Math349 Homework 1, Fall 2008

Due Thursday, September 18

1. For the given vectors, compute $3\mathbf{u} - 2\mathbf{v}$, $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $(\mathbf{u} \cdot \mathbf{v})$ and the angle θ between $\mathbf{u}$ and $\mathbf{v}$. Normalize $\mathbf{u}$ (i.e. compute the unit vector $\frac{\mathbf{u}}{\|\mathbf{u}\|}$).

   (a)
$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \qquad \text{in } \mathbb{R}^3.$$

   (b)
$$\mathbf{u} = [3,\ 2,\ 0,\ -1,\ 1]\,, \qquad \mathbf{v} = [-5,\ 0,\ 0,\ 2,\ 4] \qquad \text{in } \mathbb{R}^5.$$

2. Using vector algebra, solve the vector $\mathbf{x} \in \mathbb{R}^n$ in terms of vectors $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n$
$$\mathbf{x} + 2\mathbf{a} - \mathbf{b} = 3(\mathbf{x} + \mathbf{a}) - 2(2\mathbf{a} - \mathbf{b}).$$

3. Let $\mathbf{u}, \mathbf{v}$ be two vectors in $\mathbb{R}^n$.

   (a) Prove that
$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\,(\mathbf{u} \cdot \mathbf{v})\,.$$
   Use this relation to compute $\|\mathbf{u} + \mathbf{v}\|$ if $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = \sqrt{3}$ and $(\mathbf{u} \cdot \mathbf{v}) = 1$.

   (b) Prove that
$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.$$

4. Find all values of k for which the two vectors are orthogonal
$$\mathbf{u} = [3,\ -3,\ 2,\ 1]\,, \qquad \mathbf{v} = \left[-2,\ k,\ 1,\ k^2\right] \qquad \text{in } \mathbb{R}^4.$$

5. Find the projection of $\mathbf{v}$ onto $\mathbf{u}$ and the distance of $\mathbf{v}$ from $\mathbf{u}$
$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \qquad \text{in } \mathbb{R}^3.$$

6. (a) Find the equation of the line in $\mathbb{R}^2$ passing through $P = (1,\ 2)$ with normal vector $\mathbf{n} = [5,\ -3]$.

(b) Find the equation of the plane in $\mathbb{R}^3$ passing through $P = (0,\ 0,\ 0)$ with normal vector $\mathbf{n} = [2,\ -3,\ 1]$.

7. (a) Find the equation of the line in $\mathbb{R}^2$ passing through $P = (1,\ -1)$ with directional vector $\mathbf{d} = [1,\ -2]$.

   (b) Find the equation of the plane in $\mathbb{R}^3$ passing through $P = (1,\ 2,\ 1)$ with directional vectors $\mathbf{u} = [1,\ -1,\ 1]$ and $\mathbf{v} = [1,\ 0,\ -1]$.

8. **For Honor's Credit** Follow the following steps to prove the Cauchy-Schwartz Inequality

$$|(\mathbf{u} \cdot \mathbf{v})| \leq \|\mathbf{u}\|\|\mathbf{v}\|, \quad \text{for any two vectors } \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$$

   - **Step 1** Let x be an arbitrary scalar. Show that

$$0 \leq \|x\mathbf{u} + \mathbf{v}\|^2 = x^2\|\mathbf{u}\|^2 + 2x\,(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^2 \qquad (*)$$

   - **Step 2** Consider the right hand side of $(*)$ as a binomial for x that does not change sign (note that a binomial $y = ax^2 + 2bx + cx$ does not change sign, i.e. has no real zeros or only one zero, if and only if $b^2 - 4ac \leq 0$). Then conclude that

$$4\,(\mathbf{u} \cdot \mathbf{v})^2 - 4\|\mathbf{u}\|^2\|\mathbf{v}\|^2 \leq 0$$

     and from here deduce the Cauchy-Schwartz Inequality.

   Are there vectors $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$ such that $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = 3$ and $(\mathbf{u} \cdot \mathbf{v}) = 7$? Justify your answer.

9. **For Honor's Credit** Prove that $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal in $\mathbb{R}^n$ if and only if $\|\mathbf{u}\| = \|\mathbf{v}\|$.