

Math 349 Homework 3, Spring 2008

Due Tuesday, October 14

1. Given the matrices $A_{4 \times 4}$, $B_{3 \times 4}$, $C_{4 \times 3}$, $D_{3 \times 3}$, $F_{2 \times 3}$, $G_{2 \times 4}$. Determine whether the following operations make sense. If yes what is the size of the resulting matrix?

$$A \cdot B, \quad B \cdot A, \quad A \cdot B^T - D, \quad F \cdot B + 4G$$
$$(A \cdot C \cdot D)^T + 2B, \quad A \cdot (B^T + C) \cdot D, \quad (A \cdot G^T) - (C \cdot D \cdot F^T).$$

2. Given

$$A = \begin{pmatrix} 3 & 0 \\ -1 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 4 & 2 \end{pmatrix} \quad E = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Compute (if it is possible)

$$B - C, \quad B - C^T, \quad BA, \quad B^T C^T - (CB)^T, \quad DAF, \quad (I_2 - A)^2.$$

3. Verify $(ABC)^T = (C^T B^T A^T)$ where

$$A = \begin{pmatrix} 5 & 3 & 0 \end{pmatrix} \quad B = \begin{pmatrix} -2 & 1 \\ 6 & 4 \\ 5 & 3 \end{pmatrix} \quad C = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

4. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. Show that $A^2 = 5I_3 + 2 \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

5. Consider

$$M_1 = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ \sin \theta & -\cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Compute $P = M_1 M_2$. Show that $P^2 = I_3$.

6. (a) Give an example of non-zero 2×2 matrix A such that $A^2 = O$ where O is the zero 2×2 matrix.
- (b) Let $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$. Find nonzero 2×2 matrices B, C such that $AB = AC$ but $B \neq C$.
7. A matrix is symmetric if it is equal to its transpose. Let A be an $m \times n$ matrix.
- (a) Explain why the matrix multiplications $A^T A$ and AA^T are always possible.
- (b) Show that $A^T A$ and AA^T are both symmetric.
8. Find the inverse A^{-1} of the following matrices

$$(a) A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 0 & 3 & -1 \end{pmatrix} \qquad (b) B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{pmatrix}$$

9. The trace of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of the entries on its main diagonal and is denoted by $tr(A)$, i.e.

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

Show that if A and B are $n \times n$ matrices, then $tr(AB) = tr(BA)$.

10. (**Honor's Credits**) If A is an $n \times n$ matrix, then A is called **idempotent** if $A^2 = A$. Let now P and Q be $n \times n$ idempotent matrices.
- (a) Find a 2×2 idempotent matrix.
- (b) Show that P^T is idempotent.
- (c) Show that PQ is idempotent if $PQ = QP$.
- (d) Is $P + Q$ idempotent? Justify your answer.
- (e) Prove that the only invertible idempotent $n \times n$ matrix is the identity matrix.
11. (**Honor's Credits**) Show that there are no 2×2 matrices A and B such that $AB - BA = I_2$.
12. (**Honor's Credits**) A square matrix is called **skew-symmetric** if $A^T = -A$.

- (a) Prove that if A is an $n \times n$ matrix, then $A - A^\top$ is skew-symmetric.
- (b) Use part (a) and Theorem 3.5 in the book to prove that any matrix A can be written as the sum of a symmetric matrix and antisymmetric matrix.
- (c) Illustrate part (b) for the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$