

# Mixed Boundary Value Problems in Inverse Electromagnetic Scattering

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# Introduction

The inverse scattering problem we are considering is to **determine the shape and physical properties of an obstacle** from a knowledge of the scattered field due to the scattering of an incident time-harmonic electromagnetic wave at fixed frequency.

- A solution is needed in real time.
- The scattering obstacle may be either penetrable, a perfect conductor or partially coated but such information is not known a priori.
- Often only partial information on the scatterer is needed.

# Mixed Boundary Value Problems

**Mixed boundary value problems** in electromagnetic scattering theory arise when the scattering object is a composite material such that parts of the scatterer have different electrical properties.

Such scattering objects can be:

- Partially coated perfect conductors.
- Thin objects with one side a perfect conductor and the other side an imperfect conductor or dielectric.
- Partially coated dielectrics.

# Mixed Boundary Value Problems

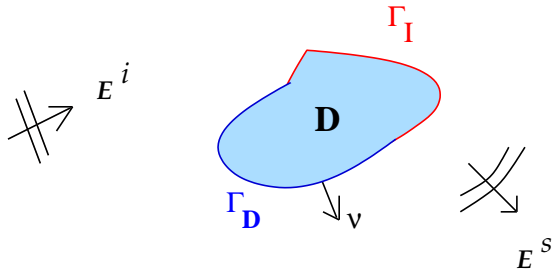
## The **direct scattering problem**

- The **mathematical analysis** of mixed boundary value problems is difficult due to the non-standard solution space.
- No matter how smooth the boundary data is, the change of boundary conditions causes the scattered field to be singular at the interface. This gives rise to **numerical** difficulties.

## The **inverse scattering problem**

- Since the **physical structure of the composite medium is not known** a priori, the use of weak scattering approximations and/or nonlinear optimization techniques are problematic.

# Partially Coated Perfect Conductors



Let  $\lambda \in L_\infty(\Gamma_I)$ . Under appropriate conditions the electric scattered field  $E^s$  satisfies

$$\nabla \times \nabla \times E^s - k^2 E^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}$$

$$\nu \times E^s = f \quad \text{on } \Gamma_D$$

$$\nu \times (\nabla \times E^s) - i\lambda(\nu \times E^s) \times \nu = h \quad \text{on } \Gamma_I$$

$$\lim_{|x| \rightarrow \infty} (\nabla \times E^s \times x - ik|x|E^s) = 0$$

where  $f := -(\nu \times E^i)|_{\Gamma_D}$      $h := -[\nu \times (\nabla \times E^i) - i\lambda(\nu \times E^i) \times \nu]$

$$E^i(x) := ik(d \times p) \times d e^{ikx \cdot d}, \quad \partial D = \overline{\Gamma_D} \cup \overline{\Gamma_I}$$

# Partially Coated Perfect Conductors

Let

$$X_{loc}(\mathbb{R}^3 \setminus \overline{D}, \Gamma_I) := \{U \in H_{loc}(\text{curl}, \mathbb{R}^3 \setminus \overline{D}), \nu \times U|_{\Gamma_I} \in L_t^2(\Gamma_I)\}.$$

**Theorem (Cakoni-Colton-Monk):** Assume  $f = \nu \times U|_{\Gamma_D}$  for  $U \in X_{loc}(\mathbb{R}^3 \setminus \overline{D}, \Gamma_I)$  and  $h \in L_t^2(\Gamma_I)$ . Then there exists a unique solution  $E^s \in X_{loc}(\mathbb{R}^3 \setminus \overline{D}, \Gamma_I)$  of the above scattering problem.

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The scattered field  $E^s$  has the asymptotic behaviour

$$E^s(x) = \frac{e^{ik|x|}}{|x|} \left\{ E_\infty(\hat{x}, d, p) + O\left(\frac{1}{|x|}\right) \right\} \text{ as } |x| \rightarrow \infty \text{ uniformly in all directions } \hat{x} = x/|x|.$$

# Electric Dipoles

The radiating solution to Maxwell's equations

$$E_e(x, z, q) := \frac{i}{k} \nabla_x \times \nabla_x \times q \Phi(x, z)$$

with

$$\Phi(x, z) := \frac{1}{4\pi} \frac{e^{ik|x-z|}}{|x-z|}, \quad q \in \mathbb{R}^3$$

is called the **electric dipole** located at  $z$  and polarized in the direction  $q \in \mathbb{R}^3$ .

$E_{e,\infty}(\hat{x}, z, q)$  denotes the **far field pattern** of the corresponding electric field.

# Partially Coated Perfect Conductors

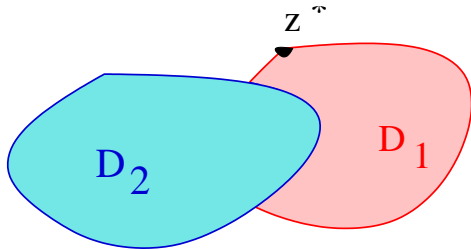
The **inverse scattering problem** is

Determine both  $D$  and  $\lambda$  from a knowledge of  $E_\infty(\hat{x}, d, p)$  for  $\hat{x} \in \Omega_0 \subset \Omega$ ,  $d \in \Omega_1 \subset \Omega$  and three linearly independent polarizations  $p$ .

Uniqueness of the inverse problem:

$E_\infty(\hat{x}, d, p)$  for  $\hat{x} \in \Omega_0 \subset \Omega$ ,  $d \in \Omega_1 \subset \Omega$  and three linearly independent polarizations  $p$  **uniquely** determines  $D$  and  $\lambda \in L^\infty(\Gamma_I)$ .

# Uniqueness of Solution to Inverse Problem



A sketch of the uniqueness proof for  $D$ . It is based on two steps: (idea due to Kirsch and Kress 1996)

- The coincidence of the *far field pattern* for all  $\hat{x}$ ,  $d \in \Omega$  and  $p \in \mathbb{R}^3$  implies that the scattered fields  $E_1^s = E_2^s$  due to the scattering of a dipole  $E^i := E_e(x, z, q)$  for  $z \in \mathbb{R}^3 \setminus (\overline{D_1} \cup \overline{D_2})$  coincide in  $\mathbb{R}^3 \setminus (\overline{D_1} \cup \overline{D_2})$
- $E_1^s = E_2^s$  depends continuously on  $E_e(x, z, q)|_{\partial D_1}$  and  $E_e(x, z, q)|_{\partial D_2}$ . A contradiction arises since

$$\lim_{z \rightarrow z^*} E_e(x, z, q)|_{\partial D_2} \text{ exists, while } \lim_{z \rightarrow z^*} E_e(x, z, q)|_{\partial D_1} \rightarrow \infty.$$

# Far Field Operator

We define the **far field operator**  $F : L_t^2(\Omega) \rightarrow L_t^2(\Omega)$  by

$$(Fg)(\hat{x}) := \int_{\Omega} E_{\infty}(\hat{x}, d, g(d)) ds(d).$$

Given  $g \in L_t^2(\Omega)$ ,  $Fg$  is the far field pattern of the scattered field corresponding to the incident field being a Herglotz wave function with kernel  $g$  given by

$$E_g(x) := ik \int_{\Omega} e^{ikx \cdot d} g(d) ds(d).$$

# The Far Field Equation

Let  $z \in D$  and consider the far field equation

$$(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q).$$

Suppose that  $g$  solves the far field equation.

Rellich's Lemma  $\implies E_g^s(x) = E_e(x, z, q)$  in  $\mathbb{R}^3 \setminus \overline{D}$

In particular, the mixed traces of  $E_g$  and  $-E_e(x, z, q)$  coincides on the boundary.

As  $z \in D \rightarrow \partial D$ ,  $\nu \times E_e(\cdot, z, q) \rightarrow \infty$  and so does  $\nu \times E_g$   
 $\implies \|g\|_{L^2} \rightarrow \infty$ .

# The Far Field Equation

Unfortunately, no solution exists to the far field equation, i.e. the solution of the **interior mixed boundary value problem** for  $z \in D$

$$\begin{aligned}\nabla \times \nabla \times E - k^2 E &= 0 && \text{in } D \\ \nu \times E &= f && \text{on } \Gamma_D \\ \nu \times (\nabla \times E) - i\lambda(\nu \times E) \times \nu &= h && \text{on } \Gamma_I\end{aligned}$$

where

$$f := -\nu \times E_e(\cdot, z, q), \quad \text{and}$$

$$h := -\nu \times (\nabla \times E_e(\cdot, z, q)) + i\lambda(\nu \times E_e(\cdot, z, q))^s \times \nu$$

is not a Herglotz wave function!

# The Linear Sampling Method

The **linear sampling method** for solving the inverse scattering problem is based on the fact that although the far field equation is in general not solvable, for every  $\epsilon > 0$  there does exist a "solution" with discrepancy  $\epsilon$  and this approximate solution tends to infinity as  $z \rightarrow \partial D$  for  $z \in D$ .

**Theorem** *The unique solution of the interior mixed boundary value problem can be approximated by an Herglotz wave function with respect to the  $H(\text{curl}, D) \cap L_t^2(\Gamma_I)$  norm.*

# Solving the Far Field Equation

We can write

$$Fg = \mathcal{B}(-\mathcal{T}_m E_g) \quad \text{and} \quad \mathcal{B}(-\mathcal{T}_m E_g) = E_{e,\infty}(\hat{x}, z, q)$$

where  $\mathcal{B}$  maps  $(f, h)$  to the far field of the corresponding radiating solution  $E_\infty$ .

It can be shown that

- $\mathcal{B}$  is compact with dense range.
- $E_{e,\infty}(\hat{x}, z, q) \in \text{Range } \mathcal{B} \iff z \in D$ .

# Solving the Far Field Equation

Using the approximation properties of Herglotz wave functions and the theory of ill-posed problems we can prove the following theorem:

**Theorem:** For every  $\epsilon > 0$  there exists  $g_z^\epsilon$  such that

$$\|F g_z^\epsilon - E_{e,\infty}(\cdot, z, q)\|_{L^2(\Omega)} \leq \epsilon \quad \text{and}$$

- For  $z \in D$ ,  $\lim_{\epsilon \rightarrow 0} \|E_{g_z^\epsilon}\|_{X(D, \Gamma_I)} < \infty$
- For each  $\epsilon > 0$ ,  $\lim_{z \rightarrow \partial D} \|E_{g_z^\epsilon}\|_{X(D, \Gamma_I)} = \infty$
- For  $z \in \mathbb{R}^3 \setminus \overline{D}$ ,  $\lim_{\epsilon \rightarrow 0} \|E_{g_z^\epsilon}\|_{X(D, \Gamma_I)} = \infty$ .

# Linear Sampling Method

The **linear sampling method** determines  $g$  from the far field equation  $Fg = E_{e,\infty}$ .

The support  $D$  can be determined by the behavior of  $g$ . In particular,  $\|E_g\|_{X(D,\Gamma_I)} \rightarrow \infty$  implies  $\|g\|_{L^2(\Omega)} \rightarrow \infty$ .

**Open Problem:** In practice  $g$  is obtained by using a regularization method such as Tikhonov regularization. Does this regularized solution behave in the same way as the approximate solution  $g$  whose existence is given by the previous theorem?

This question has been answered positively in certain cases by *Arens (2004)* using the ideas of the factorization method developed by *Kirsch (1998)*.

# Open Problem

Let  $\mathcal{F} : X \rightarrow X$ ,  $\mathcal{H} : X \rightarrow Y$  and  $\mathcal{B} : Y \rightarrow X$  be compact linear operators such that  $\mathcal{F}$  has dense range and  $\mathcal{F} = \mathcal{B}\mathcal{H}$ .

The equation

$$\mathcal{F}g = \varphi_0$$

has no solution. However, for all  $\delta > 0$  there exists  $g_\delta$  such that

$$\mathcal{F}g_\delta \rightarrow \varphi_0 \quad \text{as } \delta \rightarrow 0$$

and  $\mathcal{H}g_\delta$  converges to the solution  $v$  of  $\mathcal{B}v = \varphi_0$ .

The goal is to determine an approximation to  $g_\delta$ . We do this from

$$\min (\|\mathcal{F}\tilde{g}_\alpha - \varphi_0\|_X + \alpha\|\tilde{g}_\alpha\|_X)$$

for "appropriately" chosen  $\alpha$ . Is it possible to choose  $\alpha = \alpha(\delta)$  such that

$$\|\tilde{g}_\alpha - g_\delta\|_X = O(\delta^p) \quad \text{for some } p > 0?$$

# Limited Aperture Data

In practice we have

$$\int_{\Omega_1} E_{\infty}(\hat{x}, d, g(d)) ds(d) = E_{e,\infty}(\hat{x}, z, q) \quad \hat{x} \in \Omega_0.$$

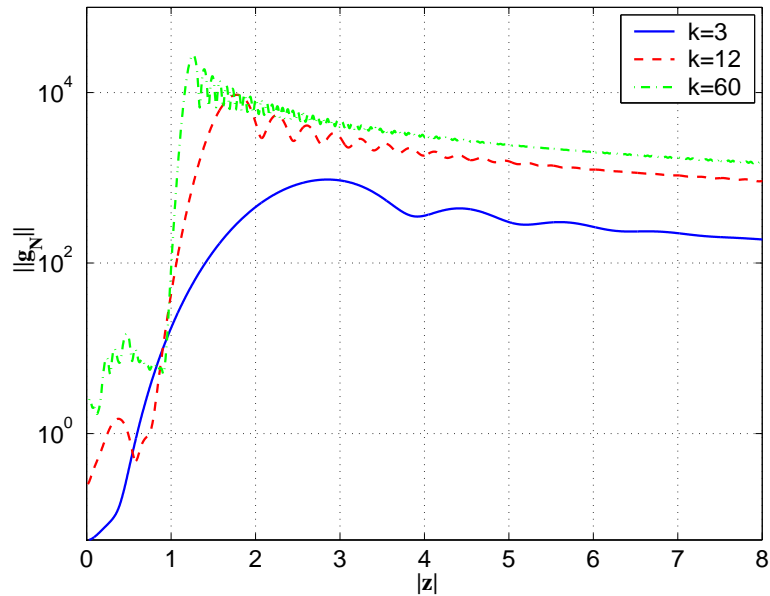
Based on

**Theorem:** *Herglotz wave functions can be approximated arbitrarily close with respect to the  $H(\text{curl}, D) \cap L_t^2(\Gamma_I)$  norm by Herglotz wave functions with kernel supported in a subset  $\Omega_0$  of  $\Omega$ .*

the above discussion is applicable to the far field equation with **limited aperture data**.

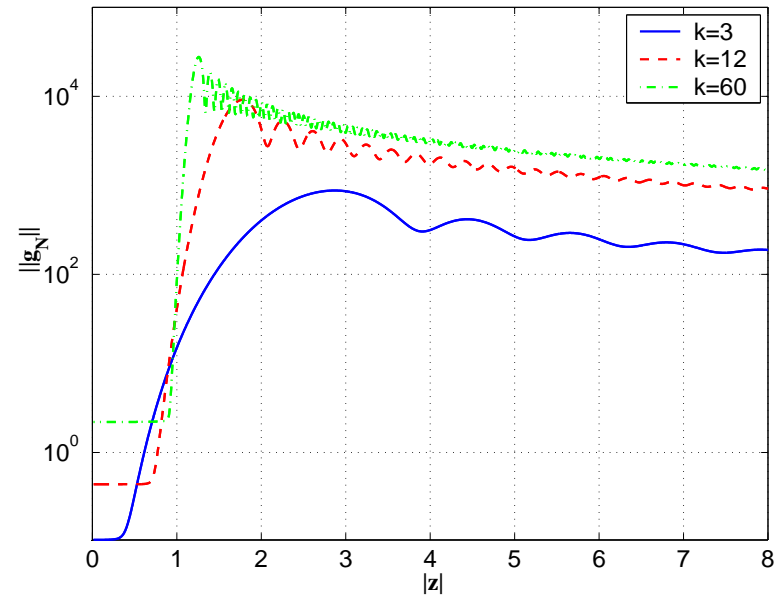
# Behavior of $\|g\|$ for Different Frequencies

*Collino-Fares-Haddar (2004)*



$\|g\|$  with respect to  $z$ :

Dirichlet boundary condition



$\|g\|$  with respect to  $z$ .

Impedance boundary condition

# The determination of $\lambda$

Having determined  $D$ , we fix a point  $z \in D$ .

Let  $E_z \in X(D, \Gamma_I)$  be the unique solution of the interior mixed boundary value problem

$$\nabla \times \nabla \times E_z - k^2 E_z = 0 \quad \text{in } D$$

$$\nu \times [E_z + E_e(\cdot, z, q)] = 0 \quad \text{on } \Gamma_D$$

$$\nu \times \nabla \times [E_z + E_e(\cdot, z, q)] - i\lambda \nu \times [E_z + E_e(\cdot, z, q)] \times \nu \quad \text{on } \Gamma_I$$

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- The set of functions  $f := \nu \times W_z|_{\Gamma_I}$  for  $z$  in a ball  $B_r$  contained in  $D$  is complete in  $L_t^2(\Gamma_I)$ .

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- The set of functions  $f := \nu \times W_z|_{\Gamma_I}$  for  $z$  in a ball  $B_r$  contained in  $D$  is complete in  $L_t^2(\Gamma_I)$ .
- Let  $z_1, z_2 \in D$ . Then

$$2 \int_{\Gamma_I} (\nu \times W_{z_1}) \cdot \lambda (\nu \times \overline{W}_{z_2}) ds = -\|q\|^2 A(z_1, z_2, k, q) \\ + k(q \cdot E_{z_1}(z_2) + q \cdot \overline{E}_{z_2}(z_1))$$

where  $A(z_1, z_2, k, q)$  is a computable number.

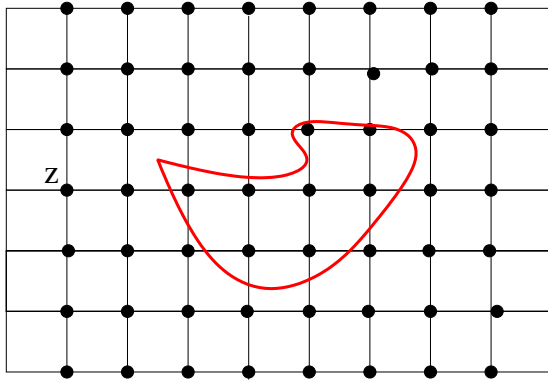
# The Determination of $\lambda$

For  $z \in D$ ,  $E_z$  can be approximated in  $X(D, \Gamma_I)$  by the Herglotz wave function  $E_{g_z}$  with kernel  $g_z$  the (regularized) solution of the **far field equation**.

- The above properties of  $W_z$  provide a method for approximating  $\|\lambda\|_{L_\infty(\Gamma_1)}$ .
- In particular, if  $\lambda$  is a **constant** we have

$$\lambda = \frac{-\frac{k^2}{6\pi} \|q\|^2 + k\Re(q \cdot E_{z_0})}{\|\nu \times W_{z_0}\|_{L_t^2(\partial D)}^2} \quad z_0 \in D.$$

# Numerical Implementation



- Construct a grid  $\mathcal{G}$ .
- For  $z_i \in \mathcal{G}$ , solve the regularized far field equation

$$(\alpha I + F^* F) g_{z_i, q} = E_{e, \infty}(\hat{x}, z_i, q)$$

- Evaluate

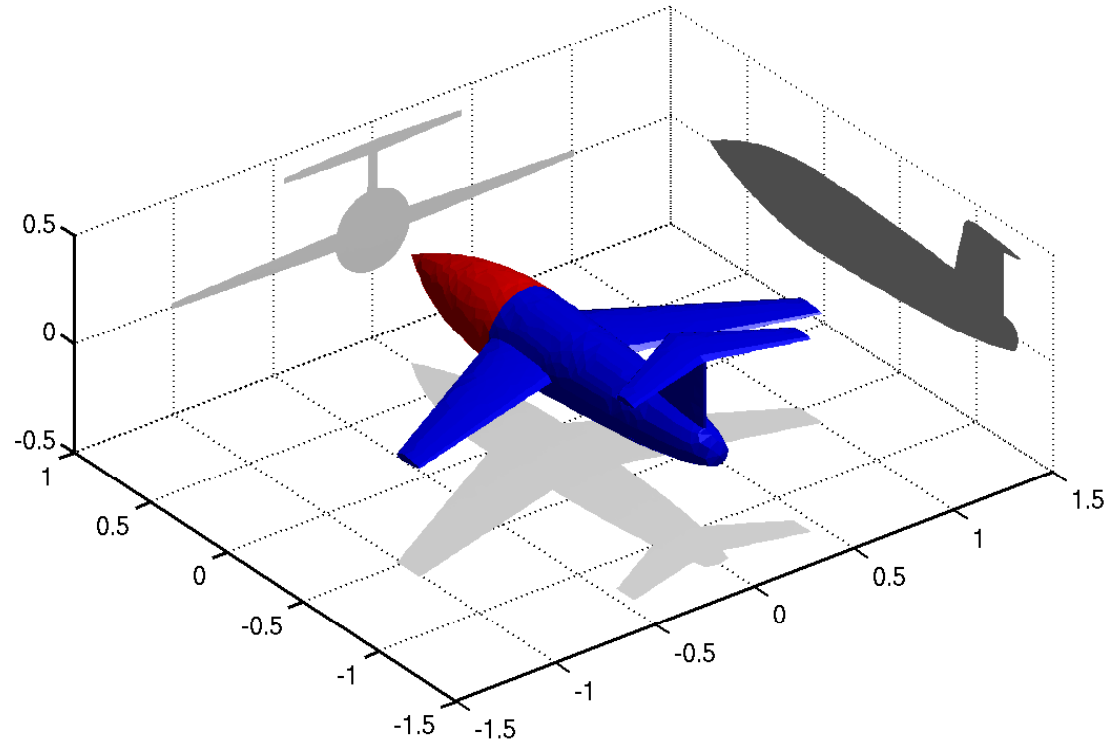
$$G(z_i) = \frac{1}{3} \left( \|g_{z_i, q_1}\|_{\ell^2}^{-1} + \|g_{z_i, q_2}\|_{\ell^2}^{-1} + \|g_{z_i, q_3}\|_{\ell^2}^{-1} \right)$$

for  $z_i \in \mathcal{G}$  and three linearly independent vectors  $q_1, q_2, q_3 \in \mathbb{R}^3$ .

- Fix  $C > 0$  and visualize the boundary by plotting

$$G(z) = C \max_{z_i \in \mathcal{G}} G(z_i).$$

# Examples of Reconstruction

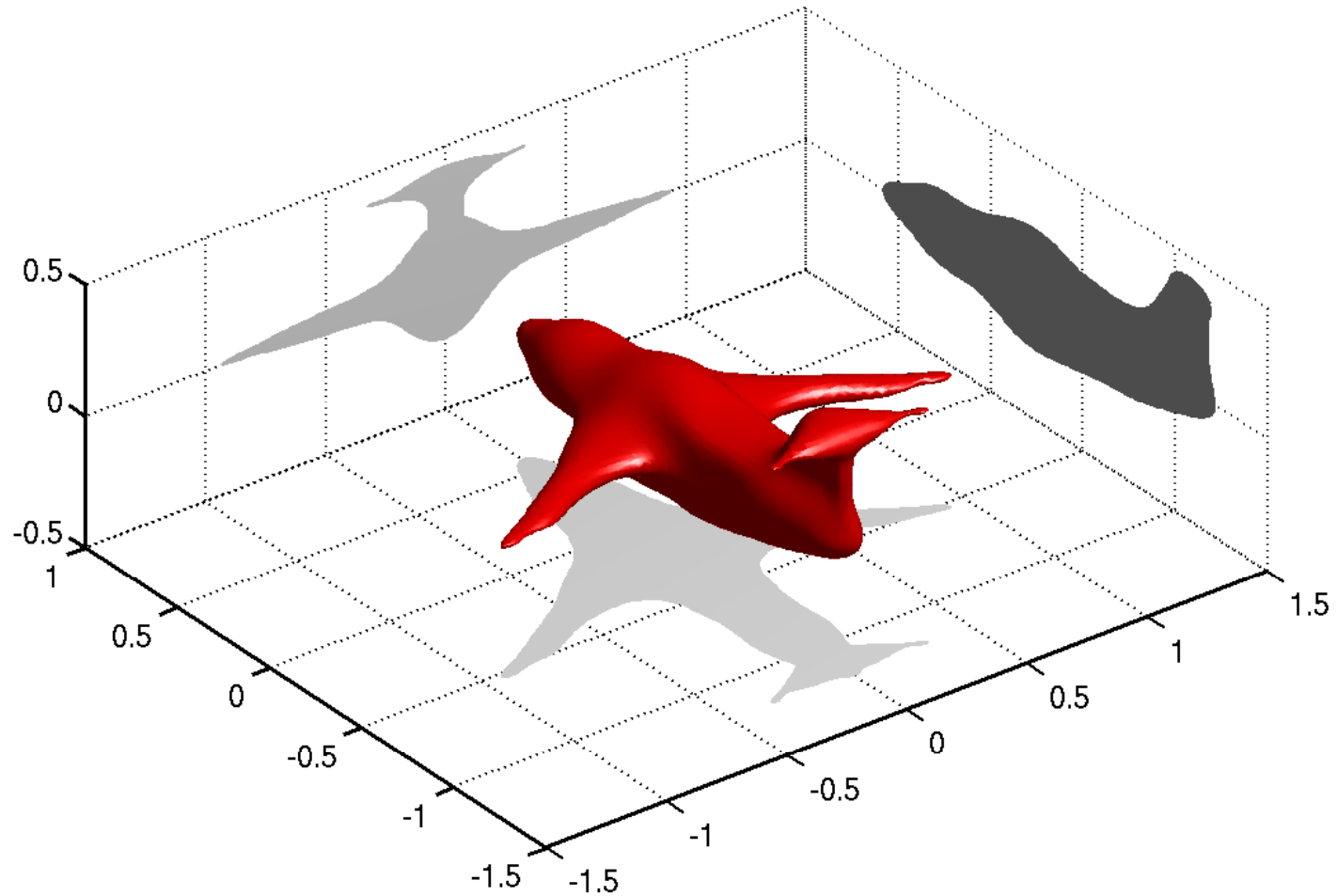


*The exact geometry*

*Impedance boundary condition with  $\lambda = 1$  is the red region*

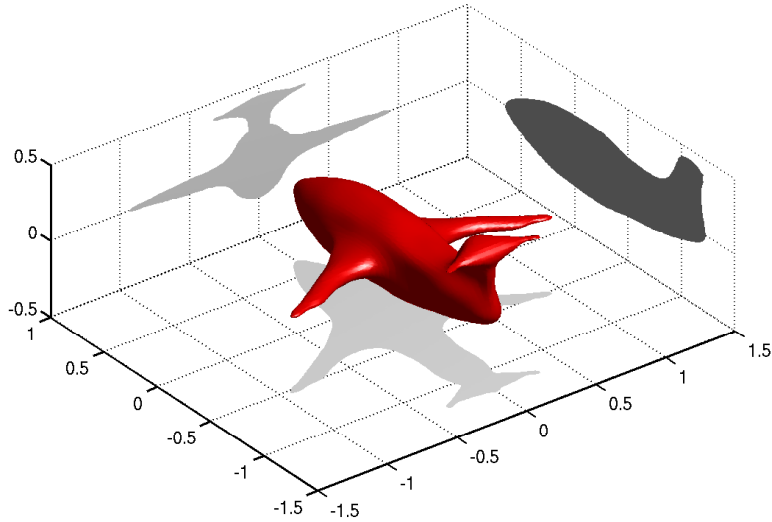
*Perfectly conducting boundary condition is the blue region*

# Examples of Reconstruction

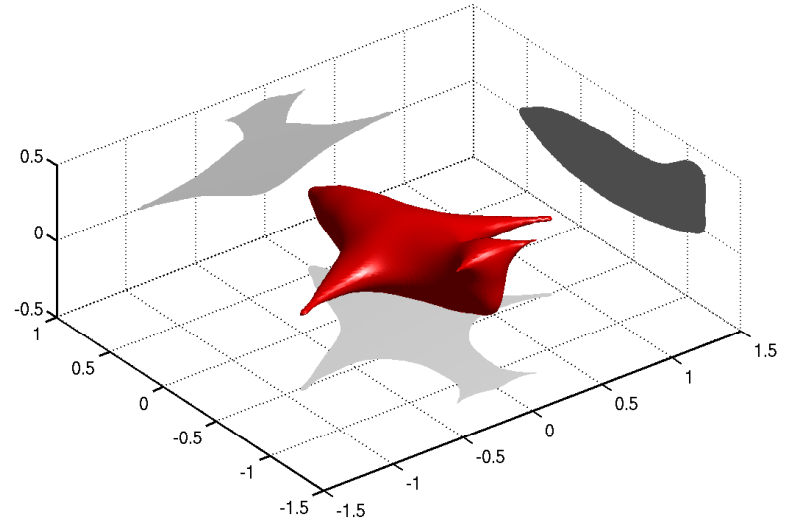


*The reconstructed partially coated airplane (wavelength=0.7)*

# Examples of Reconstruction

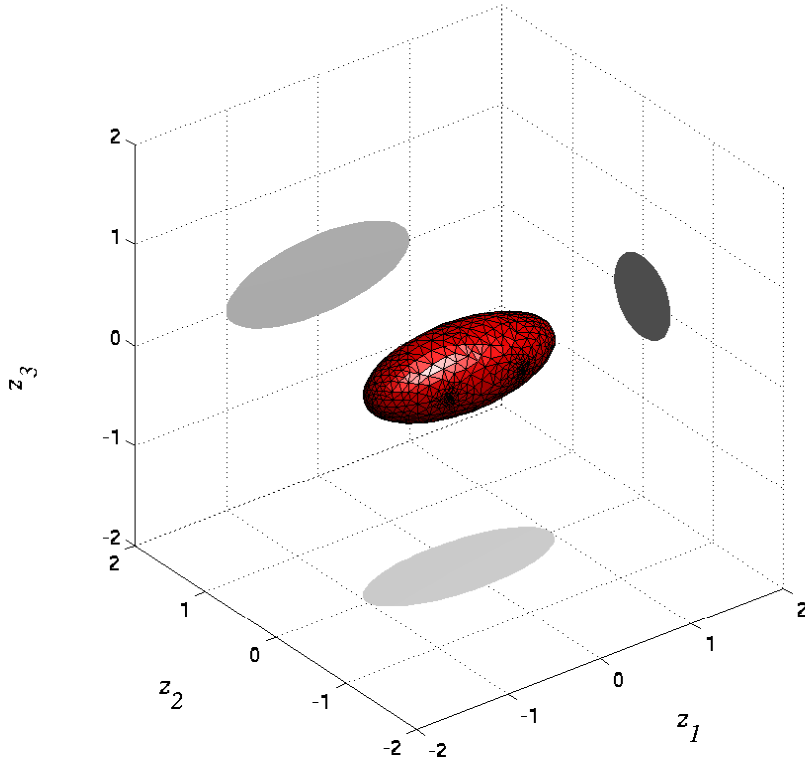


The perfectly conducting airplane

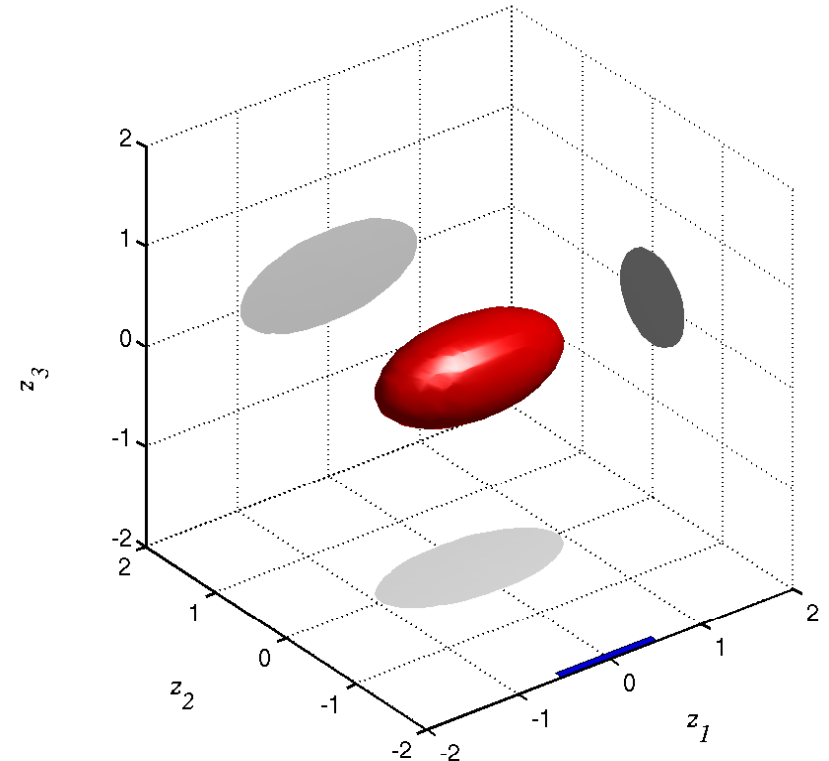


The imperfectly conducting airplane

# Examples of Reconstruction



Exact Geometry



Reconstruction

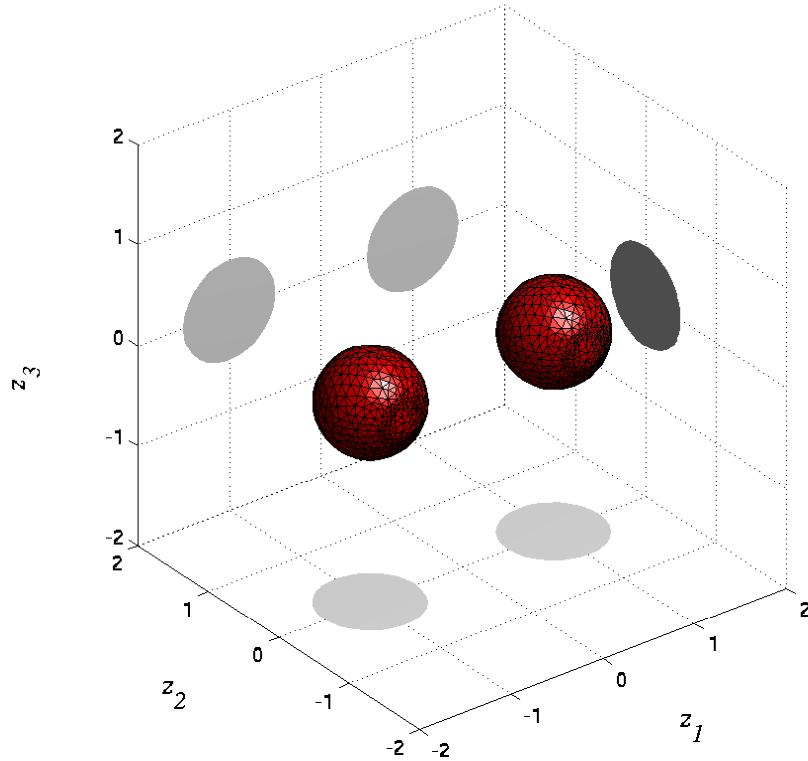
Reconstruction of a fully coated ellipsoid with  $\lambda = 1$  and  $k = 6$ .

# Examples of Reconstruction

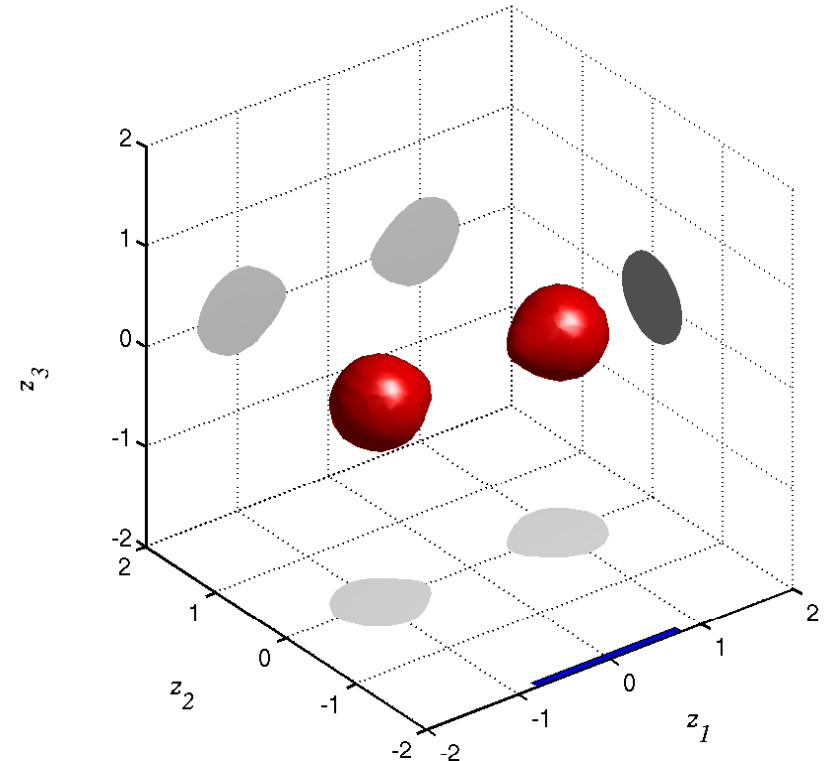
Conducting boundary condition: reconstruction of $\lambda$			
Exact	Exact $\partial D$	LSM	LSM/bound
0.1	0.102	0.099	0.095
1	0.99	1.04	0.80
1.22	1.21	1.25	0.88
2	1.97	1.46	0.89

Reconstruction of  $\lambda$  for the fully coated ellipsoid. Here  $k = 6$ .

# Examples of Reconstruction



Exact Geometry



Reconstruction

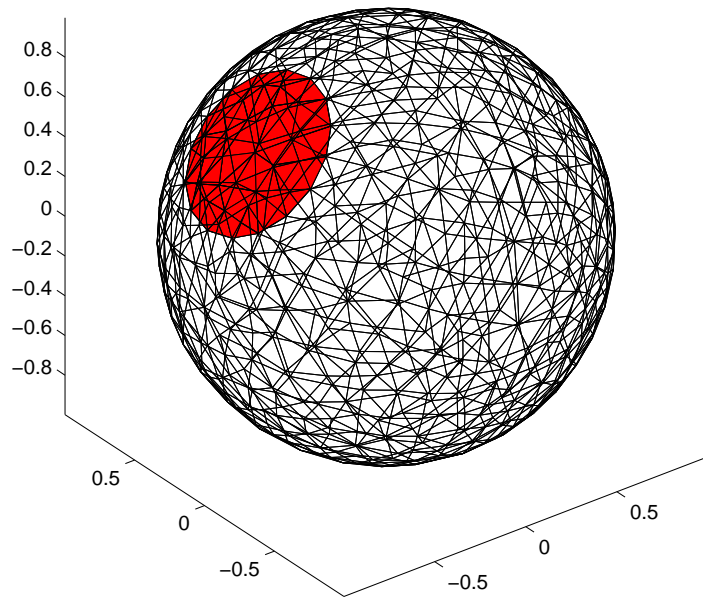
Reconstruction of fully coated two ball with  $\lambda = 1$  and  $k = 4$ .

# Examples of Reconstruction

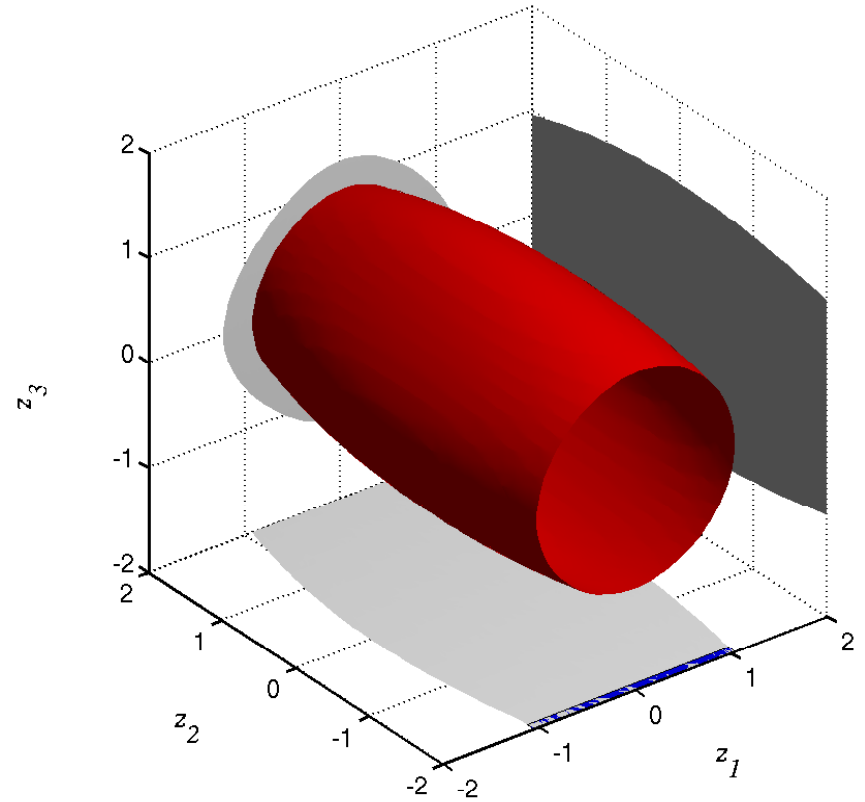
Conducting boundary condition: reconstruction of $\lambda$			
Exact	Exact $\partial D$	LSM	LSM/bound
0.1	0.104	0.11	0.099
1	0.99	0.98	0.65
1.22	1.21	1.18	0.70
2	1.97	1.46	0.71

Reconstruction of  $\lambda$  for two balls. Here  $k = 4$ .

# Limited Aperture

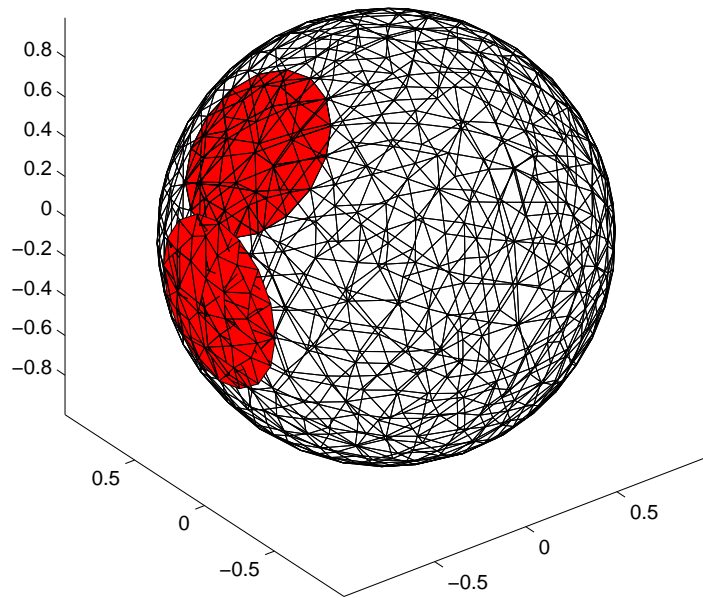


The aperture

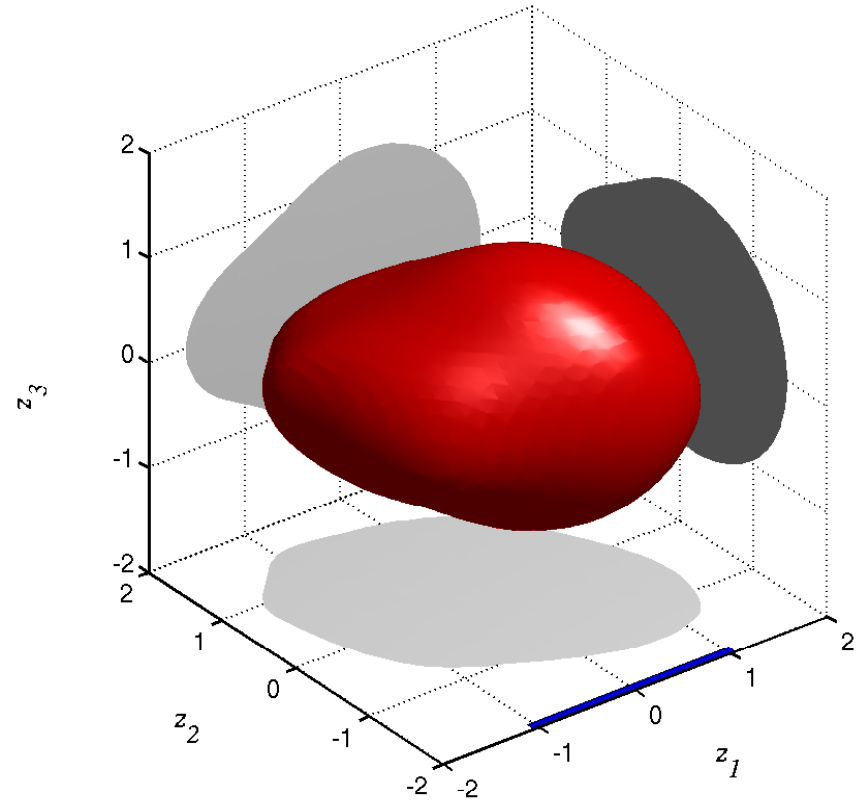


Reconstruction of the coated sphere with  $\lambda = 0.1$  with limited aperture data;  $k = 3$

# Limited Aperture

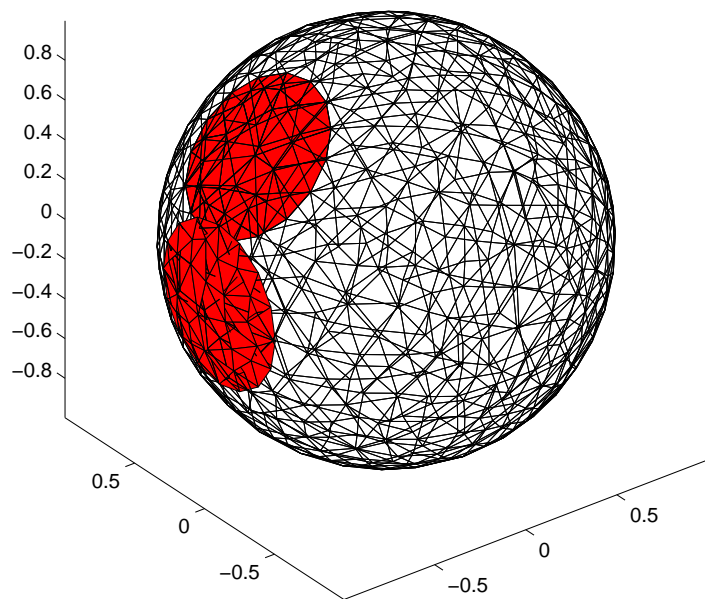


The aperture

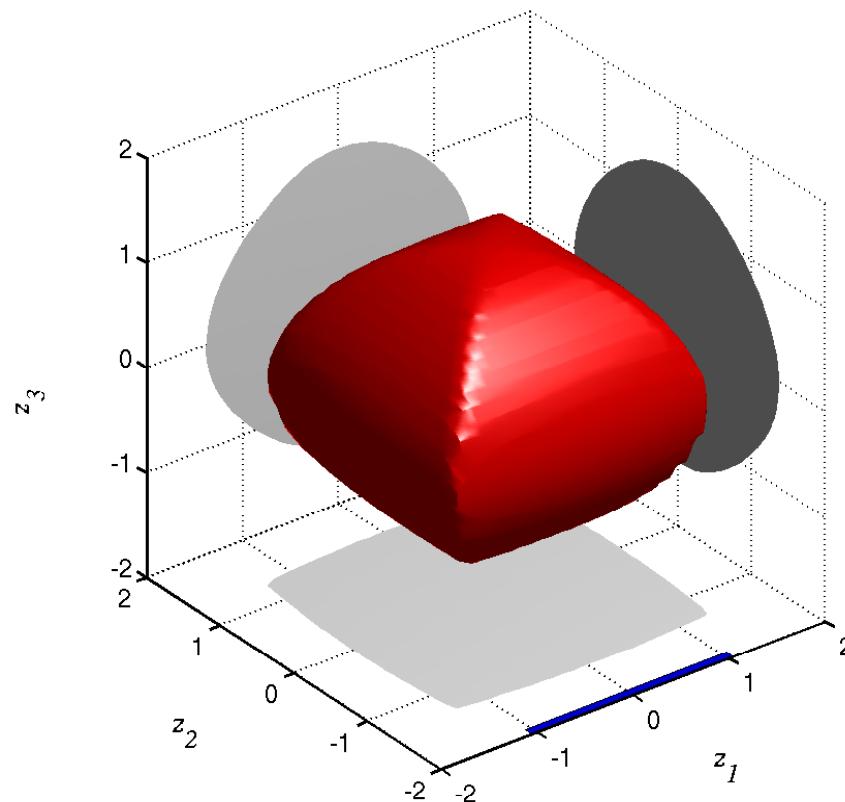


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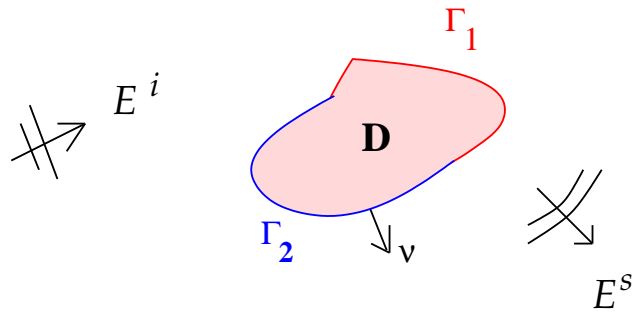


The aperture



Reconstruction of the same sphere using  
the intersection of two reconstructions from  
different single aperture

# Scattering by a Coated Dielectric



The scattered field  $E^s$ ,  $H^s$  and the interior field  $E^{int}$ ,  $H^{int}$  satisfy the equations

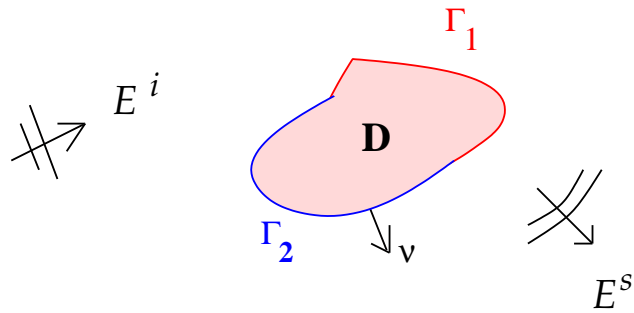
$$\begin{cases} \nabla \times E^s - ikH^s = 0 \\ \nabla \times H^s + ikE^s = 0 \end{cases} \quad \text{in } D_e := \mathbb{R}^3 \setminus \overline{D}$$

$$\begin{cases} \nabla \times E^{int} - ikH^{int} = 0 \\ \nabla \times H^{int} + ikN(x)E^{int} = 0 \end{cases} \quad \text{in } D$$

$$\lim_{|x| \rightarrow \infty} (H^s \times x - |x|E^s) = 0,$$

$N$  is a symmetric matrix,  $\bar{\xi} \cdot \Re(N)\xi \geq \gamma|\xi|^2$ ,  $\gamma > 0$   
and  $\bar{\xi} \cdot \Im(N)\xi \geq 0$ .

# Scattering by a Coated Dielectric



Let  $\eta \in L^\infty(\Gamma_2)$ ,  $\eta > 0$ , be the surface conductivity and  $\Gamma$  a Lipschitz boundary.

Then

$$\nu \times E^s - \nu \times E^{int} = -\nu \times E^i \quad \text{on } \Gamma = \Gamma_1 \cup \Gamma_2$$

$$\nu \times H^s - \nu \times H^{int} = \nu \times H^i \quad \text{on } \Gamma_1$$

$$\nu \times H^s - \nu \times H^{int} = -\nu \times H^i + \eta(x) [\nu \times (E^s + E^i)] \times \nu \quad \text{on } \Gamma_2$$

where  $E^i = \frac{i}{k} \nabla \times \nabla \times p e^{ikx \cdot d}$ ,  $H^i = \nabla \times p e^{ikx \cdot d}$ ,  $p \in \mathbb{R}^3$ .

# Inverse Problem

**Theorem:** *There exists a unique solution  $E^{int}, H^{int} \in H(\text{curl}, D)$ ,  $E^s, H^s \in H_{loc}(\text{curl}, D_e)$  such that  $\nu \times E^{int}, \nu \times E^s \in L^2(\Gamma_2)$  of the scattering problem.*

Again that the scattered electric field  $E^s$  has the asymptotic behavior

$$E^s(x) = \frac{e^{ikr}}{r} \left\{ E_\infty(\hat{x}, d, p) + O\left(\frac{1}{r}\right) \right\}, \quad r = |x|, \hat{x} = x/r.$$

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The **inverse scattering problem** reads

Determine  $D$  and  $\eta$  from a knowledge of  $E_\infty(\hat{x}, d, p)$  for  $\hat{x} \in \Omega_0 \subset \Omega$ ,  $d \in \Omega_1 \subset \Omega$  and three linearly independent polarizations  $p$ , where  $\Omega := \{x, |x| = 1\}$ .

# Uniqueness Theorems

**Theorem (Cakoni-Colton):**  $D$  is uniquely determined by the electric far field pattern  $E_\infty(\hat{x}, d, p)$  for  $\hat{x} \in \Omega_0$  and  $d \in \Omega_1$  and three linearly independent polarizations  $p_1, p_2, p_3$ .

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**Remark:** The matrix  $N(x)$  is **not** uniquely determined by the far field pattern!

**Theorem (Cakoni-Colton-Monk):** Let  $E_\infty^j(\hat{x}, d, p)$  be the electric far field pattern corresponding to a fixed matrix  $N(x)$  but different coatings  $\eta = \eta_j, j = 1, 2$ . Assume that  $k$  is not a Maxwell eigenvalue for  $D = \{x : I - N(x) \neq 0\}$ . Then if  $E_\infty^1(\hat{x}, d, p) = E_\infty^2(\hat{x}, d, p)$  for  $\hat{x} \in \Omega_0, d \in \Omega_1$  and three linearly independent polarizations  $p \in \mathbb{R}^3$ , we have that  $\eta_1(x) = \eta_2(x)$  for  $x \in \Gamma_2$

# Determination of $D$

The support of the anisotropic inhomogeneity can be determined from the above behaviour of an approximate solution of **far field equation**

$$(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q).$$

In the analysis of the far field equation the well-posedness of the following **interior transmission problem** is needed:

# Interior Transmission Problem

$$\left\{ \begin{array}{l} \nabla \times (\nabla \times E_z^{int}) - k^2 N(x) E_z^{int} = 0 \\ \nabla \times (\nabla \times E_z) - k^2 E_z = 0 \end{array} \right. \quad \text{in } D$$

$$\nu \times (E_z^{int} - E_z) = \nu \times E_e(\cdot, z, q) \quad \text{on } \Gamma = \Gamma_1 \cup \Gamma_2$$

$$\nu \times [\nabla \times (E_z^{int} - E_z)] = \nu \times (\nabla \times E_e(\cdot, z, q)) \quad \text{on } \Gamma_1$$

$$\nu \times [\nabla \times (E_z^{int} - E_z)] = \left\{ \begin{array}{l} \nu \times (\nabla \times E_e(\cdot, z, q)) \\ -ik\eta(x)\nu \times (E_z + E_e(\cdot, z, q)) \times \nu \end{array} \right. \quad \text{on } \Gamma_2$$

# Interior Transmission Problem

It is easy to show that if the **homogeneous interior transmission problem** (i.e.  $E_e(\cdot, z, q) = 0$ ) has only the trivial solution then the far field operator  $F : L_t^2(\Omega) \rightarrow L_t^2(\Omega)$  is **injective with dense range**.

Values of  $k$  for which the homogeneous interior transmission problem has nonzero solutions are called **transmission eigenvalues**.

**Nothing is known about transmission eigenvalues!**

# Determination of $\eta$

**Theorem (Cakoni-Colton-Monk):** For  $z \in D$  and  $q \in \mathbb{R}^3$  we have that

$$\eta = \frac{-\frac{k^2}{6\pi} \|q\|^2 + \Re(q \cdot E_z(z))}{\|\nu \times (E_z(\cdot) + E_e(\cdot, z, q))\|_{L_t^2(\Gamma_2)}^2}$$

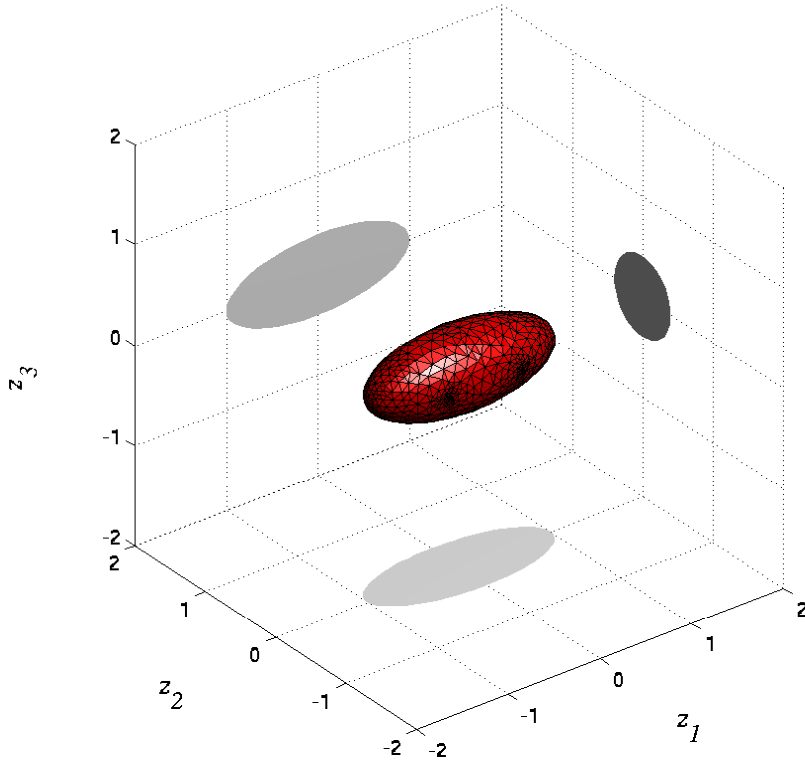
where  $E_z^{int}$ ,  $E_z$  is the solution of the interior transmission problem (if it exists!).

**Corollary:** For  $z \in D$ ,  $q \in \mathbb{R}^3$ , we have that

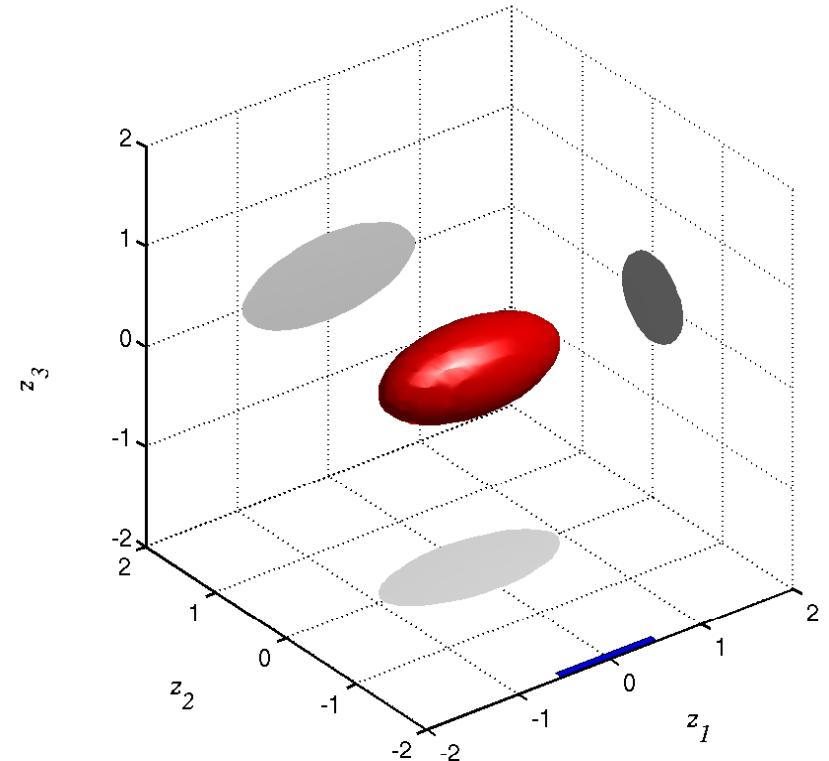
$$\eta \geq \frac{-\frac{k^2}{6\pi} \|q\|^2 + \Re(q \cdot E_z(z))}{\|E_z(\cdot) + E_e(\cdot, z, q)\|_{L^2(\Gamma)}^2}.$$

**Note** that  $E_z$  can be approximated by a Herglotz wave function with kernel  $g_z^\epsilon$  and this  $g_z^\epsilon$  is an approximate solution to the far field equation!

# Examples of Reconstruction



Exact Geometry



Reconstruction

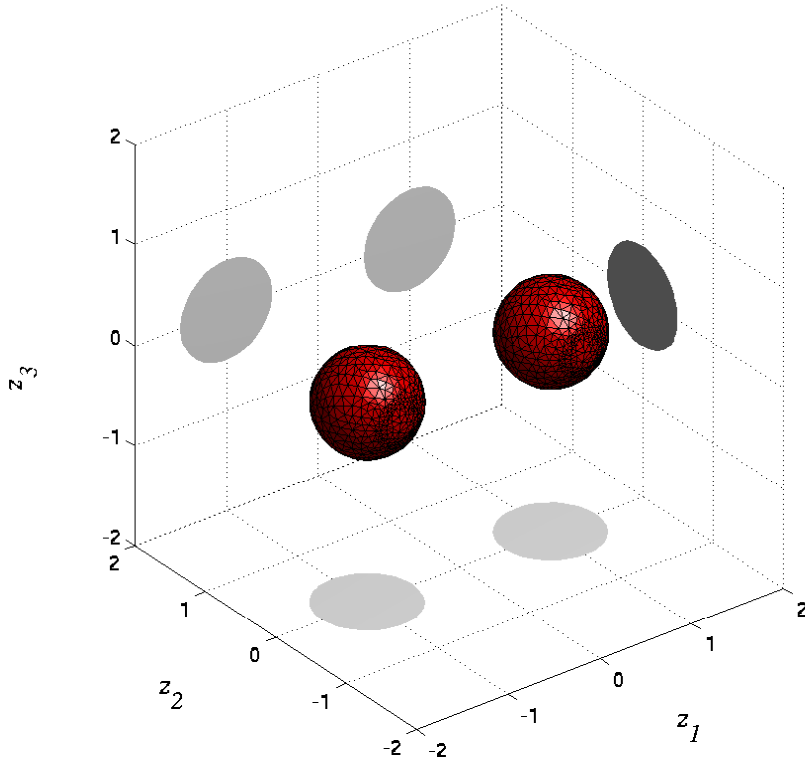
Reconstruction of a fully coated ellipsoid with  $\eta = 1$  and  $k = 6$ .

# Examples of Reconstruction

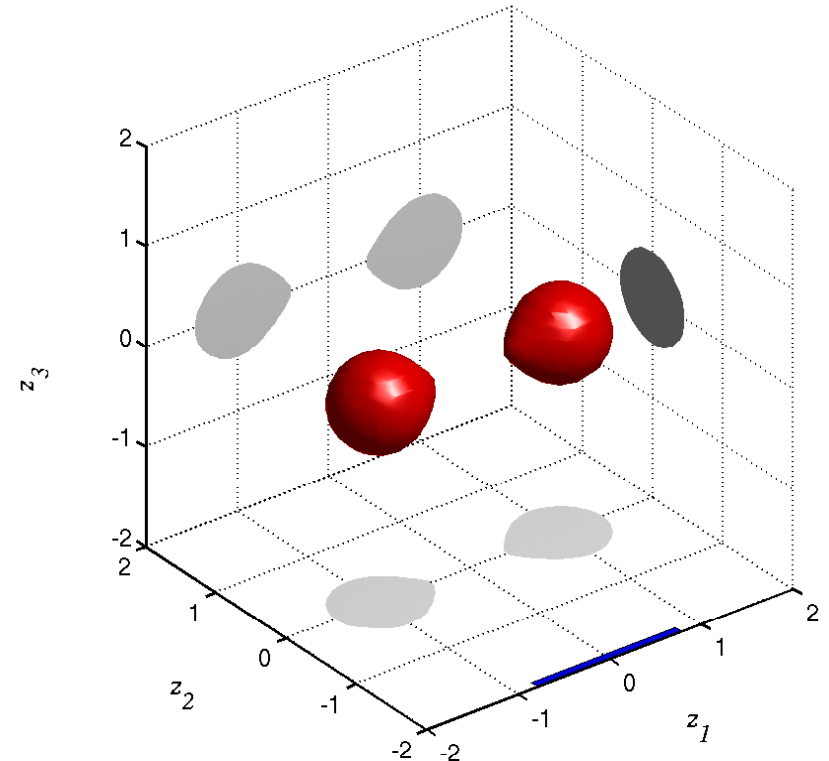
Conducting boundary condition: reconstruction of $\eta$			
Exact	Exact $\Gamma$	LSM	LSM/bound
0.0	-0.005	-0.01	-0.004
0.1	0.09	0.16	0.07
1	0.96	0.79	0.58
2	1.15	0.94	0.66

Reconstruction of  $\eta$  for the fully coated ellipsoid. Here  $k = 6$ .

# Examples of Reconstruction



Exact Geometry



Reconstruction

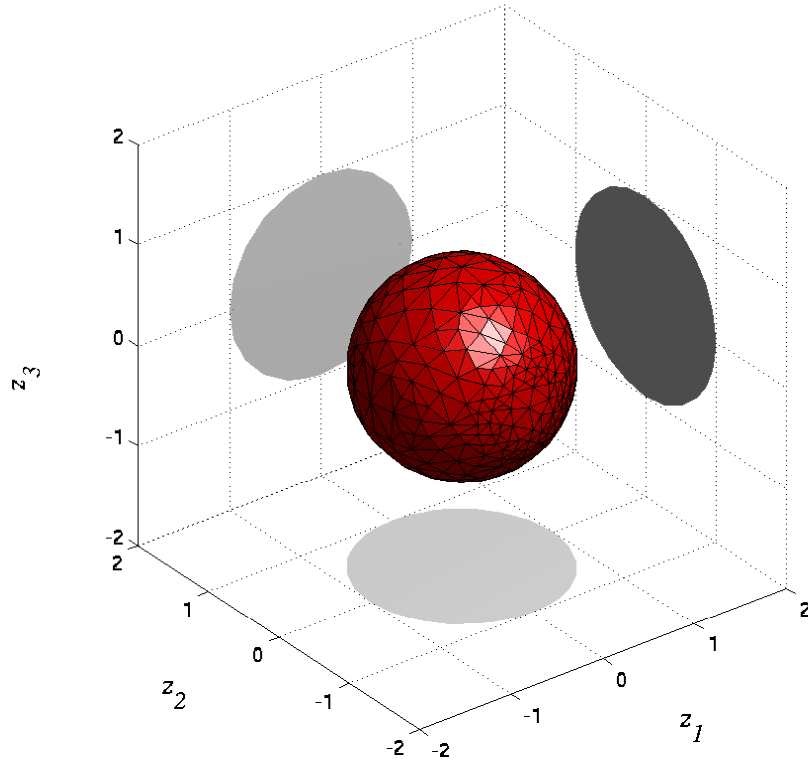
Reconstruction of fully coated two spheres with  $\eta = 1$ ,  $k = 4$ .

# Examples of Reconstruction

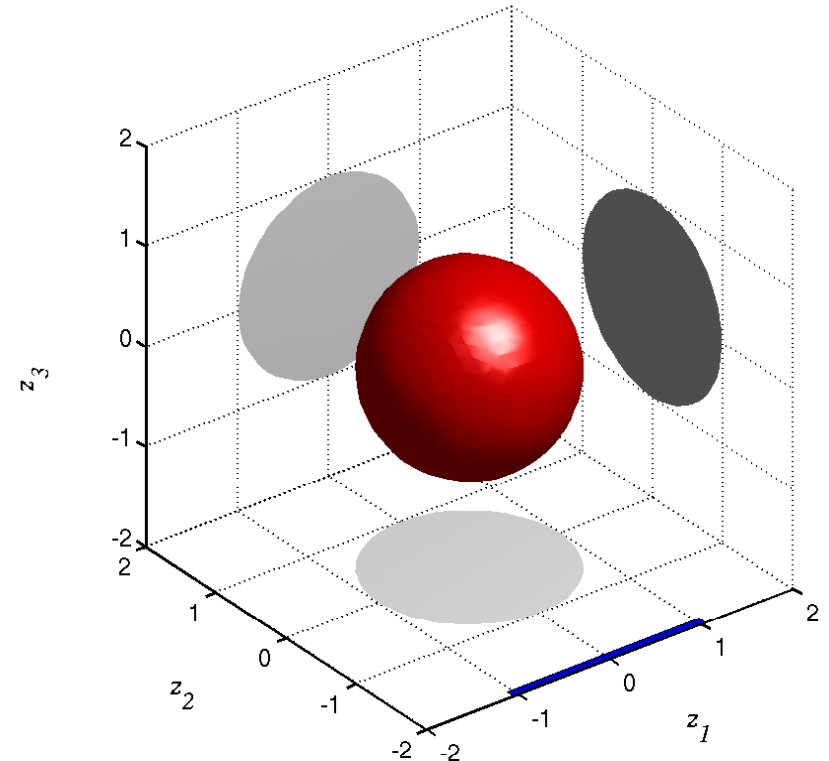
Conducting boundary condition: reconstruction of $\eta$			
Exact	Exact $\Gamma$	LSM	LSM/bound
0.1	0.11	0.13	0.011

Reconstruction of  $\eta$  for the fully coated two spheres,  $k = 6$ .

# Examples of Reconstruction



Exact Geometry



Reconstruction

Reconstruction of a partially coated sphere. The coated portion  $\Gamma_2$  is the hemisphere  $x_2 > 0$ . Here  $\eta = 1$  and  $k = 3$ .

# Examples of Reconstruction

Conducting boundary condition: reconstruction of $\eta$			
Exact	Exact $\Gamma_2$	LSM ( $\Gamma$ )	LSM/bound
0.1	0.045	0.037	0.027
1	0.94	0.52	0.43
2	2.00	0.81	0.65

Reconstruction of  $\eta$  for the partially coated sphere,  $k = 3$ .

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Qualitative Methods in Inverse Scattering Theory

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An Introduction

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