Qualitative Methods in Inverse Scattering Theory

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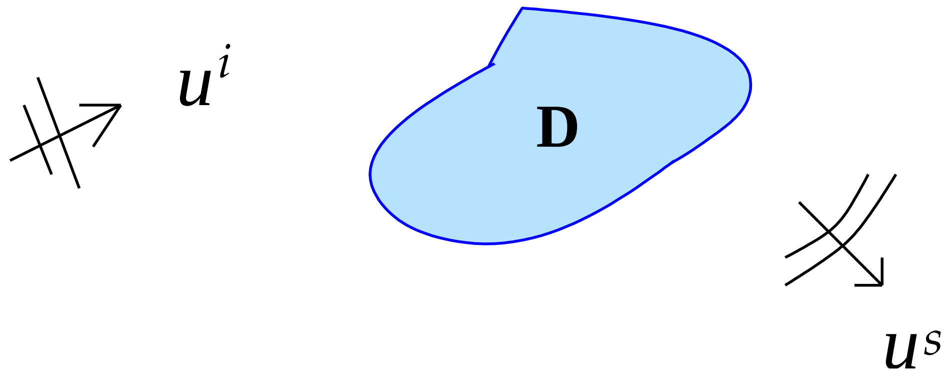
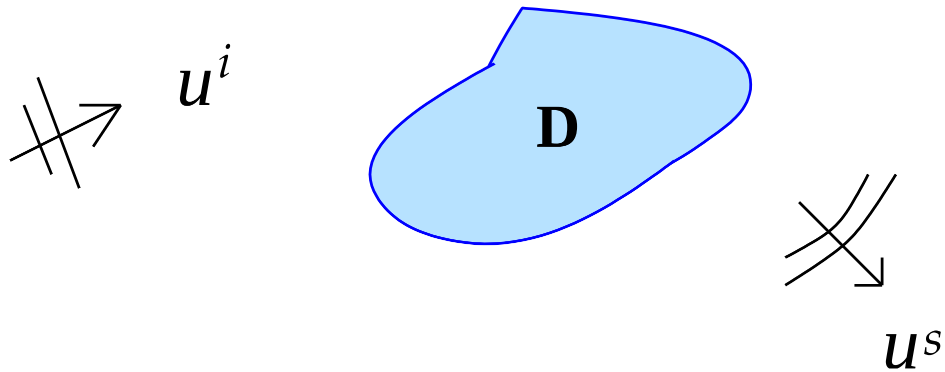
The scattering phenomena



- Direct scattering problem: Given the time-harmonic incident field u^i and the scattering object D find u^s where $u = u^i + u^s$ is the total field.

  u^s *solves a boundary value problem for a set of partial differential equations*

The scattering phenomena



- Inverse scattering problem: Given the incident field u^i and the (measured) scattered field u^s, determine geometric or/and physical properties of the scattering object D, i.e. determine
  - *coefficients in the differential equation, or/and*
  - *the domain D or/and*
  - *the boundary conditions, satisfied by $u = u^i + u^s$.*

Inverse Problems

The inverse scattering problem is nonlinear and ill-posed.

If $\mathcal{A}$ is the operator from the space X of the unknowns to the space Y of the measured data, then the inverse problem can be formulated as a nonlinear equation

$$\boxed{\mathcal{A}\varphi = h, \qquad \varphi \in X \text{ and } h \in Y}$$

In general $\mathcal{A}$ is a completely continuous operator and therefore its Frechet derivative (if it exists) has not bounded inverse.

Ill-posed problems

- If the data space Y is defined as the set of solutions to the direct problem, the existence of a solution to the inverse problem is clear; however a solution may fail to exist if the data are perturbed by noise!
- Uniqueness of a solution to an inverse problem is often not an easy task and obviously important - if uniqueness is not guaranteed by the given data, then either additional data have to be observed or the set of admissible solutions has to be restricted using a-priori information on the solution.
- The failure of stability is the most delicate to deal with. Hence one must use regularization theory which has its origin in the fundamental work by Tikhonov and Miller in the mid- 60s.

Approaches to inverse problems

- Born or weak scattering approximation:
  - multiple scattering is ignored, hence the problem is linear
  - a priori information is needed

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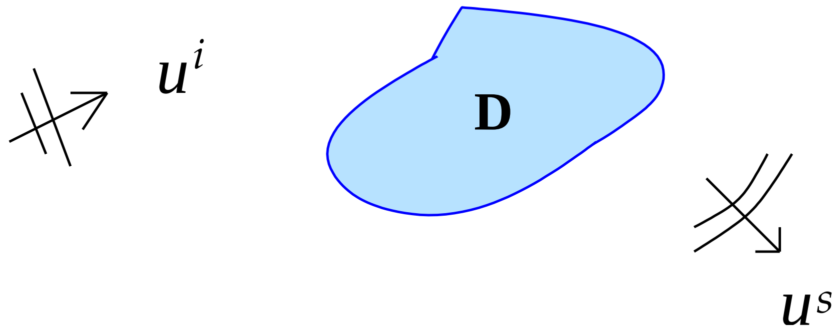
Approaches to inverse problems

- Born or weak scattering approximation:
  - multiple scattering is ignored, hence the problem is linear
  - a priori information is needed
- Optimization techniques:
  - multiple scattering is included, hence the problem is nonlinear
  - a priori information is needed
- Qualitative methods:
  - multiple scattering is included
  - the problem is linear
  - essentially no a priori information is needed

"A lack of information cannot be remedied by any mathematical trickery" - C. Lanczos

F. Cakoni - D. Colton *"Qualitative Methods in Inverse Scattering"* , Springer, 2006.

Scattering by an inhomogeneous cylinder



$$\Delta_2 u + k^2 n(x) u = 0 \qquad \text{in} \qquad \mathbb{R}^2$$

$$u(x) = e^{ikx\cdot d} + u^s(x) \qquad \text{in} \qquad \mathbb{R}^2$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

where $d \in \Omega := \{x : |x| = 1\}$, $r = |x|$ and $k > 0$ is the wave number.

We assume that $m = 1 - n$ has compact support $\overline{D}$ and n is a piecewise C^1 function.

The inverse scattering problem

The scattered field u^s has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O\left(r^{-3/2}\right)$$

as $r \to \infty$, where $\hat{x} = x/|x|$ and $u_\infty(\hat{x}, d)$ is the far field pattern of the scattered field u^s.

The inverse scattering problem is to determine D and $n(x)$ from a knowledge of $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$.

The Inverse Scattering Problem

Theorem: D is uniquely determined by $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ and a fixed value of the wave number k.

Theorem: $n(x)$ is uniquely determined by $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ and an interval of values of k.

Open Problem: Is D uniquely determined by $u_\infty(\hat{x}, d)$ for one or finitely many d, all $\hat{x} \in \Omega$ and a fixed wave number k?

Open Problem: Is $n(x)$ uniquely determined by $u_\infty(\hat{x}, d)$ for one or finitely many d, all $\hat{x} \in \Omega$ and a fixed wave number k?

The Far Field Operator

The far field operator $F : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$(Fg)(\hat{x}) := \int\limits_{\Omega} u_\infty(\hat{x}, d)g(d)ds(d).$$

$(Fg)(\hat{x})$ is the far field pattern corresponding to the incident field being a Herglotz wave function v_g defined by

$$v_g(x) := \int\limits_{\Omega} e^{ikx\cdot d}g(d)ds(d).$$

The Far Field Operator

Theorem: The far field operator F is injective with dense range if and only if there does not exist a solution $v, w \in H^1(D)$ of the interior transmission problem

$$
\begin{aligned}
\Delta_2 w + k^2 n(x) w &= 0 && \text{in} \quad D \\
\Delta_2 v + k^2 v &= 0 && \text{in} \quad D \\
w &= v && \text{on} \quad \partial D \\
\frac{\partial w}{\partial \nu} &= \frac{\partial v}{\partial \nu} && \text{on} \quad \partial D
\end{aligned}
$$

such that v is a Herglotz wave function with kernel $g \neq 0$.

Transmission Eigenvalues

Definition: If $k > 0$ is such that the interior transmission problem has a nontrivial solution then k is called a transmission eigenvalue.

Theorem: If $\Im(n) \neq 0$ then k is not a transmission eigenvalue.

Theorem: Let $m := 1 - n$ and assume that $m(x) > 0$ or $m(x) < 0$ for $x \in \overline{D}$. Then the set of transmission eigenvalues is a discrete set.

Open Problem: Show that the transmission eigenvalues exist for a non-spherically stratified inhomogenous medium.

Determination of D

Let $\Phi(x,z) := \frac{i}{4} H_0^{(1)}(k|x-z|)$ be the radiating fundamental solution to the Helmholtz equation

$$\Delta_2 v + k^2 v = 0.$$

$\Phi(\hat{x},z)$ has the far field pattern $\Phi_\infty(\hat{x},z) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot z}$.

Define the far field equation

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x},z).$$

In general, there is no solution to the far field equation. However, we have the following theorem:

The Linear Sampling Method

Theorem: Assume that k is not a transmission eigenvalue. Then

- For $z \in D$ and a given $\epsilon > 0$ there exists a $g_\epsilon^z \in L^2(\Omega)$ such that

$$\|F g_\epsilon^z - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} < \epsilon$$

  and the Herglotz wave function $v_{g_\epsilon^z}$ converges in $L^2(D)$ to v where v, w is a solution of an inhomogeneous interior transmission problem.

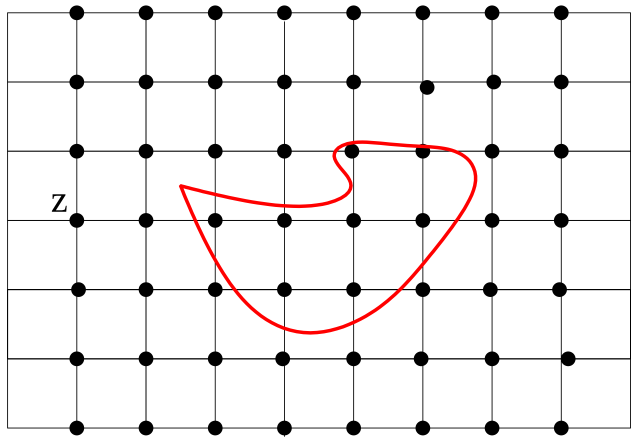
- For $z \in D$ and a fixed $\epsilon > 0$ we have that

$$\lim_{z \to \partial D} \|v_{g_\epsilon^z}\|_{L^2(D)} = \infty \text{ and } \lim_{z \to \partial D} \|g_\epsilon^z\|_{L^2(\Omega)} = \infty$$

- For $z \in \mathbb{R}^2 \setminus \overline{D}$ and a given $\epsilon > 0$, every $g_\epsilon^z \in L^2(\Omega)$ that satisfies

$$\|F g_\epsilon^z - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} < \epsilon \quad \text{is such that} \quad \lim_{\epsilon \to 0} \|v_{g_\epsilon^z}\|_{L^2(D)} = \infty.$$

Numerical Implementation



- Construct a grid $\mathcal{G}$.

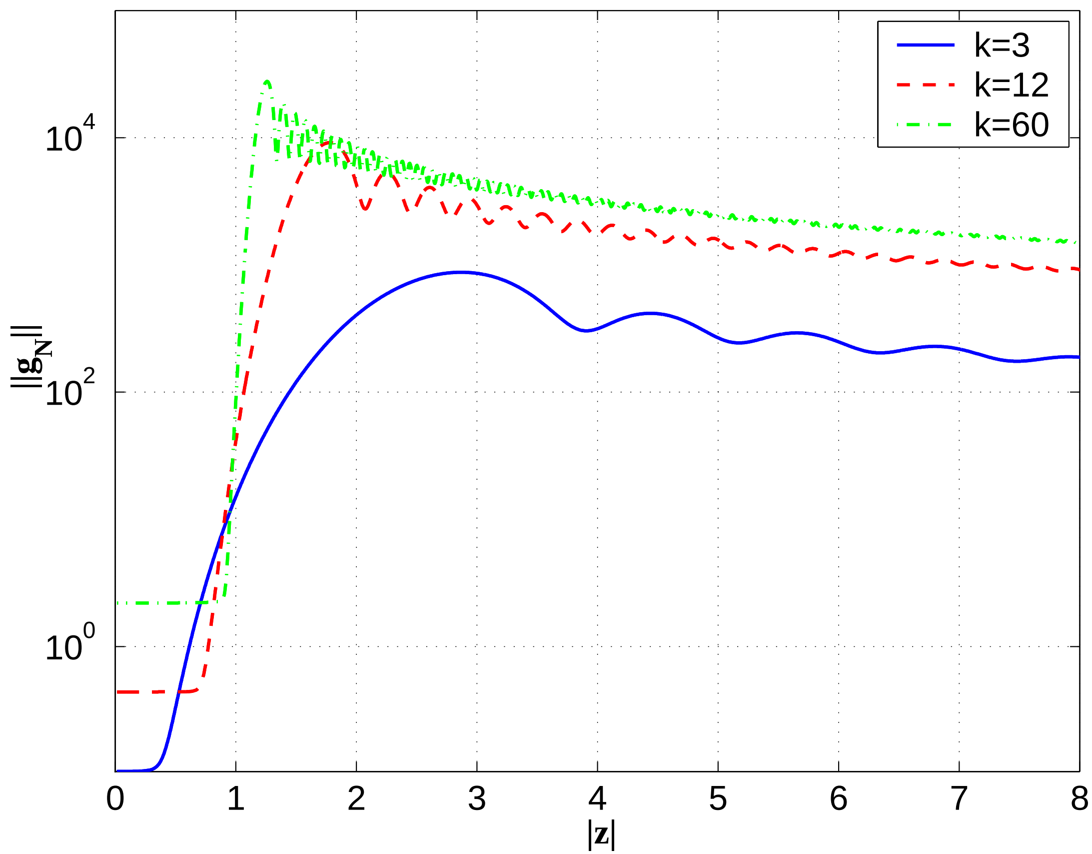
- For $z_i \in \mathcal{G}$, solve the regularized far field equation

$$(\alpha I + F^*F)g_{z_i} = \Phi_\infty(\hat{x}, z)$$

- Choose a cut-off value C and assert that $z_i \in D$ if and only if

$$\|g_{z_i}\|_{\ell^2} \leq C.$$

Behavior of $||g||$ for Different Frequencies



$||g||$ with respect to z.

Open Problem

Let $\mathcal{F} : X \to X$, $\mathcal{H} : X \to Y$ and $\mathcal{B} : Y \to X$ be compact linear operators such that $\mathcal{F}$ has dense range and $\mathcal{F} = \mathcal{B}\mathcal{H}$. The equation

$$\mathcal{F} g = \varphi_0$$

has no solution. However, for all $\delta > 0$ there exists g_δ such that

$$\mathcal{F} g_\delta \to \varphi_0 \qquad \text{as} \quad \delta \to 0$$

and $\mathcal{H} g_\delta$ converges to the solution v of $\mathcal{B} v = \varphi_0$.
The goal is to determine an approximation to g_δ. We do this from

$$\min \left(\|\mathcal{F}\tilde{g}_\alpha - \varphi_0\|_X + \alpha \|\tilde{g}_\alpha\|_X \right).$$

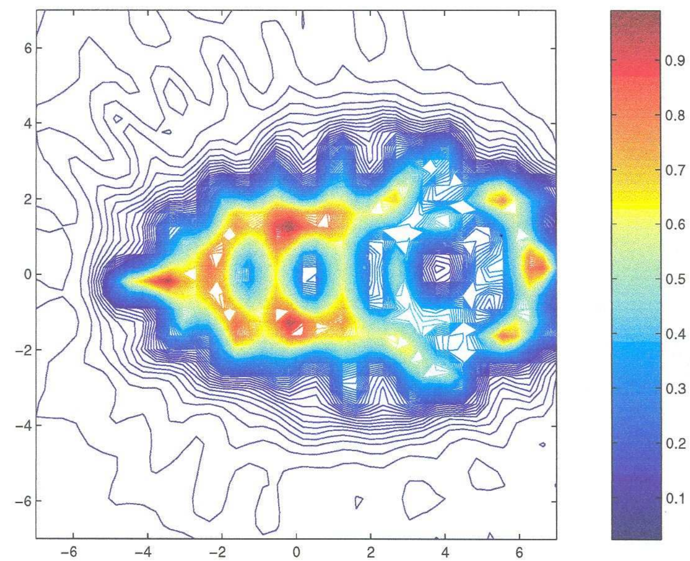
Problem: Is it possible to choose $\alpha = \alpha(\delta)$, δ is the noise level, such that

$$\|\tilde{g}_\alpha - g_\delta\|_X = O(\delta^p) \qquad \text{for some } p > 0?$$

Reconstruction of D with Real Data

Ipswich data*

Target ips010: Plexiglass Triangle
Penetrable



Plot of $\frac{1}{\|g\|_{L^2(\Omega)}}$
Sampling region: 40×40 grid
Delta 0.22, $\lambda = 3$

*Measured data provided by:
Electromagnetics Technology Division, AFRL/SNH
31 Grenier Street
Hanscom AFB, MA 01731-3010

Determination of a Lower Bound for $n(x)$

The following theorem shows that if $n(x) > 1$ for $x \in \overline{D}$ and D is known, the (smallest) transmission eigenvalue provides a lower bound for $\sup_D n$.

Theorem: Suppose $n(x) > 1$ for $x \in \overline{D}$. Then

$$\sup_D n > \frac{\lambda_0(D)}{k^2}$$

where k is a transmission eigenvalue and $\lambda_0(D)$ is the first Dirichlet eigenvalue of $-\Delta_2$ in D.

Determination of a Lower Bound for $n(x)$

Remark: If $0 < n(x) < 1$ for $x \in \overline{D}$ and k is a transmission eigenvalue, all that can be said is that

$$k^2 > \lambda_0(D).$$

Open Problem: Obtain a lower bound for $\sup_D n$ when $0 < n(x) < 1$ for $x \in \overline{D}$.

Remark: The linear sampling method can be expected to fail when k is a transmission eigenvalue.
In particular, since D contains the origin, the norm of the (regularized) solution to

$$(Fg)(\hat{x}) = 1/4\pi$$

should be large for such values of k.

Scattering by an Anisotropic Medium

$$\nabla \cdot A\nabla u + k^2 v = 0 \qquad \text{in} \quad D$$

$$\Delta u^s + k^2 u^s = 0 \qquad \text{in} \quad \mathbb{R}^2 \setminus \overline{D}$$

$$v - u^s(x) = e^{ikx\cdot d} \qquad \text{on} \quad \partial D$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial u^s}{\partial \nu} = \frac{\partial e^{ikx\cdot d}}{\partial \nu} \qquad \text{on} \quad \partial D$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

A is a symmetric matrix with entries $a_{jk} \in C^1(\overline{D})$,

$$\overline{\xi} \cdot A\xi \geq \gamma|\xi|^2, \gamma > 0, \quad \text{and} \quad \frac{\partial v}{\partial \nu_A} = \nu \cdot A\nabla v.$$

Scattering by an Anisotropic Medium

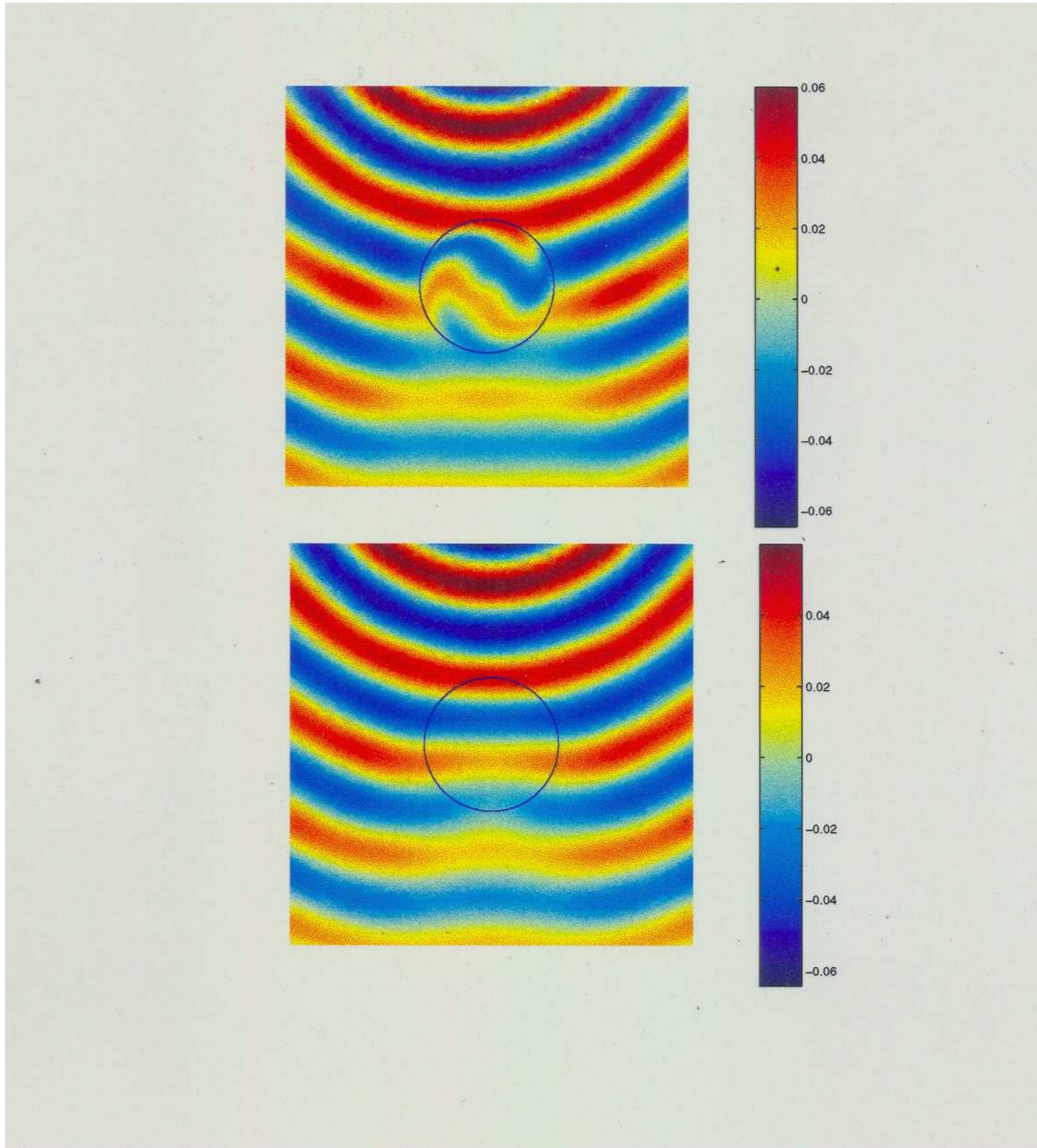
The scattered field u^s again has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O\left(r^{-3/2}\right)$$

as $r \to \infty$.

However $u_\infty(\hat{x}, d)$ does not uniquely determine A even if it is known for an interval of values of k!

Non-uniqueness for anisotropic media



Scattering by an Anisotropic Medium

Theorem: D is uniquely determined by $u_\infty(\hat{x}, d)$ for $x, d \in \Omega$ and a fixed value of the wave number k.

D can again be determined from u_∞ by using the linear sampling method, i.e. by solving the ill-posed first kind equation

$$\int_\Omega u_\infty(\hat{x}, d) g(d) ds(d) = \Phi_\infty(\hat{x}, z).$$

What, if anything, can be said about A from a knowledge of u_∞?

Scattering by an Anisotropic Medium

Theorem: Suppose that $\overline{\xi} \cdot A^{-1}\xi \geq \beta|\xi|^2, \; \beta > 1$. Then

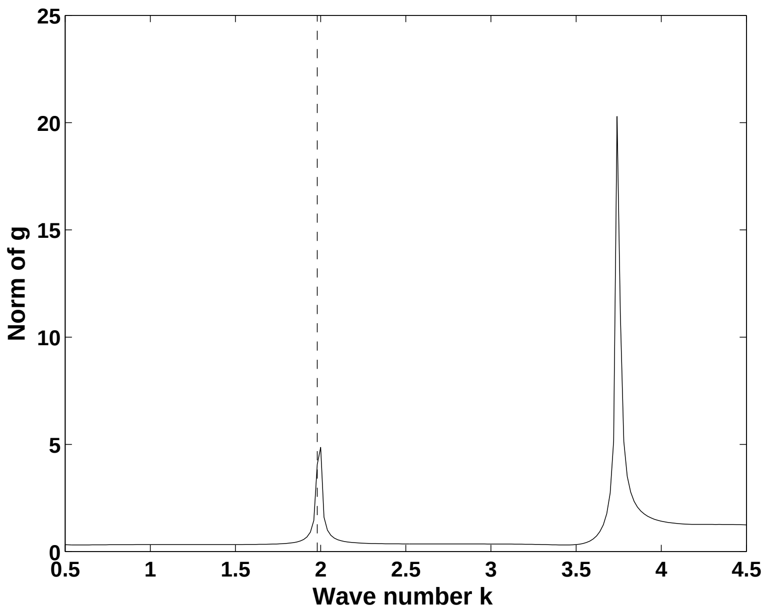
$$\text{sup}_D \|A^{-1}\|_2 > \frac{\lambda_0(D)}{k^2}$$

where k is a transmission eigenvalue and $\lambda_0(D)$ is the first Dirichlet eigenvalue of $-\Delta_2$ in D.

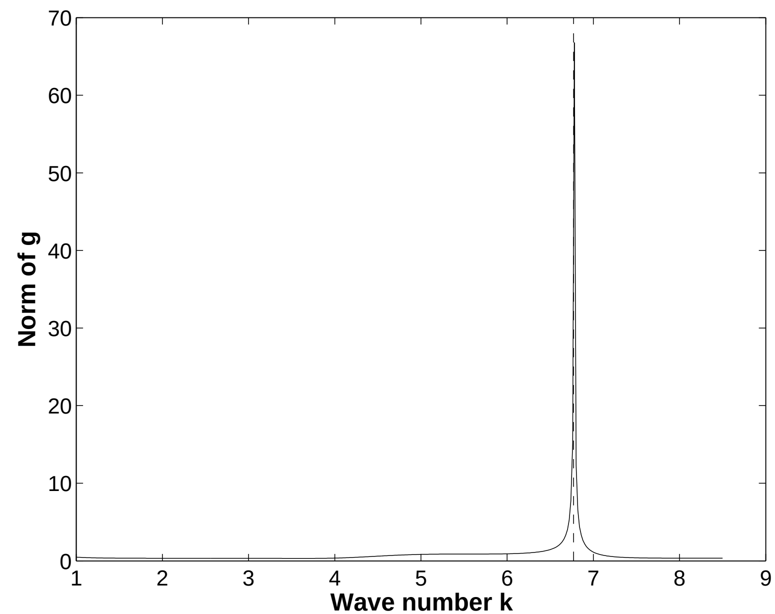
Note that $\lambda_0(D)$ can be computed since D is reconstructed and the first transmission eigenvalue k can be estimated from the far field equation.

Numerical Examples

D is a circle of diameter 1



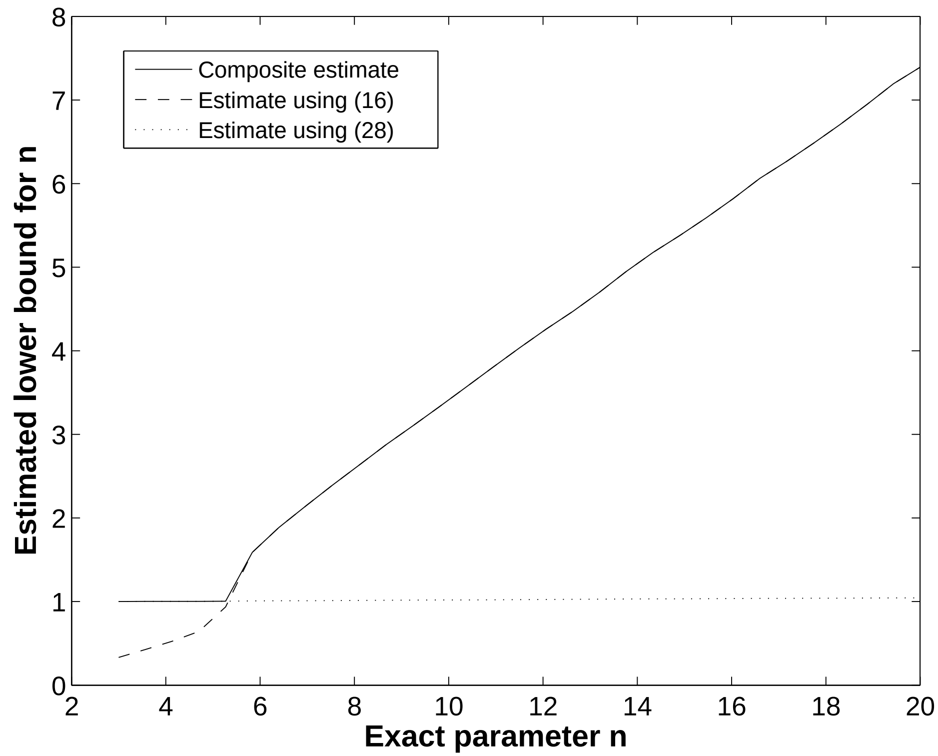
$n = 16$



$n = 4$

Numerical Examples

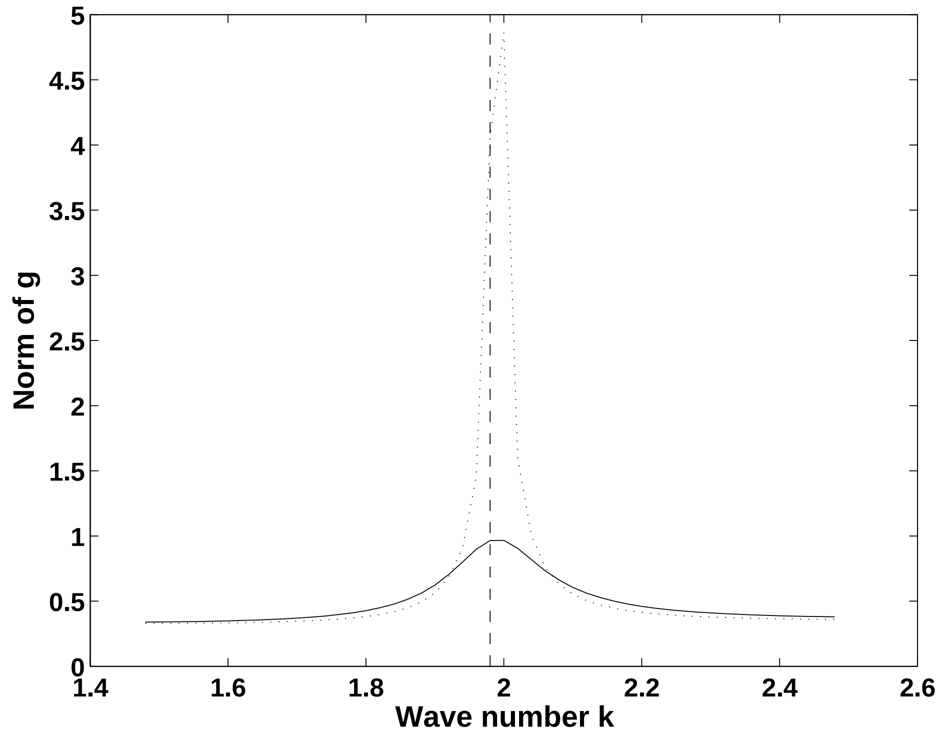
D is a circle of diameter 1



Solid line is max$\{1, \lambda_0(D)/k^2\}$.

Numerical Examples

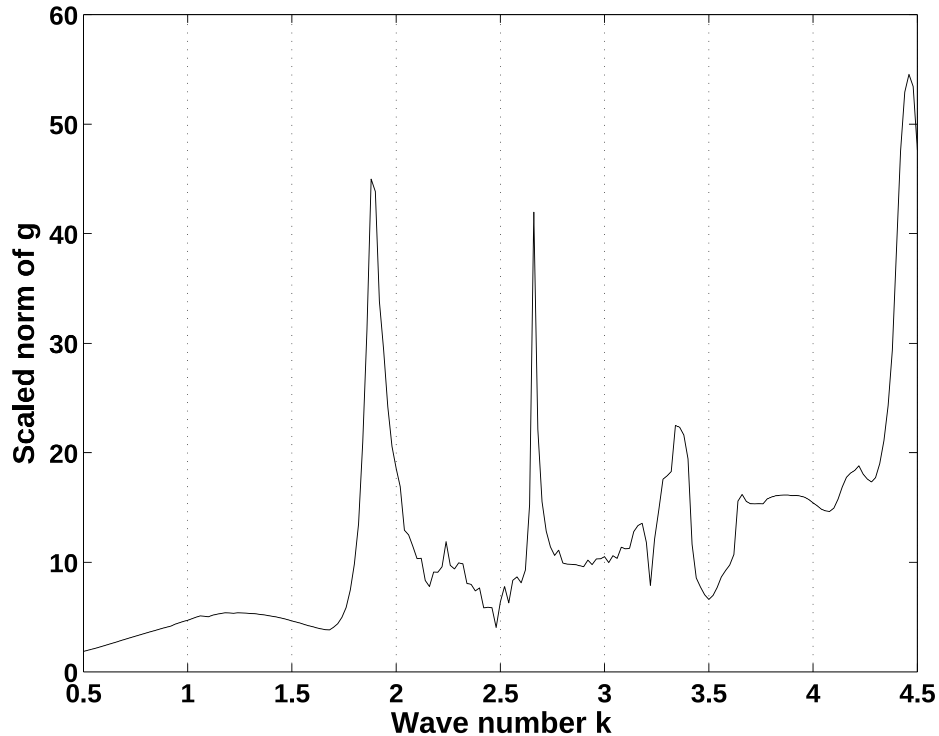
D is a circle of diameter 1



$$n = 16 + i$$

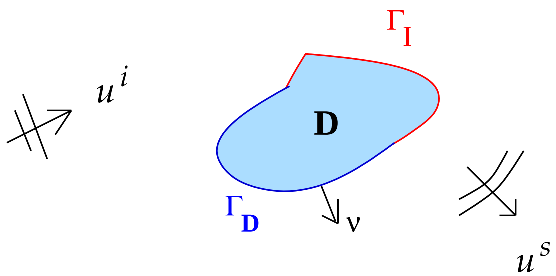
Numerical Examples

D is the rectangle $[-0.5,\ 0.5] \times [-0.4,\ 0.4]$, $\lambda_0(D) \approx 25.3$.



$n = 16$. Our estimate is $n \geq \dfrac{25.3}{1.88^2} \approx 7.1$

Obstacle Scattering



Let $\lambda \in L^\infty(\Gamma_I)$.

The total field u satisfies

$$\Delta_2 u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{D}$$

$$u = 0 \quad \text{on} \quad \Gamma_D$$

$$\frac{\partial u}{\partial \nu} + \lambda(x) u = 0 \quad \text{on} \quad \Gamma_I$$

$$u(x) = e^{ikx\cdot d} + u^s(x)$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

The inverse obstacle scattering problem

The scattered field u^s once more has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O\left(r^{-3/2}\right)$$

as $r \to \infty$, where $\hat{x} = x/|x|$ and $u_\infty(\hat{x}, d)$ is the far field pattern of the scattered field u^s .

The inverse scattering problem is to determine D and λ from a knowledge of $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$.

Solution of the inverse problem

D can again be determined from u_∞ by using the linear sampling method, i.e. by solving the ill-posed first kind equation

$$\int_\Omega u_\infty(\hat{x}, d) g(d) ds(d) = \Phi_\infty(\hat{x}, z).$$

The Herglotz function with kernel the solution g of the far field equation for z in D approximate the solution w of

$$\begin{aligned}
\Delta_2 w + k^2 w = 0 & \qquad \text{in} \quad D \\
w + \Phi(\cdot,\, z) = 0 & \qquad \text{on} \quad \Gamma_D \\
\frac{\partial(w + \Phi(\cdot,\, z))}{\partial \nu} + \lambda(x)(w + \Phi(\cdot,\, z)) = 0 & \qquad \text{on} \quad \Gamma_I
\end{aligned}$$

Determination of λ

Let $\tilde{\lambda}(x)$ is the extension by 0 of $\lambda(x)$ to the whole boundary ∂D. Define

$$\mathcal{V} := \{ f \in L^2(\Gamma_I) : \; f = w_z + \Phi(\cdot\,, z)|_{\Gamma_I}, \;\; z \in B_r \}$$

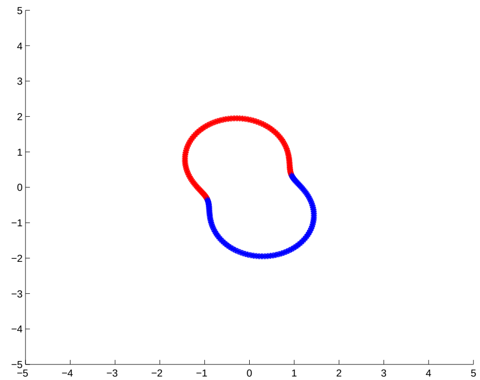
where $B_r \subset D$. Two important properties of $W_z := w_z + \Phi(\cdot\,, z)$ are:

- $\mathcal{V}$ is complete in $L^2(\Gamma_I)$.
- For any $z_1, z_2 \in D$

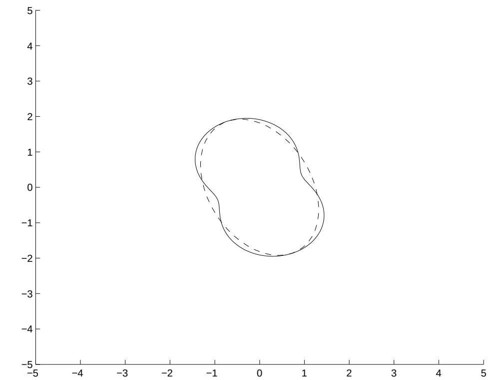
$$2 \int_{\partial D} \tilde{\lambda}(x) W_{z_1} \overline{W}_{z_2} ds = -\frac{1}{2} J_0(k|z_1 - z_2|) + i w_{z_1}(z_2) - i \overline{w}_{z_2}(z_1).$$

Remember that $v_g \approx w_z$ in $L^2(\Gamma_I)$, and g is computable form the data.

Examples of Reconstructions for λ



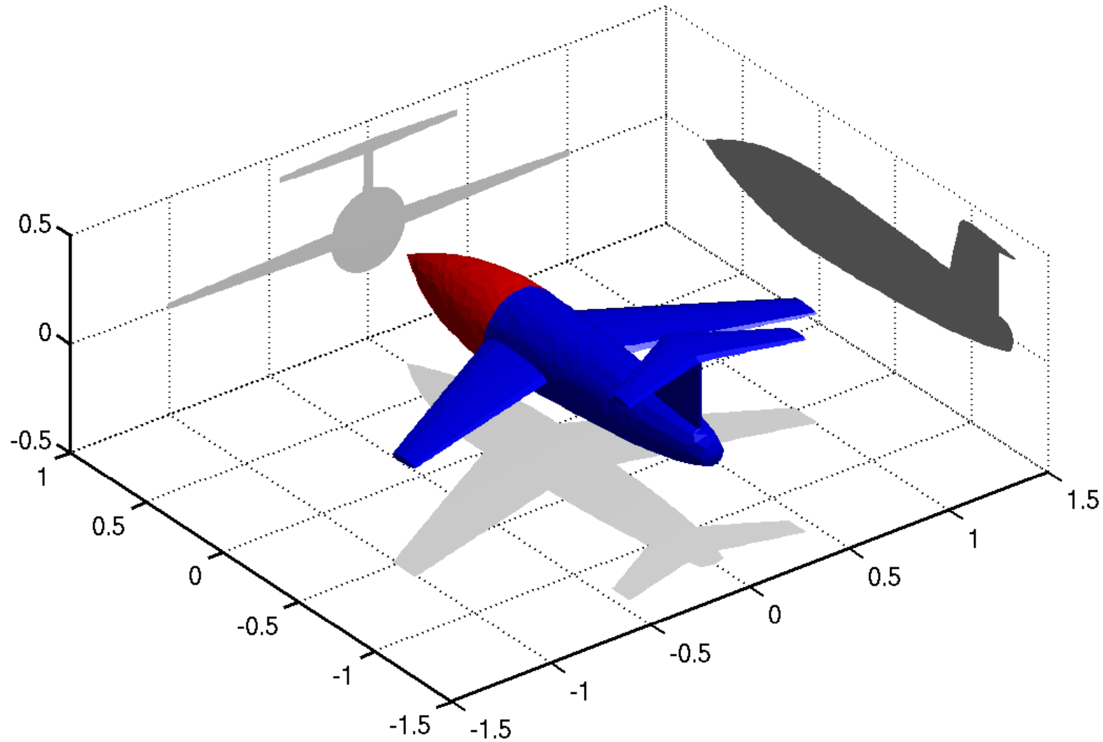
Boundary of the Scatterer



Approximated Boundary

	Maximum	Average	Median
λ=2 impedance	2.192	1.992	1.979
λ=2 imped., approx. bound.	2.395	1.823	1.886
λ=2 mixed conditions	2.595	2.207	2.257
λ=5 impedance	5.689	4.950	5.181
λ=5 imped., approx. bound.	5.534	4.412	4.501
λ=5 mixed conditions	5.689	4.950	5.180

$\mathbb{R}^3$-Examples of Reconstruction

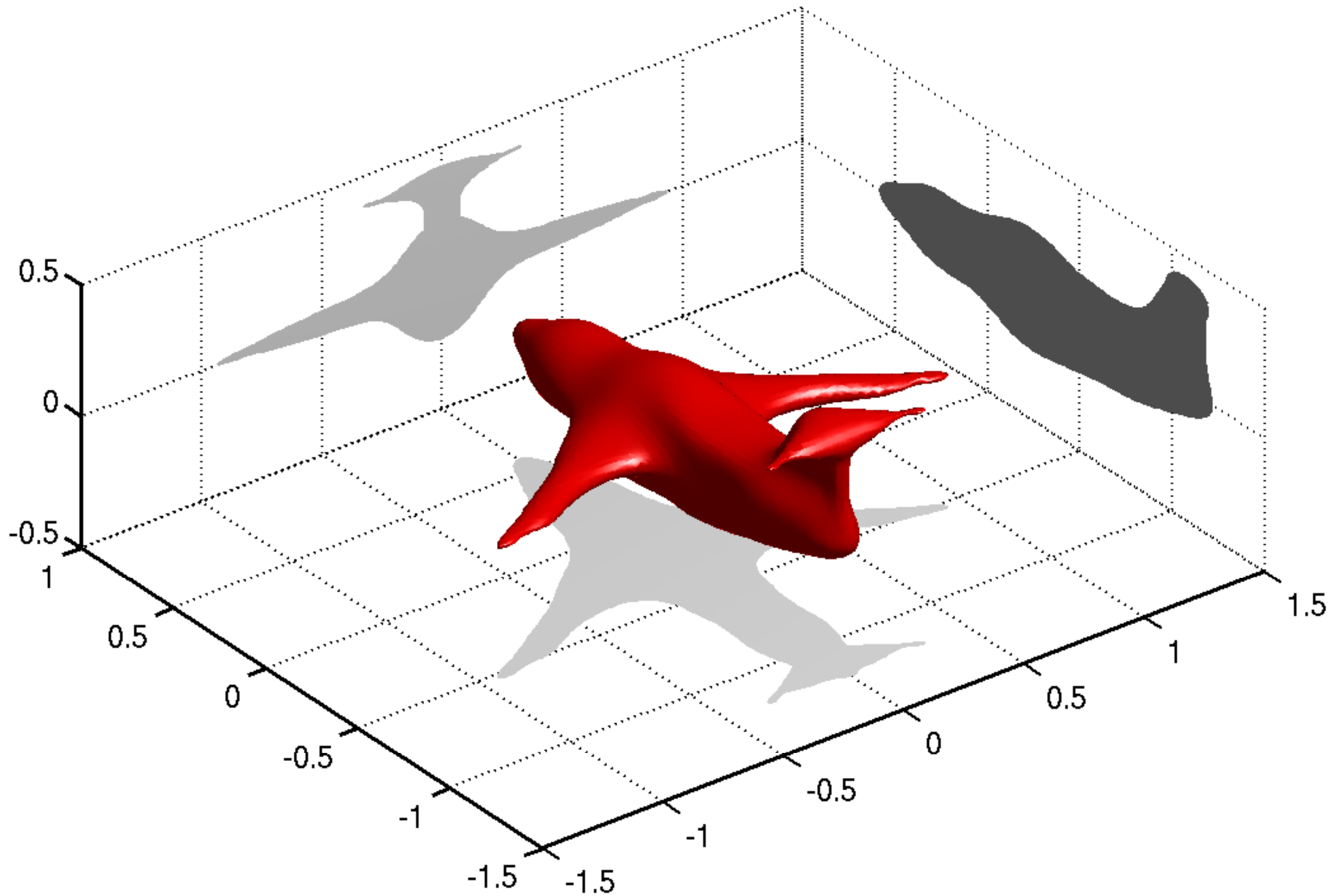


The exact geometry

Impedance boundary condition with $\lambda = 1$ is the red region

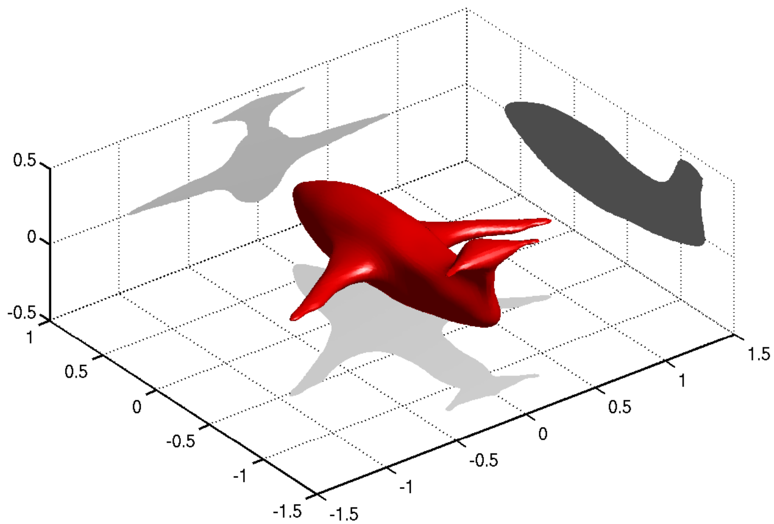
Perfectly conducting boundary condition is the blue region

$\mathbb{R}^3$-Examples of Reconstruction

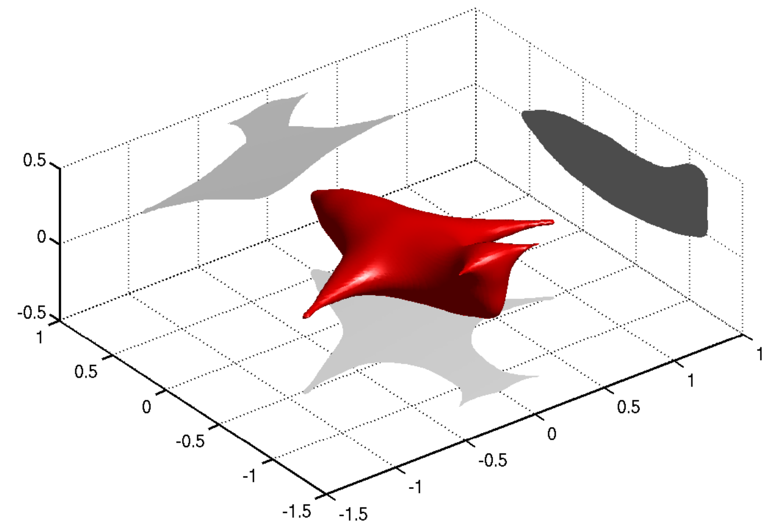


The reconstructed partially coated airplane (wavelength=0.7)

$\mathbb{R}^3$-Examples of Reconstruction

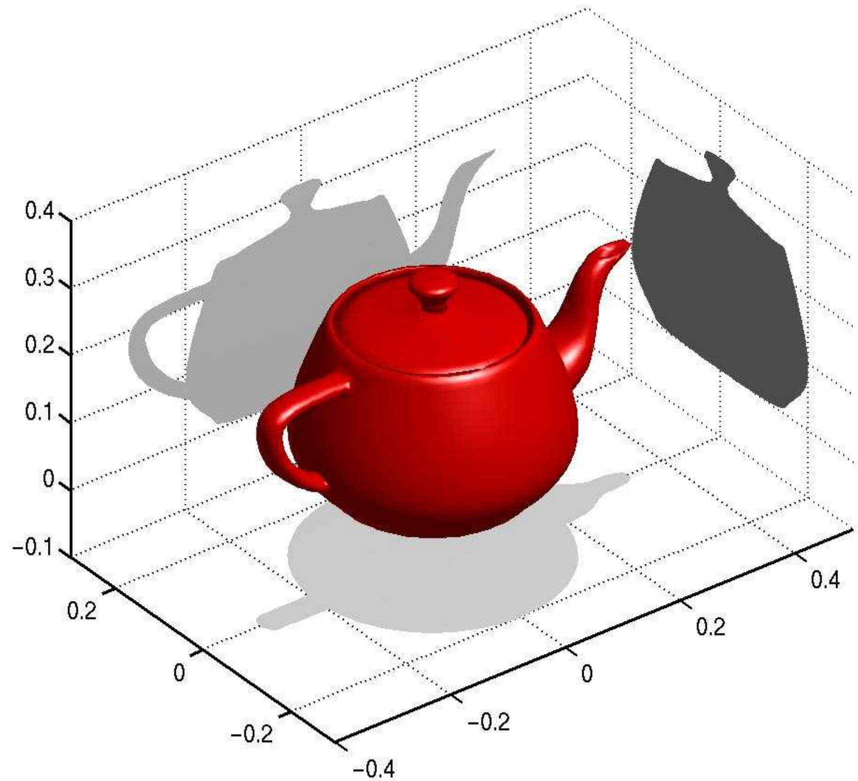


The perfectly conducting airplane



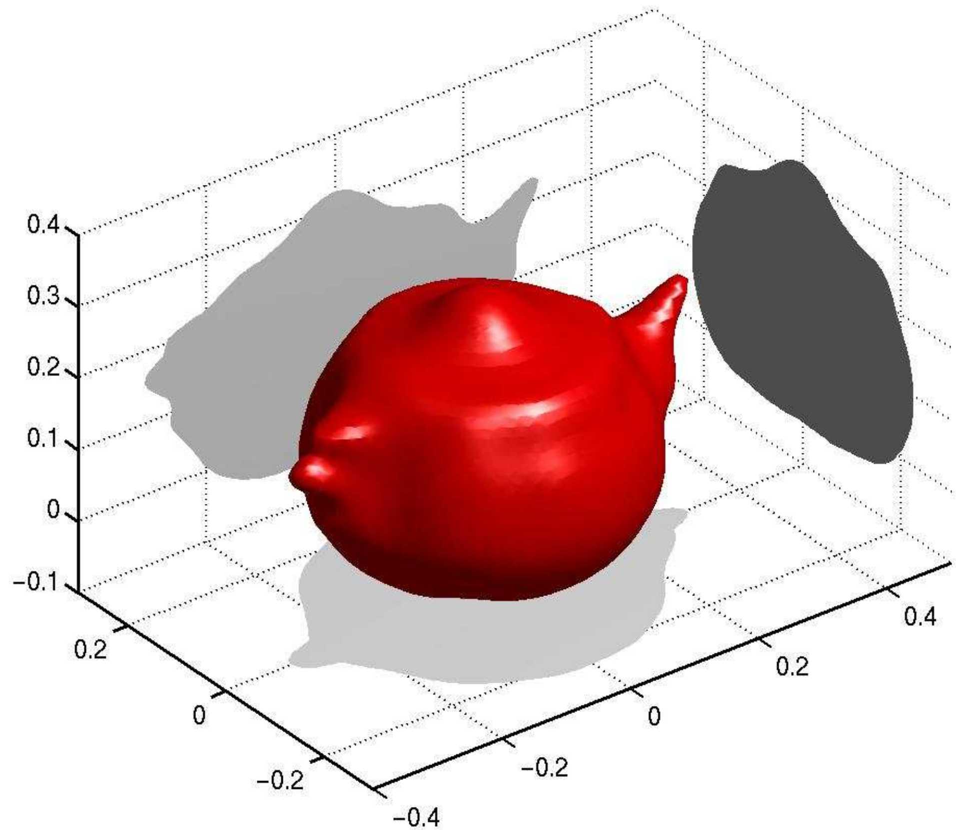
The imperfectly conducting airplane

$\mathbb{R}^3$-Examples of Reconstruction



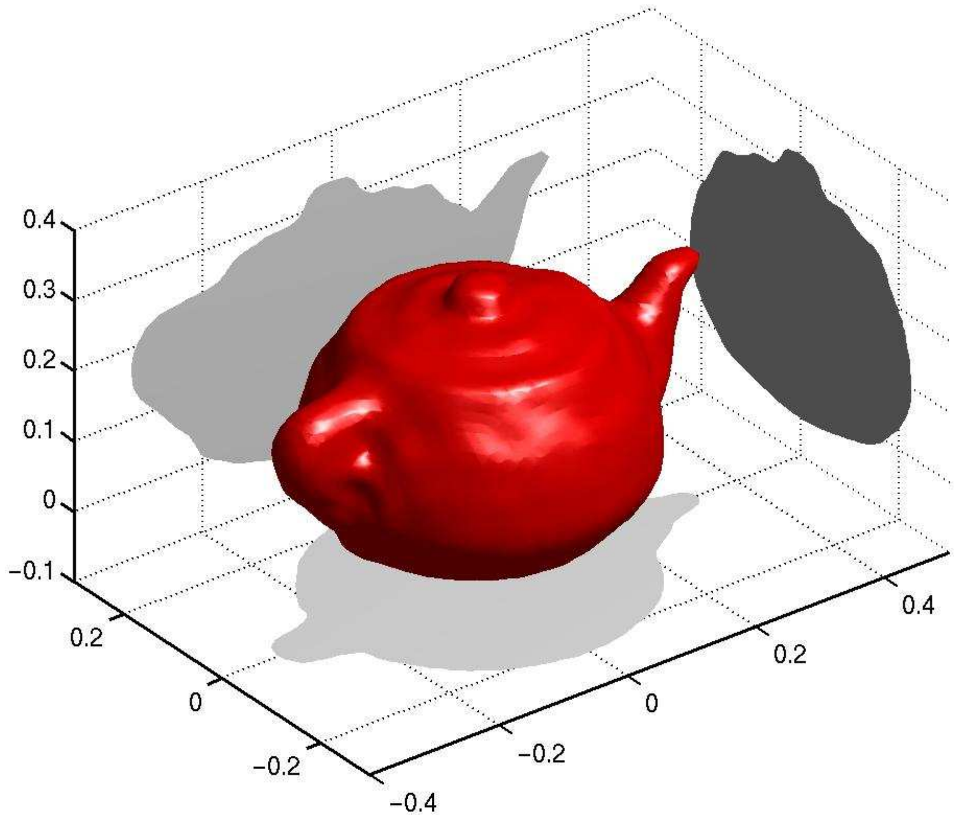
Perfectly conducting teapot, exact geometry

$\mathbb{R}^3$-Examples of Reconstruction



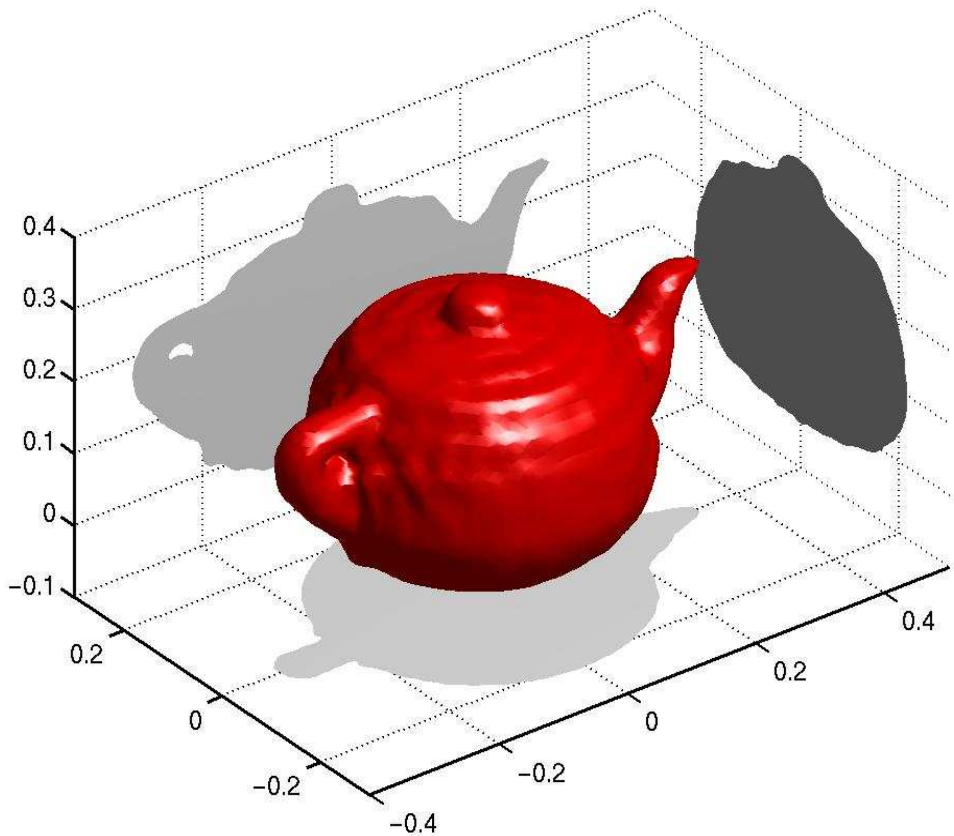
Reconstruction for low frequency

$\mathbb{R}^3$-Examples of Reconstruction



Reconstruction for intermediate frequency

$\mathbb{R}^3$-Examples of Reconstruction



Reconstruction for high frequency