Homework 1, Math 602, Spring 2009, due Wednesday, February 25.

1. Suppose f is a continuous function on $[a,b]$ such that $\int_a^b f(x)x^n\,dx = 0$ (Riemann integral) for every non-negative integer n. Prove that $f = 0$.

2. Suppose $k(x,y)$ is a continuous function on $[a,b]\times[c,d]$. For any Riemann Integrable function f on $[c,d]$, define the function

$$(Kf)(x) = \int_c^d k(x,y)\,f(y)\,dy \qquad x \in [a,b].$$

   (a) Prove that Kf is continuous.

   (b) Prove that $\overline{E}$ is a compact subset of $C[a,b]$ where

$$E = \{\, Kf \;:\; f \text{ Riemann Integrable on } [c,d],\ \int_c^d |f| \le 1 \,\}.$$

3. Consider the space $C^1[a,b]$ of continuously differentiable functions on $[a,\,b]$ equipped with the distance

$$d(f,g) = \max_{x\in[a,\,b]} \{|f(x)-g(x)| + |f'(x)-g'(x)|\}\,.$$

   (a) Prove that $C^1[a,b]$ is a complete metric space.

   (b) Let E be a subset of $C^1[a,b]$. Then $\overline{E}$ is a compact subset of the metric space $C^1[a,b]$ if and only if

   - ("Uniform boundedness") There is a real number M so that for all $f \in E$,

$$\sup_{x\in[a,b]} |f(x)| + |f'(x)| \le M.$$

     and

   - ("Equicontinuity") Given $\epsilon > 0$, there is a $\delta > 0$, so that for all $x, y \in [a,b]$ and all $f \in E$,

$$|f(x)-f(y)| + |f'(x)-f'(y)| < \epsilon \qquad \text{if } |x-y| < \delta.$$

   Recall: A subset K of a metric space (X,d) is compact if and only if it is sequentially compact, e.g. every sequence in K has a limit point in K.

4. Exercises 10.2, 10.3 and 10.4 from Apostol.