times H^s - \nu \times H^{int} &= \nu \times H^i \quad \text{on} \quad \Gamma_1 \\
\nu \times H^s - \nu \times H^{int} &= -\nu \times H^i + \eta(x) \left[\nu \times \left(E^s + E^i\right)\right] \times \nu \text{ on } \Gamma_2
\end{aligned}
$$

where $E^i = \frac{i}{k} \nabla \times \nabla \times p\, e^{ikx \cdot d}$, $H^i = \nabla \times p\, e^{ikx \cdot d}$, $p \in \mathbb{R}^3$.

Inverse Problem

The scattered electric field E^s has the asymptotic behavior

$$E^s(x) = \frac{e^{ikr}}{r}\left\{E_\infty(\hat{x}, d, p) + O\left(\frac{1}{r}\right)\right\}, \quad r = |x|,\ \hat{x} = x/r.$$

The inverse scattering problem reads

Determine D and η from a knowledge of $E_\infty(\hat{x}, d, p)$ for $\hat{x} \in \Omega_0 \subset \Omega$, $d \in \Omega_1 \subset \Omega$ and three linearly independent polarizations p, where $\Omega := \{x,\ |x| = 1\}$.

Uniqueness Theorems

Theorem *(Cakoni-Colton)*: D *is uniquely determined by the electric far field pattern* $E_\infty(\hat{x}, d, p)$ *for* $\hat{x} \in \Omega_0$ *and* $d \in \Omega_1$ *and three linearly independent polarizations* $p_1,\ p_2,\ p_3$.

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Theorem *(Cakoni-Colton-Monk)*: *Let $E^j_\infty(\hat{x}, d, p)$ be the electric far field pattern corresponding to a fixed matrix $N(x)$ but different coatings $\eta = \eta_j$, $j = 1, 2$. Assume that k is not a Maxwell eigenvalue for $D = \{x : I - N(x) \neq 0\}$. Then if $E^1_\infty(\hat{x}, d, p) = E^2_\infty(\hat{x}, d, p)$ for $\hat{x} \in \Omega_0$, $d \in \Omega_1$ and three linearly independent polarizations $p \in \mathbb{R}^3$, we have that $\eta_1(x) = \eta_2(x)$ for $x \in \Gamma_2$*

Electric Dipoles

The radiating solution to Maxwell's equations

$$E_e(x,z,q) := \frac{i}{k} \nabla_x \times \nabla_x \times q\, \Phi(x,z)$$

$$H_e(x,z,q) := \nabla_x \times q\, \Phi(x,z)$$

with

$$\Phi(x,z) := \frac{1}{4\pi} \frac{e^{ik|x-z|}}{|x-z|}, \; q \in \mathbb{R}^3$$

is called the electric dipole located at z and polarized in the direction $q \in \mathbb{R}^3$.

$E_{e,\infty}(\hat{x}, z, q)$ denotes the far field pattern of the corresponding electric field.

Far Field Operator

We define the far field operator $F : L^2_t(\Omega) \to L^2_t(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))\, ds(d).$$

Given $g \in L^2_t(\Omega)$, Fg is the far field pattern of the scattered field corresponding to the incident field being a Herglotz wave function with kernel g given by

$$E_g(x) := ik \int_\Omega e^{ikx\cdot d} g(d) ds(d).$$

The Far Field Equation

We consider the far field equation

$$(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q).$$

and look for solution $g \in L^2(\Omega)$ to the far field equation.

Solving the Far Field Equation

Theorem *(Cakoni-Monk)*: *Assume that k is not a transmission eigenvalue. Then, for every $\epsilon > 0$ there exists g_z^ϵ such that*

$$\|Fg_z^\epsilon - E_{e,\infty}(\cdot, z, q)\|_{L^2(\Omega)} \leq \epsilon \quad \textit{and}$$

- *For* $z \in D$, $\quad \lim_{\epsilon \to 0} \|E_{g_z^\epsilon}\|_{X(D,\Gamma_2)} < \infty$
- *For each* $\epsilon > 0$, $\quad \lim_{z \to \partial D} \|E_{g_z^\epsilon}\|_{X(D,\Gamma_2)} = \infty$
- *For* $z \in \mathbb{R}^3 \setminus \overline{D}$, $\quad \lim_{\epsilon \to 0} \|E_{g_z^\epsilon}\|_{X(D,\Gamma_2)} = \infty$.

Here $X(D, \Gamma_2) := \left\{ u \in H(curl, D),\ \nu \times u|_{\Gamma_2} \in L_t^2(\Gamma_2) \right\}$.

Linear Sampling Method

The linear sampling method determines g from the far field equation $Fg = E_{e,\infty}$.

The support D can be determined by the behavior of g. In particular, $\|E_g\|_{X(D,\Gamma_2)} \to \infty$ implies $\|g\|_{L^2(\Omega)} \to \infty$.

Open Problem: In practice g is obtained by using a regularization method such as Tikhonov regularization. Does this regularized solution behave in the same way as the approximate solution g whose existence is given by the previous theorem?

This question has been answered positively in certain cases by *Arens (2004)* using the ideas of the factorization method developed by *Kirsch (1998)*.

Limited Aperture Data

In practice we have

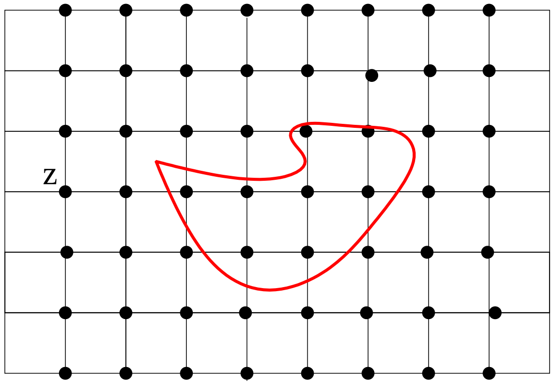
$$\int_{\Omega_1} E_\infty(\hat{x}, d, g(d))\, ds(d) = E_{e,\infty}(\hat{x}, z, q) \quad \hat{x} \in \Omega_0.$$

Based on

Theorem *(Cakoni-Colton)*: *With respect to the $X(D, \Gamma_2)$ norm the set of Herglotz wave functions with kernel supported in a subset Ω_1 of Ω is dense in $\mathbb{H} = \left\{u \in X(D, \Gamma_2) : \nabla \times \nabla \times u - k^2 u = 0 \text{ in } D\right\}$.*

the above discussion is applicable to the far field equation with limited aperture data.

Numerical Implementation



- Construct a grid $\mathcal{G}$.
- For $z_i \in \mathcal{G}$, solve the regularized far field equation

$$(\alpha I + F^* F) g_{z_i,q} = E_{e,\infty}(\hat{x}, z_i, q)$$

- Evaluate

$$G(z_i) = \frac{1}{3}\left(\|g_{z_i,q_1}\|_{\ell^2}^{-1} + \|g_{z_i,q_2}\|_{\ell^2}^{-1} + \|g_{z_i,q_3}\|_{\ell^2}^{-1}\right)$$

  for $z_i \in \mathcal{G}$ and three linearly independent vectors $q_1, q_2, q_3 \in \mathbb{R}^3$.
- Fix $C > 0$ and visualize the boundary by plotting

$$\mathcal{G}(z) = C \max_{z_i \in \mathcal{G}} G(z_i).$$

Interior Transmission Problem

Let η be a constant and let E_z, E^{int} be the unique solution of the interior transmission problem (Haddar (2004) for $\eta = 0$, Cakoni-Haddar (in preparation) for $\eta \neq 0$)

$$\left\{ \begin{array}{c} \nabla \times (\nabla \times E_z^{int}) - k^2 N(x) E_z^{int} = 0 \\ \\ \nabla \times (\nabla \times E_z) - k^2 E_z = 0 \end{array} \right. \qquad \textit{in} \quad D$$

$$\nu \times (E_z^{int} - E_z) = \nu \times E_e(\cdot, z, q) \qquad \textit{on} \quad \Gamma = \Gamma_1 \cup \Gamma_2$$

$$\nu \times [\nabla \times (E_z^{int} - E_z)] = \nu \times (\nabla \times E_e(\cdot, z, q)) \qquad \textit{on} \quad \Gamma_1$$

$$\nu \times [\nabla \times (E_z^{int} - E_z)] = \left\{ \begin{array}{c} \nu \times (\nabla \times E_e(\cdot, z, q)) \\ -ik\eta(x) \nu \times (E_z + E_e(\cdot, z, q)) \times \nu \end{array} \right. \qquad \textit{on} \ \Gamma_2$$

Determination of η

Theorem: *For $z \in D$ and $q \in \mathbb{R}^3$ we have that*

$$\eta = \frac{-\frac{k^2}{6\pi}\|q\|^2 + \Re\left(q \cdot E_z(z)\right)}{\|\nu \times \left(E_z(\,\cdot\,) + E_e(\cdot\,, z, q)\right)\|^2_{L^2_t(\Gamma_2)}}$$

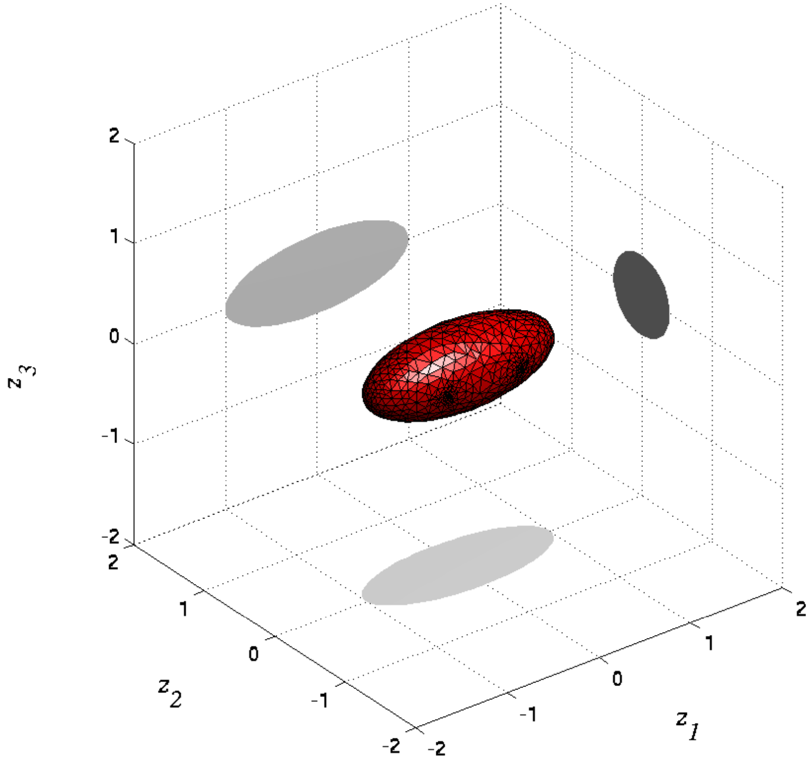
where E_z^{int}, E_z is the solution of the interior transmission problem (if it exists!).

Corollary: For $z \in D$, $q \in \mathbb{R}^3$, we have that

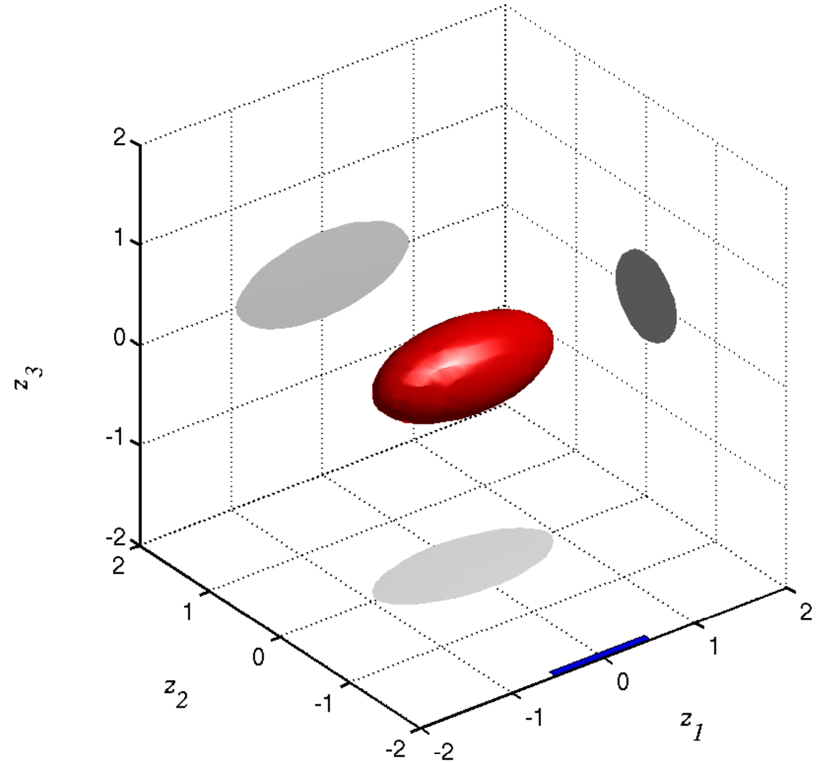
$$\eta \geq \frac{-\frac{k^2}{6\pi}\|q\|^2 + \Re\left(q \cdot E_z(z)\right)}{\|E_z(\,\cdot\,) + E_e(\cdot\,, z, q)\|^2_{L^2(\Gamma)}}.$$

Note that E_z can be approximated by a Herglotz wave function with kernel g_z^ϵ and this g_z^ϵ is an approximate solution to the far field equation!

Examples of Reconstruction



Exact Geometry



Reconstruction

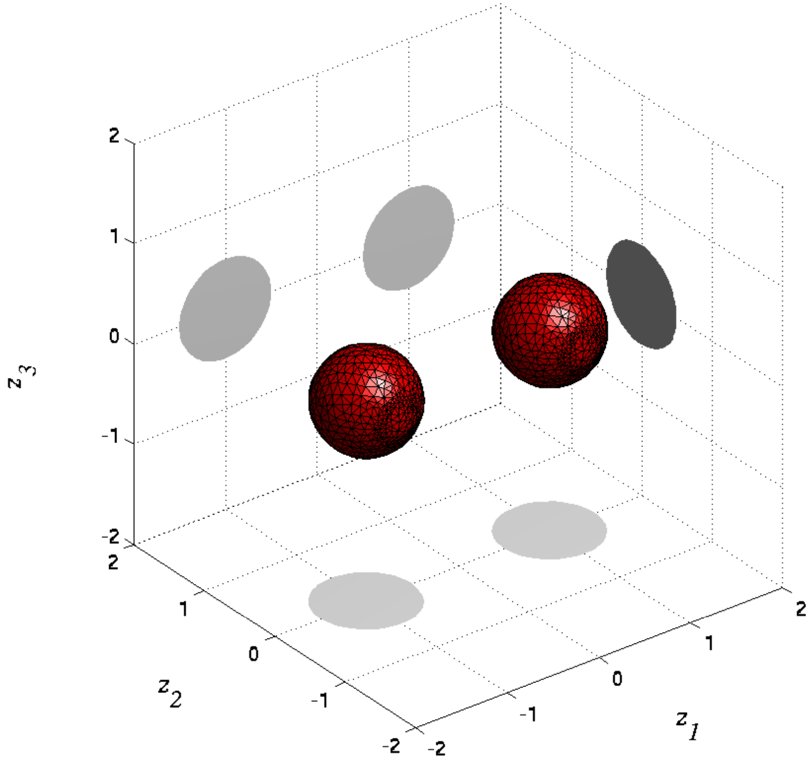
Reconstruction of a fully coated ellipsoid with $\eta = 1$ and $k = 6$.

Examples of Reconstruction

Conducting boundary condition: reconstruction of η			
Exact	Exact Γ	LSM	LSM/bound
0.0	-0.005	-0.01	-0.004
0.1	0.09	0.16	0.07
1	0.96	0.79	0.58
2	1.15	0.94	0.66

Reconstruction of η for the fully coated ellipsoid. Here $k = 6$.

Examples of Reconstruction



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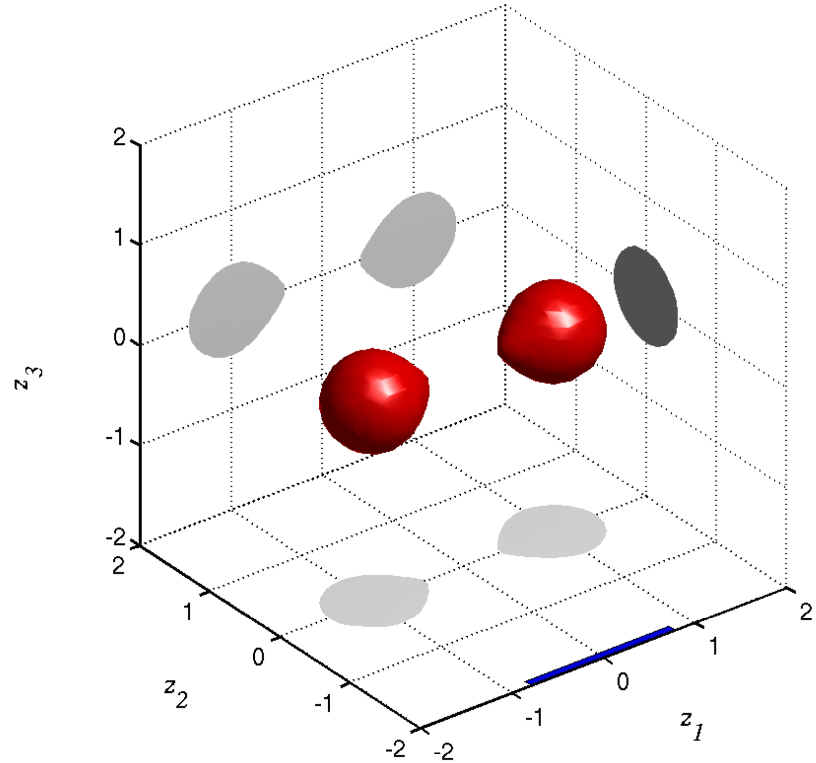


Reconstruction

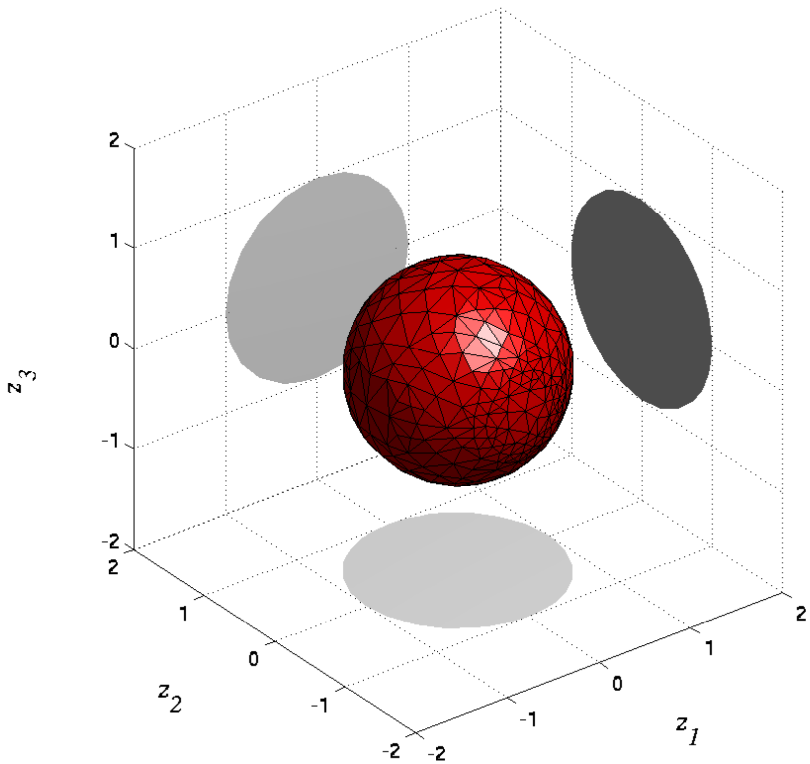
Reconstruction of fully coated two spheres with $\eta = 1$, $k = 4$.

Examples of Reconstruction

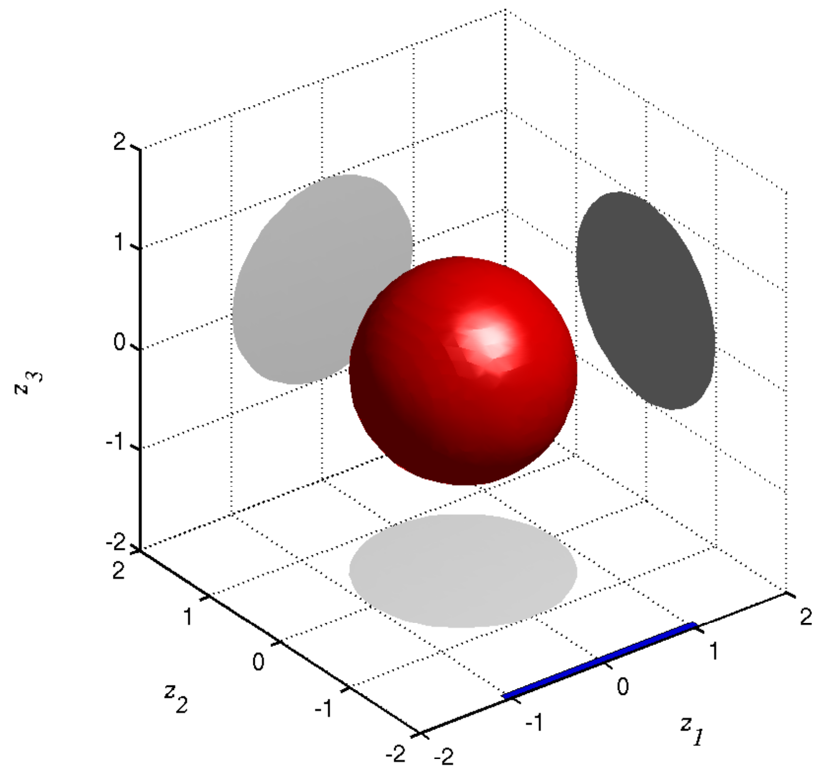
Conducting boundary condition: reconstruction of η			
Exact	Exact Γ	LSM	LSM/bound
0.1	0.11	0.13	0.011

Reconstruction of η for the fully coated two spheres, $k = 6$.

Examples of Reconstruction



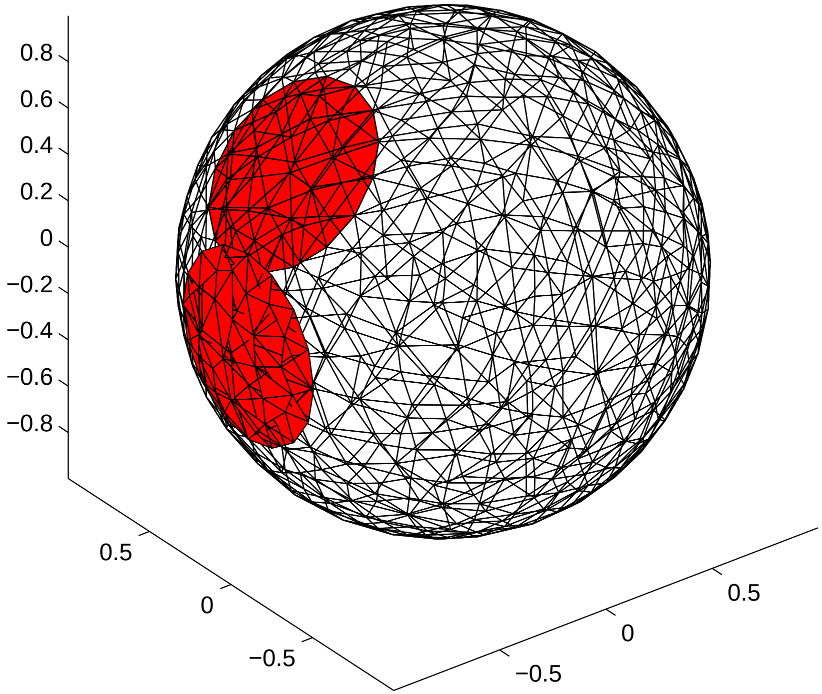
Exact Geometry



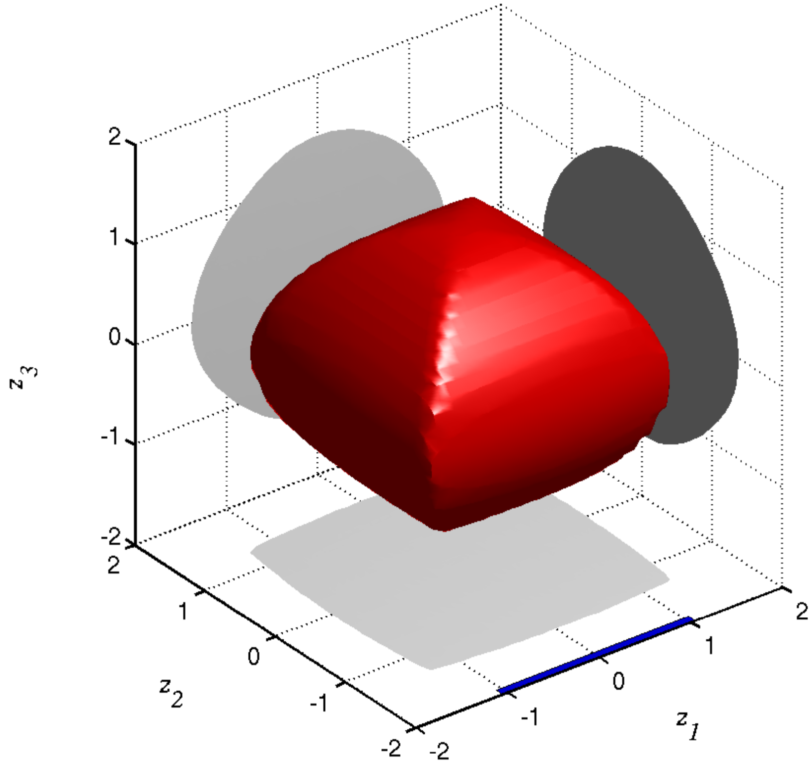
Reconstruction

Reconstruction of a partially coated sphere. The coated portion Γ_2 is the hemisphere $x_2 > 0$. Here $\eta = 1$ and $k = 3$.

Limited Aperture



The aperture



Reconstruction of the same sphere with limited aperture data

Examples of Reconstruction

Conducting boundary condition: reconstruction of η			
Exact	Exact Γ_2	LSM (Γ)	LSM/bound
0.1	0.045	0.037	0.027
1	0.94	0.52	0.43
2	2.00	0.81	0.65

Reconstruction of η for the partially coated sphere, $k = 3$.