Inequalities in Inverse Scattering for Anisotropic Media

Fioralba Cakoni

cakoni@math.udel.edu

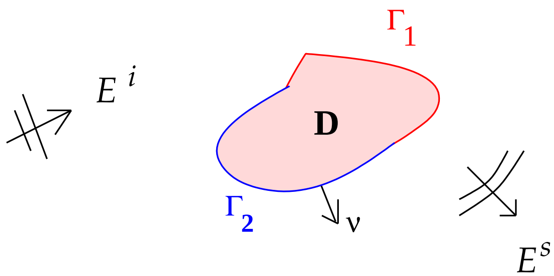
Department of Mathematical Sciences

University of Delaware, USA

Joint work with David Colton, Houssem Haddar and Peter Monk

Research supported by AFOSR

Scattering by an Anisotropic Medium



- The scatterer is an infinitely long cylinder.
- The incident wave is such that the electric field is polarized $\perp$ to the cylinder axis.

- The dielectric is orthotropic, i.e. the index of refraction is given by

$$N(x) = \frac{1}{\epsilon_0}\left(\epsilon(x) + i\frac{\sigma(x)}{\omega}\right) = \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{21} & n_{22} & 0 \\ 0 & 0 & n_{33} \end{pmatrix}.$$

Thus the incident, scattered and interior magnetic fields have the form

$$H^i(0,\ 0,\ u^i), \quad H^{int} = (0,\ 0,\ v), \quad H^s = (0,\ 0,\ u^s)$$

The Forward Problem

$$\nabla \cdot A(x)\nabla v + k^2 v = 0 \quad \text{in} \quad D$$

$$\Delta u^s + k^2 u^s = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{D}$$

$$v - (u^s + u^i) = 0 \quad \text{on} \quad \Gamma_1$$

$$v - (u^s + u^i) = -i\eta(x)\frac{\partial (u^s + u^i)}{\partial \nu} \quad \text{on} \quad \Gamma_2$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial (u^s + u^i)}{\partial \nu} = 0 \qquad \text{on} \qquad \partial D = \Gamma_1 + \Gamma_2$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

$$u^i(x) = e^{ikx\cdot d}, \quad \frac{\partial v}{\partial \nu_A} := \nu \cdot A\nabla v, \quad d \in \Omega := \{x : \ |x| = 1\}$$

The Forward Problem

We assume:

- ∂D is piecewise smooth.
- A is a symmetric matrix-valued function with piecewise C^1 entries in $\overline{D}$.
- $\mathcal{R}e\left(\overline{\xi} \cdot A\xi\right) \geq \gamma|\xi|^2$ and $\mathcal{R}e\left(\overline{\xi} \cdot A^{-1}\xi\right) \geq \gamma|\xi|^2$, $\gamma > 0$
- $\mathcal{I}m\left(\overline{\xi} \cdot A\xi\right) \leq 0$
- $\eta \in L_\infty(\Gamma_2)$, $\eta(x) \geq \eta_0 > 0$.

Inverse Scattering Problem

The scattered field u^s has the asymptotic behaviour

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O\left(r^{-3/2}\right)$$

as $r \to \infty$ where $r = |x|$, $\hat{x} = x/r$, k is fixed and u_∞ is the far field pattern of the scattered field u^s.

The inverse scattering problem is to determine D, A or/and η from a knowledge of $u_\infty(\hat{x}, d)$ for $\hat{x}, -d \in \Omega_0 \subset \Omega$ and a range of frequencies k.

Uniqueness Theorems

Theorem: (Uniqueness of D) Assume that either $\mathcal{R}e\left(\overline{\xi}\cdot A\xi\right)\geq\gamma|\xi|^2$ or $\mathcal{R}e\left(\overline{\xi}\cdot A^{-1}\xi\right)\geq\gamma|\xi|^2$ for some $\gamma>1$. Then D is uniquely determined by $u_\infty(\hat{x},d)$ for $\hat{x},d\in\Omega$ and a fixed value of the wave number k.

Proof *by Hähner for* $\eta=0$*, Cakoni-Colton for* $\eta>0$

Uniqueness Theorems

Theorem: (Uniqueness of η) Given A and D, suppose that for an arbitrary $\Gamma_0 \subset \partial D$ there exists an incident direction d such that $\partial u / \partial \nu \neq 0$ in Γ_0. Then $\eta \in C(\overline{\Gamma_2})$ is uniquely determined from $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ and a fixed value of the wave number k.

Proof *by Cakoni-Colton-Monk. Under more regularity assumptions O'Dell has proved uniqueness for η without the assumption that A is fixed*.

However $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ does not uniquely determine A even if it is known for an interval of values of k.

The Far Field Operator

The far field operator $F : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d) ds(d).$$

$(Fg)(\hat{x})$ is the far field pattern corresponding to the incident field being a Herglotz wave function v_g defined by

$$v_g(x) := \int_\Omega e^{ikx \cdot d} g(d) ds(d).$$

The Far Field Operator

The far field operator F is injective with dense range if and only if there does not exist a solution $v, w \in H^1(D)$ such that $\partial v/\partial \nu \in L^2(\Gamma_2)$ of the interior transmission problem

$$\begin{aligned} \nabla \cdot A \nabla w + k^2 w &= 0 && \text{in} \quad D \\ \Delta v + k^2 v &= 0 && \text{in} \quad D \\ w &= v && \text{on} \quad \Gamma_1 \\ w &= v - i\eta \frac{\partial v}{\partial \nu} && \text{on} \quad \Gamma_2 \\ \frac{\partial w}{\partial \nu_A} &= \frac{\partial v}{\partial \nu} && \text{on} \quad \partial D \end{aligned}$$

such that v is a Herglotz wave function with kernel $g \neq 0$.

Transmission Eigenvalues

Definition: If $k > 0$ is such that the interior transmission problem has a nontrivial solution then k is called a transmission eigenvalue.

Theorem: If $\mathcal{I}m \left(\overline{\xi} \cdot A\xi\right) < 0$ in D then k is not a transmission eigenvalue.

Open Problem: Show that transmission eigenvalues do not exist if $\mathcal{I}m \left(\overline{\xi} \cdot A\xi\right) = 0$ and Γ_2 is not empty.

Theorem: If $\mathcal{I}m \left(\overline{\xi} \cdot A\xi\right) = 0$, and $\bar{\xi} \cdot \left(A^{-1} - I\right) \xi \geq \alpha|\xi|^2$ or $\bar{\xi} \cdot A^{-1}(I - A^{-1})^{-1} \xi \geq \alpha|\xi|^2$ where $\alpha > 0$ is a constant, then the set of transmission eigenvalues is a discrete set.

Proof *by Cakoni, Colton and Haddar*

Determination of D

Let $\Phi(x,z) := \frac{i}{4}H_0^{(1)}(k|x-z|)$ which has the far field pattern

$$\Phi_\infty(\hat{x},z) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}e^{-ik\hat{x}\cdot z}.$$

Define the far field equation

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x},z).$$

The support D can be determined from the behavior of the regularized solution g to the far field equation (linear sampling method).

F. Cakoni and D. Colton, Qualitative Methods in Inverse Scattering Theory, Springer Verlag, Berlin, 2006.

Determination of η

For $z \in D$, v_{g_z} (where g_z is an approximate solution of the far field equation) converges to v_z where v_z, w_z is the solution of

$$
\begin{aligned}
\nabla \cdot A \nabla v_z + k^2 v_z &= 0 && \text{in} \quad D \\
\Delta w_z + k^2 w_z &= 0 && \text{in} \quad D \\
v_z - w_z &= \Phi(\cdot, z) && \text{on} \quad \Gamma_1 \\
v_z - w_z &= \Phi(\cdot, z) - i\eta \frac{\partial}{\partial \nu} \left(w_z + \Phi(\cdot, z) \right) && \text{on} \quad \Gamma_2 \\
\frac{\partial v_z}{\partial \nu_A} - \frac{\partial w_z}{\partial \nu} &= \frac{\partial}{\partial \nu} \Phi(\cdot, z) && \text{on} \quad \partial D.
\end{aligned}
$$

provided that k is not a transmission eigenvalue.

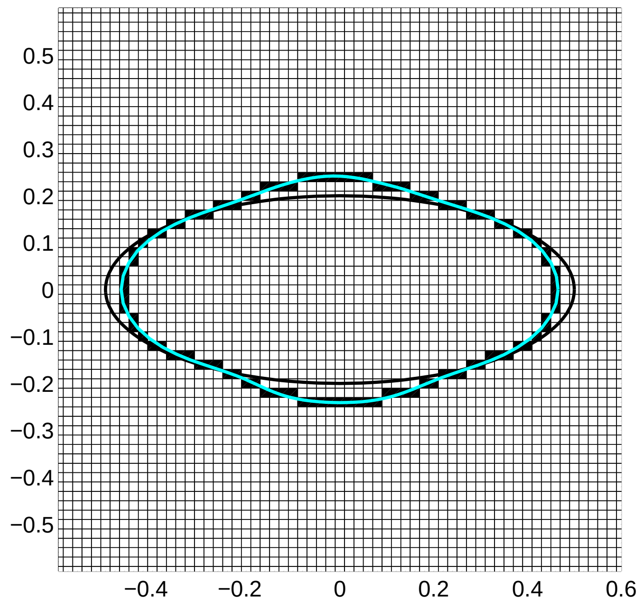
Determination of η

Assuming that $\mathcal{I}m\left(\overline{\xi}\cdot A\xi\right)=0$, one can show that (Colton, Cakoni, Monk)

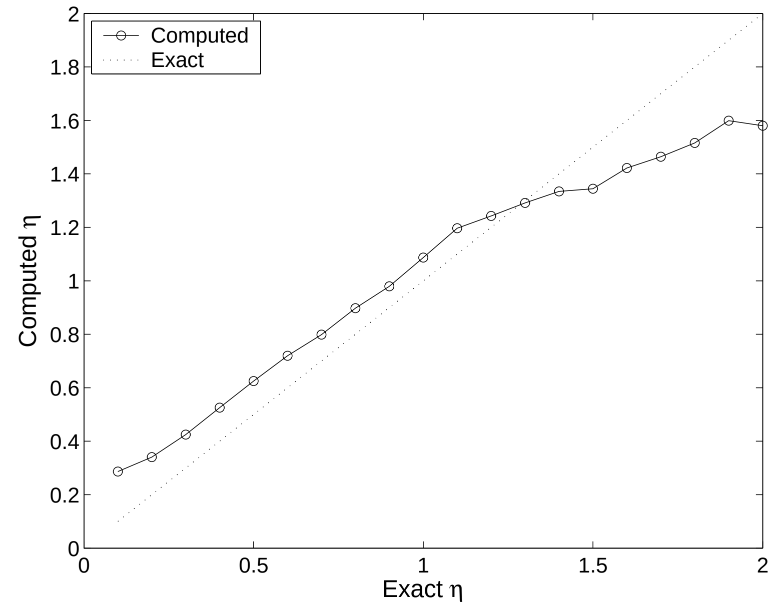
$$\eta(x)\geq\frac{-\frac{1}{4}-\mathcal{I}\text{m}\,(w_z(z))}{\|\frac{\partial}{\partial\nu}\,(w_z+\Phi(\cdot\,,z))\,\|^2_{L^2(\Gamma_2)}}\qquad z\in D,\ x\in\Gamma_2.$$

- Γ_2 can be reconstructed from the behavior of $\mathcal{I}\text{m}\,(w_z(z))$ (Cakoni, Sini, Zeev).
- Recall, w_z can be approximated in $L^2(\Gamma_2)$ by the Herglotz wave function v_{g_z} with kernel g_z an approximate solution of the far field equation.

Numerical Examples

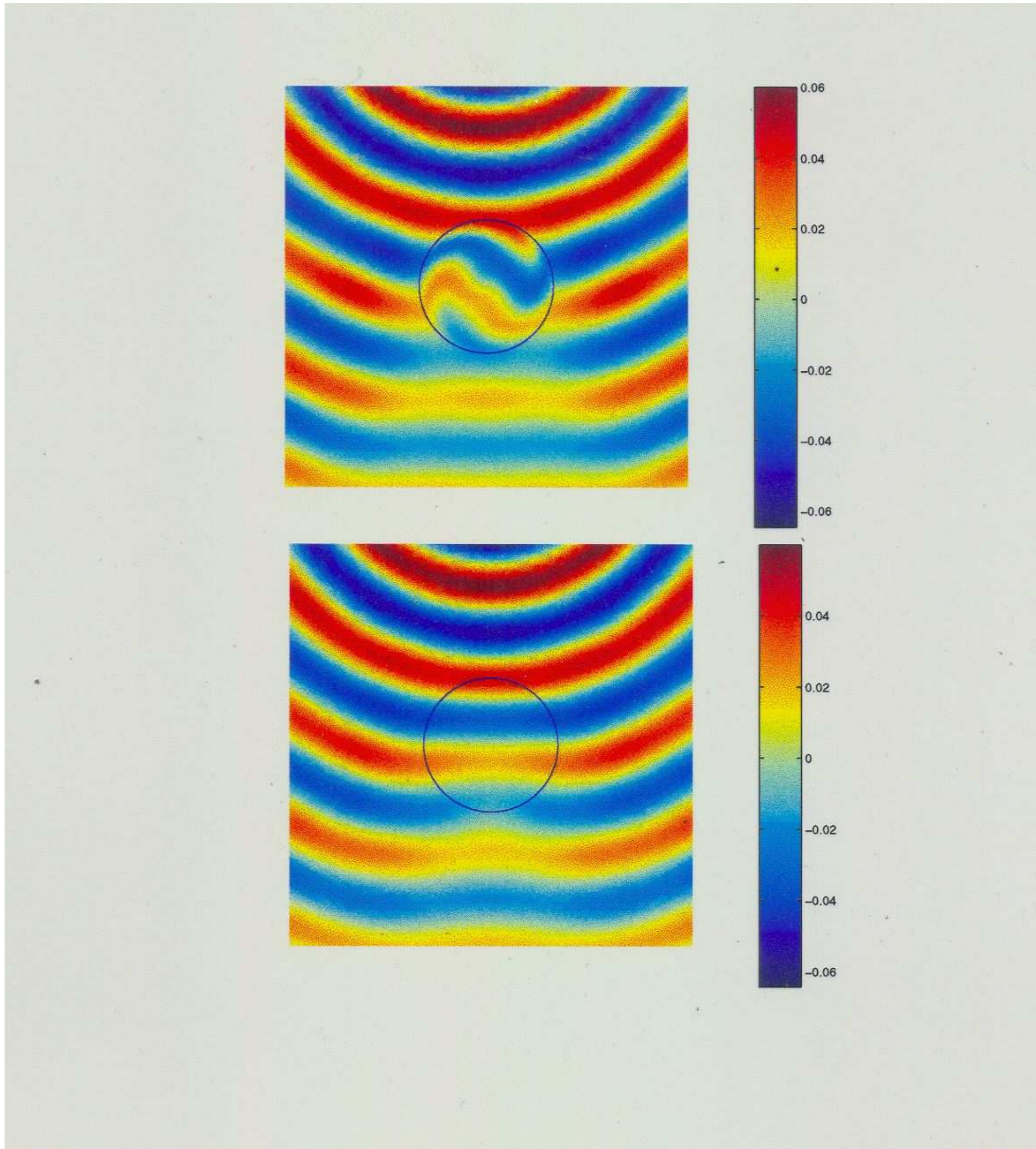


Reconstruction of D
(fully coated)



Reconstruction of η
(with reconstructed ∂D)

Non-uniqueness for anisotropic media



Interior Transmission Problem

What, if anything, can be said about A from a knowledge of u_∞?

To this end we return to the interior transmission problem, assume that $\mathcal{I}m\left(\bar{\xi}\cdot A\xi\right)=0$ and $\Gamma_2=\emptyset$, define

$$H_0(div,D):=\{u\in H(div,D):\ \nu\cdot u=0 \text{ on } \partial D\}$$

$$\mathcal{H}_0(D):=\left\{u\in H_0(div,D):\ \nabla\cdot u\in H_0^1(D)\right\},$$

and substitute

$$\mathbf{w}=A\nabla w \quad \text{and} \quad \mathbf{v}=\nabla v.$$

Interior Transmission Problem

Then the interior transmission problem is equivalent to finding $\mathbf{u} = \mathbf{w} - \mathbf{v} \in \mathcal{H}_0(D)$ that satisfies

$$\nabla(\nabla \cdot \mathbf{u}) + k^2\mathbf{u} = k^2 \left(I - A^{-1}\right) \mathbf{w} \qquad \text{in} \quad D.$$

which can be written in the variational form

$$\int_D (A^{-1} - I)^{-1} \left(\nabla\nabla \cdot \mathbf{u} + k^2\mathbf{u}\right) \cdot \left(\nabla\nabla \cdot \overline{\psi} + k^2 A^{-1}\overline{\psi}\right) \, dx = 0$$

Interior Transmission Problem

Theorem

- Assume that $\bar{\xi} \cdot (A^{-1} - I)^{-1} \xi \geq \alpha |\xi|^2$ where $\alpha > 0$ is a constant. Then the above variational form is coercive if $k^2 < \alpha\lambda(D)$.
- Assume that $\bar{\xi} \cdot A^{-1}(I - A^{-1})^{-1} \xi \geq \alpha |\xi|^2$ where $\alpha > 0$ is a constant. Then the above variational form is coercive if $k^2 < \lambda(D)$.

Here $\lambda(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D.

Proof *by Cakoni, Colton and Haddar*

Estimates for A

Since α is the smallest eigenvalue of the $(A^{-1} - I)^{-1}$, the previous theorem implies the following result:

- Assume that $\|A^{-1}(x)\|_2 \geq \delta > 1$ for all $x \in D$ and some constant δ. Then,

$$\sup_D \|A^{-1}\|_2 \geq \frac{\lambda(D)}{k^2}$$

- Assume that $0 < \beta \leq \|A^{-1}(x)\|_2 \leq \delta < 1$ for all $x \in D$ and some constants β and δ. Then,

$$k^2 > \lambda(D)$$

where k is a transmission eigenvalue and $\lambda(D)$ is the first eigenvalue of $-\Delta$ on D.

Computation of Eigenvalues

The linear sampling method can be expected to fail when k is a transmission eigenvalue.

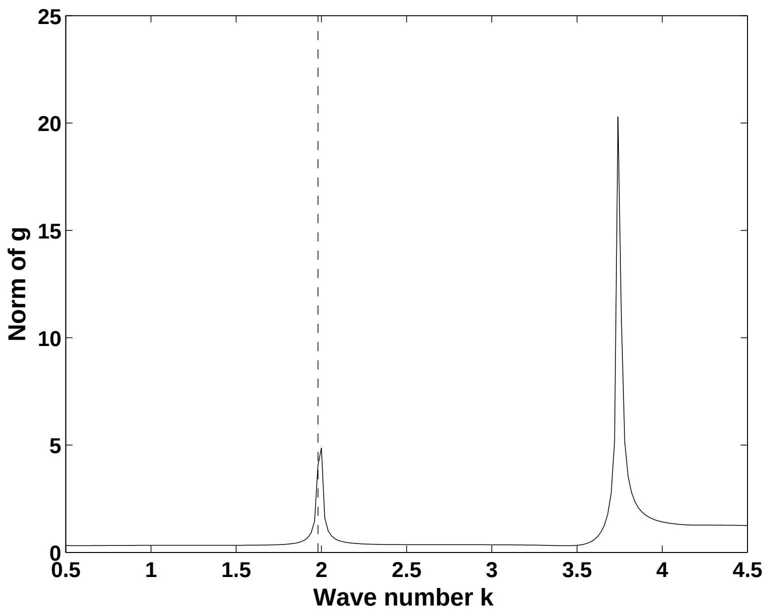
In particular, the norm of the (regularized) solution to

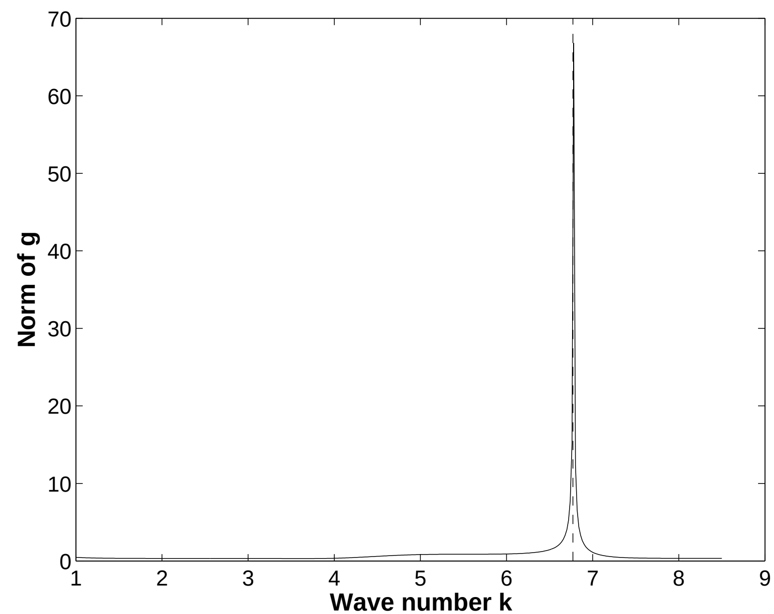
$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z_0) \qquad z_0 \in D$$

should be large for such values of k.

Numerical Examples

D is a disk of diameter 1.



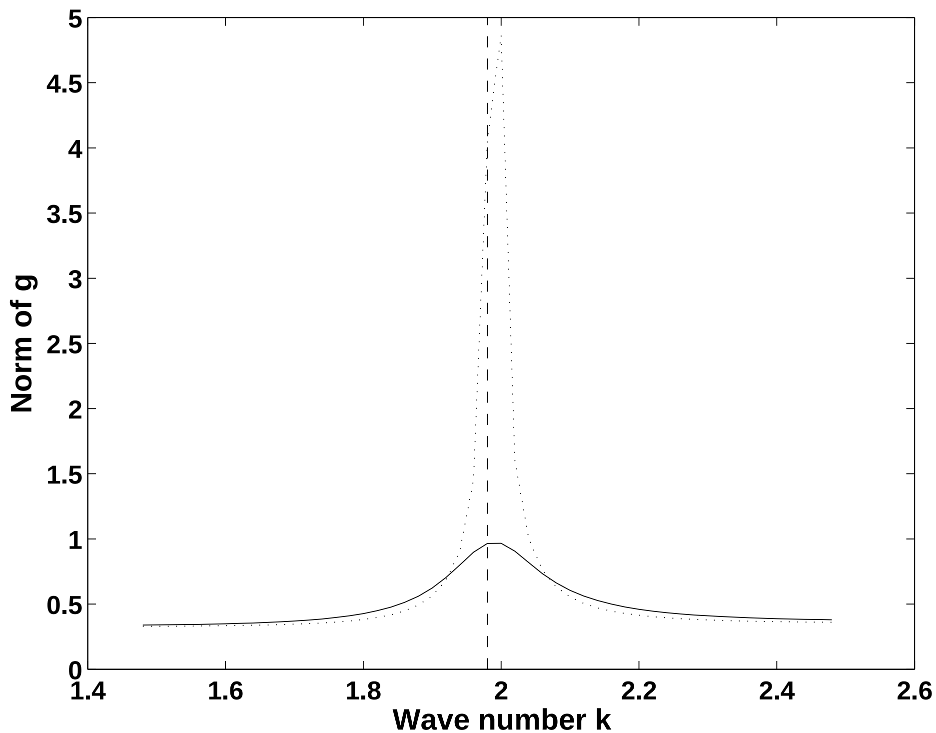


$n = 16$ $\qquad$ $n = 4$

$$\Delta u + k^2 n u = 0$$

Numerical Examples

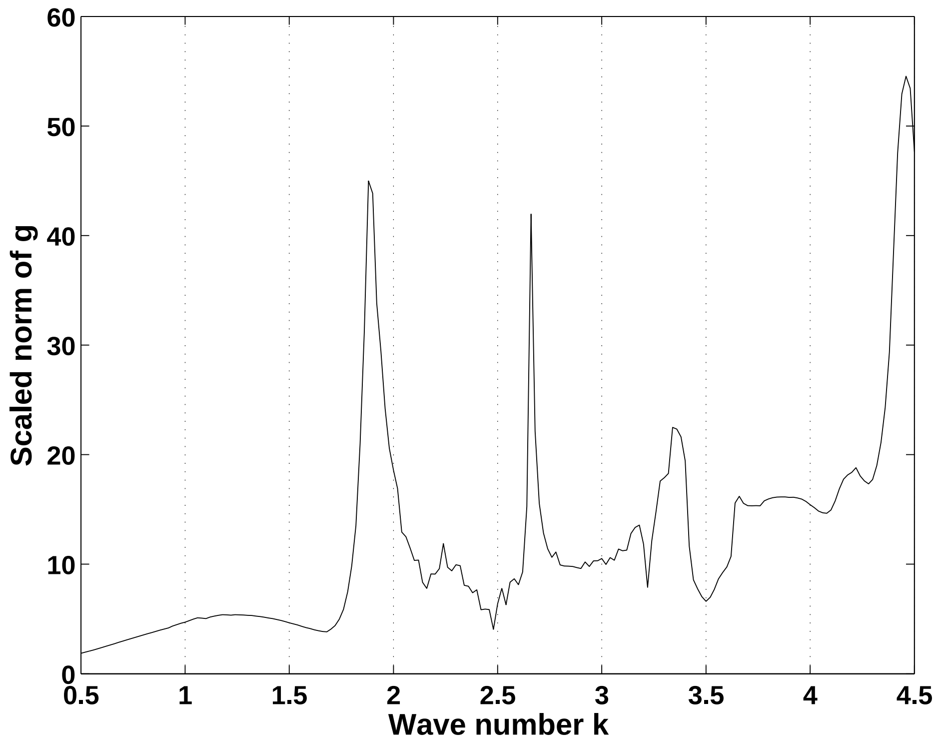
D is a disk of diameter 1.



$$n = 16 + i$$

Numerical Examples

D is the rectangle $[-0.5,\ 0.5] \times [-0.4,\ 0.4]$, $\lambda_0(D) \approx 25.3$.



$n = 16$ and the estimate is $n \geq \dfrac{25.3}{(1.88)^2} \approx 7.1$

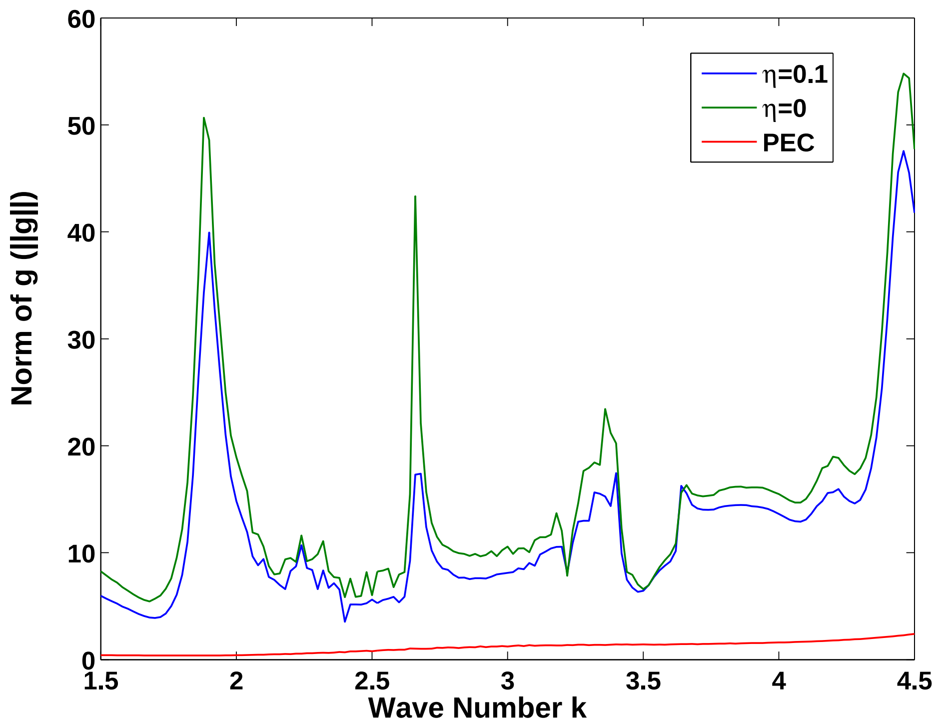
Coated Dielectric

Until now we have assumed that Γ_2 is the empty set. We now consider the case when Γ_2 is not empty, i.e. the scatterer is coated by a thin highly conducting layer.

For small η is still possible to see the transmission eigenvalues using the far field data.

Numerical Examples

D is the rectangle $[-0.5,\ 0.5] \times [-0.4,\ 0.4]$



Detection of decoys

In applications connected with the detection of decoys it is necessary to distinguished between a perfect conductor and a coated dielectric.

Recall from the previous estimate that if relative permittivity (i.e. A^{-1}) is close to one then transmission eigenvalues become large.

On the other hand if η is large the coated dielectric behaves like a perfect conductor.

Numerical Examples

D is the rectangle $[-0.5,\ 0.5] \times [-0.4,\ 0.4]$