

# Math 602 Exam 1

Must turn in on **Friday, April 10, not later than 4:30 pm**

*This is a take home exam. You are supposed to work **individually** and have **NO** discussion with your classmates and others. Please, present your work clearly.*

## Problem 1 (30 points)

1. Prove that the integral

$$\int_0^1 \frac{x \log x}{(1+x)^2} dx$$

exists as a Lebesgue integral.

2. Determine those values of  $p$  and  $q$  for which the following integral

$$\int_0^1 x^p (1-x^2)^q dx$$

exists as Lebesgue integral.

*You must show that for a range of  $p$  and  $q$  the integral exists and for the other values of  $p$  and  $q$  does not exist.*

3. Show that the integral

$$\int_0^\infty \frac{\cos x}{\sqrt{x}} dx$$

exist as a Riemann improper integral but not as Lebesgue integral.

## Problem 2 (20 points)

1. Let  $f \geq 0$  be a Lebesgue integrable function on a generalized interval  $I$ . Prove that  $\int_I f = 0$  **if and only if**  $f = 0$  almost everywhere in  $I$ .
2. Let  $\{f_n\}$  be a sequence of Lebesgue integrable nonnegative functions in on  $I$  such that  $0 \leq f_1 \leq f_2 \leq \dots$  almost everywhere on  $I$ . Show that

$$\lim_{n \rightarrow \infty} \int_I f_n = 0$$

**if and only if**  $\lim_{n \rightarrow \infty} f_n(x) = 0$  almost everywhere in  $I$ .

### Problem 3 (30 points)

1. Let

$$f_n(x) = \frac{nx}{(1+n^2x^2)} \quad x \in [0, 1]$$
$$g_n(x) = \frac{n^{3/2}x}{(1+n^2x^2)} \quad x \in [0, 1].$$

Show that the Lebesgue Bounded Convergence Theorem (Th 10.29) applies to  $\{f_n\}$  and the Lebesgue Dominated Convergence Theorem (Th 10.27) applies to  $\{g_n\}$ . Apply the respective theorems.

2. It is easy to guess the limits of

$$\int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2} dx \quad \text{and} \quad \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx$$

as  $n \rightarrow \infty$ . Prove that your guesses are correct.

3. Define the following sequence of functions on  $R = (-\infty, \infty)$

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{when } -n \leq x \leq n \\ 0 & \text{otherwise} \end{cases}$$

Show that  $\lim_{n \rightarrow \infty} f_n(x) = 0$  **uniformly** on  $R$ . Show that

$$\lim_{n \rightarrow \infty} \int_R f_n \neq \int_R \lim_{n \rightarrow \infty} f_n.$$

Which assumption of the Lebesgue dominated convergence theorem is violated? (This example shows that uniform convergence does not imply the Lebesgue dominated convergence theorem if the interval is unbounded).

### Problem 4 (20 points for part 1 + part 3)

(Application of the Levi Monotone Convergence Theorem).

1. Prove Fatou's lemma: Given a sequence  $\{f_n\}$  of **nonnegative functions** in  $L(I)$  (i.e. Lebesgue integrable) such that:

- (a)  $\{f_n\}$  converges almost everywhere on  $I$  to a limit function  $f$ .  
 (b)  $\int_I f_n \leq A$  for some constant  $A > 0$  and all  $n \geq 1$  ( $A$  is independent of  $n$ ).

Then the limit function  $f \in L(I)$  (i.e.  $f$  is Lebesgue integrable) and  $\int_I f \leq A$ .

*Hint: To prove the result define  $g_n(x) = \inf\{f_n(x), f_{n+1}(x), \dots\}$  and apply Theorem 10.24 to  $g_n$ .*

2. (Extra Credits) Prove the extension of Fatou's lemma: Given a sequence  $\{f_n\}$  of **nonnegative functions** in  $L(I)$  and denote by

$$f(x) = \liminf_{n \rightarrow \infty} f_n(x), \quad x \in I.$$

Then  $f \in L(I)$  and

$$\int_I f \leq \liminf_{n \rightarrow \infty} \int_I f_n \quad (*).$$

*Hint: Recall from Math 600 that, for a sequence  $\{c_n\}$ ,  $\liminf_{n \rightarrow \infty} c_n$  (lower limit) is the smallest limit point of  $\{c_n\}$  or the smallest of subsequential limits of  $\{c_n\}$  (see Definition 8.2). Construct the same  $g$  as in part 1. to prove the result.*

3. The following example show that strict inequality may hold in (\*). Take  $I = [0, 1]$  and

$$g(x) = \begin{cases} 0 & 0 \leq x \leq 1/2 \\ 1 & 1/2 < x \leq 1 \end{cases}$$

Define the sequence  $\{f_n\}$  on  $I$  by  $f_{2k}(x) = g(x)$  and  $f_{2k+1}(x) = g(1-x)$ . Show that

$$\liminf_{n \rightarrow \infty} \int_I f_n(x) = 0 \quad x \in [0, 1]$$

but

$$\int_{[0,1]} f_n = 1/2.$$