THE EXISTENCE OF AN INFINITE DISCRETE SET OF TRANSMISSION EIGENVALUES*

FIORALBA CAKONI †, DROSSOS GINTIDES ‡, AND HOUSSEM HADDAR§

Abstract. We prove the existence of an infinite discrete set of transmission eigenvalues corresponding to the scattering problem for isotropic and anisotropic inhomogeneous media for the Helmholtz and Maxwell's equations. Our discussion includes the case of the interior transmission problem for an inhomogeneous medium with cavities, i.e. subregions with contrast zero.

Key words. Interior transmission problem, transmission eigenvalues, inhomogeneous medium, inverse scattering.

AMS subject classifications. 35R30, 35Q60, 35J40, 78A25.

1. Introduction.

The interior transmission problem arises in inverse scattering theory for inhomogeneous media. It is a boundary value problem for a set of equations defined in a bounded domain coinciding with the support of the scattering object. Of particular interest is the spectrum associated with this boundary value problem, more specifically the existence of eigenvalues which are called transmission eigenvalues. On one hand, in the context of sampling methods for reconstructing the support of the scatterer [2], [17], one needs to avoid those frequencies that correspond to transmission eigenvalues, and hence it is important to know that the transmission eigenvalues form a discrete set. On the other hand, one can use transmission eigenvalues to obtain information about physical properties of the scattering object [1], [4], [6] and therefore it is important to know whether they exist and to understand their connection with the index of refraction. This application is based on the recent results in [3] which justify the numerical observation that the transmission eigenvalues can be computed from the far field data. Either way, the investigation of the spectral properties of the interior transmission problem has become an interesting question in inverse scattering theory.

The interior transmission problem was first introduced in [12] in connection with an inverse scattering problem for acoustic waves. Roughly speaking, two main approaches are available in the study of the well-posedness of the interior transmission problem, namely integral equation methods [10], [15], and variational methods [5], [6], [8], [14]. Until recently the only known result on transmission eigenvalues was the fact that they from at most a discrete set with infinity as the only possible accumulation point. The first result about the existence of transmission eigenvalues was announced in [18] for the case of the reduced wave equation in an isotropic inhomogeneous medium where it was shown that there exist a finite number of transmission eigenvalues provided that the index of refraction is large enough. This paper was soon followed by [9], [16] where the same result was proven for anisotropic media and Maxwell's equations. Subsequently the difficult case of a medium with cavities, i.e.

*The research of F.C. was supported in part by the U.S. Air Force Office of Scientific Research under Grant FA 9550-05-1-0127.

†Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716-2553, USA (cakoni@math.udel.edu).

‡Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece (dgindi@math.ntua.gr).

§INRIA Saclay Ile de France / CMAP Ecole Polytechnique, Route de Saclay, 91128 Palaiseau Cedex, France (haddar@cmapx.polytechnique.fr).

regions with zero contrast, was investigated in [5]. We refer the reader to [13] for a comprehensive review on the interior transmission problem. Further progress on the question of the existence of transmission eigenvalues was recently made in [7] where the assumption on the size of the index of refraction was removed and for the case of medium with constant index of refraction it was proven that there exists an infinite discrete set of transmission eigenvalues.

In this paper we will extend the ideas of [7] to show that there exist an infinite discrete set of transmission eigenvalues for inhomogeneous isotropic and anisotropic media for both the Helmholtz and Maxwell's equations including the case of media with cavities. The only assumption we impose is that the index of refraction is less then or greater than the index of refraction of the background medium. Our proof employees the analytical framework developed in [6] and [9] and makes use of transmission eigenvalues for balls with constant index of refraction first used in [7]. We will also provide lower and upper bounds for the first transmission eigenvalue.

The plan of our paper is as follows. Having set up the analytic framework, we first show the existence of infinitely many transmission eigenvalues and lower and upper bounds for the first transmission eigenvalue for the case of isotropic inhomogeneous media for the Helmholtz equation. Then in Section 2.3 we provide similar results for the case of anisotropic media for both the Helmholtz and Maxwell's equations. Finally, we discuss the case of media with voids, i.e. with subregions with zero contrast, for which we also prove the existence of infinitely many transmission eigenvalues. Of potential use in non-destractive testing [1] is a new upper bound for the first transmission eigenvalue provided by our analysis.

Although, the results of this paper provide an important step forward in understanding the spectral properties of the interior transmission problem, many questions still remain. We think that some interesting open problems in this direction are the existence of transmission eigenvalues for the case of media with contrast partly positive and partly negative, the existence of complex transmission eigenvalues and the completeness of the eigensystem of the interior transmission problem.

2. The existence of an infinite set of transmission eigenvalues. We consider the interior transmission eigenvalue problem corresponding to the scattering problem for inhomogneous isotropic and anisotropic media for the Helmholtz equation as well as for the Maxwell's equations. Our goal is to prove the existence of infinitely many transmission eigenvalues and provide some estimates for these transmission eigenvalues. Throughout this section, $D \subset \mathbb{R}^d$, $d = 2, 3$ is a bounded simply connected region with piece-wise smooth boundary ∂D and ν denotes the outward normal vector to ∂D.

2.1. Abstract analytical framework. The interior transmission eigenvalue problems we discuss in this paper can be described by the following abstract analytical framework which is introduced in [9]. In particular, let U be a separable Hilbert space with scalar product $(\cdot, \cdot)$, $\mathbb{A}$ be a bounded, positive definite and self-adjoint operator on U and let $\mathbb{B}$ be a non negative, self-adjoint and compact bounded linear operator on U. Then there exists an increasing sequence of positive real numbers $(\lambda_j)_{j\geq 1}$ and a sequence $(u_j)_{j\geq 1}$ of elements of U such that $\mathbb{A}u_j = \lambda_j \mathbb{B} u_j$. The sequence $(u_j)_{j\geq 1}$ forms a basis of $(\mathbb{A} \ker(\mathbb{B}))^\perp$ and if $\ker(\mathbb{B})^\perp$ has infinite dimension then $\lambda_j \to +\infty$ as $j \to \infty$ (see Theorem 2.1 in [9]). Furthermore these eigenvalues satisfy a min-max principle (see Corollary 2.1 [9]), namely

$$\lambda_j = \min_{W\subset\mathcal{U}_j}\left(\max_{u\in W\setminus\{0\}}\frac{(\mathbb{A}u,u)}{(\mathbb{B}u,u)}\right) \tag{2.1}$$

where $\mathcal{U}_j$ denotes the set of all j dimensional subspaces W of U such that $W\cap\ker(\mathbb{B}) = \{0\}$. These eigenvalues can be ordered in the increasing order, i.e. $\lambda_1 \le \lambda_2 \le \dots$.

Let $\tau \longmapsto \mathbb{A}_\tau$ be a continuous mapping from $]0,\infty[$ to the set of self-adjoint and positive definite bounded linear operators on U and consider the generalized eigenvalue problem

$$\mathbb{A}_\tau u - \lambda_j(\tau)\mathbb{B}u = 0, \qquad u\in U. \tag{2.2}$$

Obviously from (2.1) we have that $\lambda_j(\tau)$ for every $j\in\mathbb{N}$ is a continuous function of τ in $]0,\infty[$. The following theorem proved in [9] provides the fundamental tool in proving the existence of transmission eigenvalues.

THEOREM 2.1. *Let $\tau \longmapsto \mathbb{A}_\tau$ be a continuous mapping from $]0,\infty[$ to the set of self-adjoint and positive definite bounded linear operators on U and let $\mathbb{B}$ be a self-adjoint and non negative compact bounded linear operator on U. We assume that there exists two positive constant $\tau_0 > 0$ and $\tau_1 > 0$ such that*

1. *$\mathbb{A}_{\tau_0} - \tau_0\mathbb{B}$ is positive on U,*
2. *$\mathbb{A}_{\tau_1} - \tau_1\mathbb{B}$ is non positive on a m dimensional subspace of U.*

Then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1,\dots,m$, has at least one solution in $[\tau_0,\tau_1]$ where $\lambda_j(\tau)$ is the j^{th} eigenvalue (counting multiplicity) of $\mathbb{A}_\tau$ with respect to $\mathbb{B}$, i.e. $\ker(\mathbb{A}_\tau - \lambda_j(\tau)\mathbb{B}) \neq \{0\}$.

2.2. The scalar isotropic media. The interior transmission eigenvalue problem corresponding to the scattering problem for an isotropic inhomogenous medium in $\mathbb{R}^d$, $d = 2,3$ reads:

$$\Delta w + k^2 n(x) w = 0 \quad \text{in} \quad D \tag{2.3}$$

$$\Delta v + k^2 v = 0 \quad \text{in} \quad D \tag{2.4}$$

$$w = v \quad \text{on} \quad \partial D \tag{2.5}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial D \tag{2.6}$$

for $w\in L^2(D)$ and $v\in L^2(D)$ such that $w - v \in H^2_0(D)$ where

$$H^2_0(D) = \left\{u\in H^2(D):\ u = 0 \quad \text{and} \quad \frac{\partial u}{\partial\nu} = 0 \quad \text{on} \quad \partial D\right\}.$$

Here we assume that the positive real valued function n is such that $n(x) > \alpha > 0$ almost everywhere in D, $n\in L^\infty(D)$ and $1/|n-1| \in L^\infty(D)$. Note that these assumptions are relevant form physical point of view (see the last section for the case of media with voids, i.e. where $n = 1$ in parts of D).

DEFINITION 2.2. Values of $k > 0$ for which the homogeneous interior transmission problem (2.3)-(2.6) has nonzero solutions $w\in L^2(D)$ and $v\in L^2(D)$ such that $w - v\in H^2_0(D)$, are called transmission eigenvalues. If $k > 0$ is a transmission eigenvalue, we call $u = w - v$ the corresponding eigenfunction, where w and v is a non zero solution of (2.3)-(2.6). It is possible to write (2.3)-(2.6) as an equivalent eigenvalue problem for $u = w - v \in H_0^2(D)$ for the following fourth order equation

$$\left(\Delta + k^2 n\right) \frac{1}{n-1} \left(\Delta + k^2\right) u = 0 \tag{2.7}$$

which in variational form is formulated as finding a function $u \in H_0^2(D)$ such that

$$\int_D \frac{1}{n-1}(\Delta u + k^2 u)(\Delta \overline{v} + k^2 n \overline{v})\, dx = 0 \qquad \text{for all } v \in H_0^2(D). \tag{2.8}$$

Following [9] we set $\tau := k^2$ and define the following bounded sesquilinear forms on $H_0^2(D) \times H_0^2(D)$:

$$\mathcal{A}_\tau(u,v) = \left(\frac{1}{n-1}(\Delta u + \tau u),\, (\Delta v + \tau v)\right)_D + \tau^2\,(u,\, v)_D\,, \tag{2.9}$$

$$\begin{aligned}\tilde{\mathcal{A}}_\tau(u,v) &= \left(\frac{1}{1-n}(\Delta u + \tau n u),\, (\Delta v + \tau n v)\right)_D + \tau^2\,(nu,\, v)_D \\ &= \left(\frac{n}{1-n}(\Delta u + \tau u),\, (\Delta v + \tau v)\right)_D + (\Delta u,\, \Delta v)_D\end{aligned} \tag{2.10}$$

and

$$\mathcal{B}(u,v) = (\nabla u,\, \nabla v)_D \tag{2.11}$$

where $(\cdot\,,\,\cdot)_D$ denotes the $L^2(D)$ inner product. Using the Riesz representation theorem we now define the bounded linear operators $\mathbb{A}_\tau : H_0^2(D) \to H_0^2(D)$, $\tilde{\mathbb{A}}_\tau : H_0^2(D) \to H_0^2(D)$ and $\mathbb{B} : H_0^2(D) \to H_0^2(D)$ by

$$(\mathbb{A}_\tau u, v)_{H^2(D)} = \mathcal{A}_\tau(u,v), \ \left(\tilde{\mathbb{A}}_\tau u, v\right)_{H^2(D)} = \tilde{\mathcal{A}}_\tau(u,v) \text{ and } (\mathbb{B}u, v)_{H^2(D)} = \mathcal{B}(u,v). \tag{2.12}$$

In terms of these operators we can rewrite (2.8) as

$$(\mathbb{A}_\tau u - \tau \mathbb{B} u, v)_{H^2(D)} = 0 \quad \text{or} \quad \left(\tilde{\mathbb{A}}_\tau u - \tau \mathbb{B} u, v\right)_{H^2(D)} = 0 \qquad \text{for all } v \in H_0^2(D). \tag{2.13}$$

The following result is proven in [9].

LEMMA 2.3. *The operators* $\mathbb{A}_\tau : H_0^2(D) \to H_0^2(D)$, $\tilde{\mathbb{A}}_\tau : H_0^2(D) \to H_0^2(D)$, $\tau > 0$ *and* $\mathbb{B} : H_0^2(D) \to H_0^2(D)$ *are self-adjoint. If for some constant* $\gamma > 0$ *and for almost all* $x \in D$, $\frac{1}{n(x)-1} > \gamma > 0$ *then* $\mathbb{A}_\tau$ *is a positive definite operator, whereas if* $\frac{n(x)}{1-n(x)} > \gamma > 0$ *then* $\tilde{\mathbb{A}}_\tau$ *is a positive definite operator. In addition,* $\mathbb{B}$ *is a positive compact operator.*

Obviously, $\mathbb{A}_\tau$ and $\tilde{\mathbb{A}}_\tau$ depend continuously on $\tau \in (0, +\infty)$. From the above discussion, $k > 0$ is a transmission eigenvalue if for $\tau = k^2$ the kernel of the operator $\mathbb{A}_\tau - \tau\mathbb{B}$ (if $1/(n-1) > \gamma > 0$) and the kernel of the operator $\tilde{\mathbb{A}}_\tau - \tau\mathbb{B}$ (if $n/(1-n) > \gamma > 0$) is non trivial. In order to analyze the kernel of these operators we consider the auxiliary eigenvalue problems

$$\mathbb{A}_\tau u - \lambda(\tau)\mathbb{B} u = 0 \qquad u \in H_0^2(D) \qquad \text{if} \quad 1/(n-1) > \gamma > 0 \tag{2.14}$$

and

$$\tilde{\mathbb{A}}_\tau u - \lambda(\tau)\mathbb{B}u = 0 \qquad u \in H_0^2(D) \qquad \text{if} \quad n/(1-n) > \gamma > 0. \tag{2.15}$$

Thus, a transmission eigenvalue $k > 0$ is such that $\tau := k^2$ solves $\lambda(\tau) - \tau = 0$ where $\lambda(\tau)$ is an eigenvalue corresponding to (2.14) or (2.15) in respective cases. Our goal is now to use Theorem 2.1 to prove the existence of an infinite set of transmission eigenvalues. To this end, we first recall another result proven [9]. Let $\lambda_1(D)$ be the first Dirichlet eigenvalue for $-\Delta$ in D.

LEMMA 2.4. *If $\frac{1}{n(x)-1} > \gamma > 0$ for some constant $\gamma > 0$ and for almost all $x \in D$, then*

$$(\mathbb{A}_\tau u - \tau\mathbb{B}u, u)_{H^2} \geq \alpha\|u\|^2_{H^2} > 0 \quad \text{for all} \quad 0 < \tau < \frac{\lambda_1(D)}{\sup_D(n)} \quad \text{and} \quad u \in H_0^2(D).$$

If $\frac{n(x)}{1-n(x)} > \gamma > 0$ for some constant $\gamma > 0$ and for almost all $x \in D$, then

$$\left(\tilde{\mathbb{A}}_\tau u - \tau\mathbb{B}u, u\right)_{H^2} \geq \alpha\|u\|^2_{H^2} > 0 \quad \text{for all} \qquad 0 < \tau < \lambda_1(D) \quad \text{and} \quad u \in H_0^2(D).$$

REMARK 2.1. The multiplicity of transmission eigenvalues is finite since, if k_0 is a transmission eigenvalue, then the kernel of $I - \tau_0\mathbb{A}_{\tau_0}^{-1/2}\mathbb{B}\mathbb{A}_{\tau_0}^{-1/2}$ or $I - \tau_0\tilde{\mathbb{A}}_{\tau_0}^{-1/2}\mathbb{B}\tilde{\mathbb{A}}_{\tau_0}^{-1/2}$ where $\tau_0 := k_0^2$ is finite since the operators $\tau_0\mathbb{A}_{\tau_0}^{-1/2}\mathbb{B}\mathbb{A}_{\tau_0}^{-1/2}$ and $\tau_0\tilde{\mathbb{A}}_{\tau_0}^{-1/2}\mathbb{B}\tilde{\mathbb{A}}_{\tau_0}^{-1/2}$ are compact and self-adjoint (if $1/(n-1) > \gamma > 0$) and (if $n/(1-n) > \gamma > 0$), respectively [19]. (Here $\mathbb{A}^{-1/2}$ is defined by $\mathbb{A}^{-1/2} = \int_0^\infty \lambda^{-1/2} dE_\lambda$ where dE_λ is the spectral measure associated with the positive operator $\mathbb{A}$.)

Now let us consider the interior transmission problem corresponding to a ball B_R of radius R centered at zero with constant index of refraction $n_0 > 0$ such that $n_0 \neq 1$, i.e.

$$\Delta w + k^2 n_0 w = 0 \quad \text{in} \quad B_R \tag{2.16}$$

$$\Delta v + k^2 v = 0 \quad \text{in} \quad B_R \tag{2.17}$$

$$w = v \quad \text{on} \quad \partial B_R \tag{2.18}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial B_R \tag{2.19}$$

By a separation of variables technique, it is possible to show [13] (see also [11]) that (2.16)-(2.19) has a countable discrete set of eigenvalues. Let k_{R,n_0} be the first transmission eigenvalue corresponding to B_R and n_0. Typically, this k_{R,n_0} is the first zero of

$$W(k) := \det\begin{pmatrix} J_0(kR) & J_0(k\sqrt{n_0}R) \\ -J_0'(kR) & -\sqrt{n_0}J_0'(k\sqrt{n_0}R) \end{pmatrix} = 0 \qquad \text{in} \quad \mathbb{R}^2 \tag{2.20}$$

where J_0 is the Bessel functions of order zero, and

$$W(k) = \det\begin{pmatrix} j_0(kR) & j_0(k\sqrt{n_0}R) \\ -j_0'(kR) & -\sqrt{n_0}j_0'(k\sqrt{n_0}R) \end{pmatrix} = 0 \qquad \text{in} \quad \mathbb{R}^3 \tag{2.21}$$

where j_0 is the spherical Bessel function of order 0 (if the first zero of the above determinants is not the first transmission eigenvalue, the latter will be a zero of a similar determinant corresponding to higher order of Bessel functions or spherical Bessel functions). Let v^{B_R,n_0} and w^{B_R,n_0} be a non zero solution of (2.16)-(2.19) corresponding to k_{R,n_0}, and denote by $u^{B_R,n_0} := w^{B_R,n} - v^{B_R,n}$ the corresponding eigenfunction. We have that $u^{B_R,n_0} \in H_0^2(B_R)$ and

$$\int\limits_{B_R} \frac{1}{n_0-1}(\Delta u^{B_R,n_0} + k_{R,n_0}^2 u^{B_R,n_0})(\Delta \overline{u}^{B_R,n_0} + k_{R,n_0}^2 n_0 \overline{u}^{B_R,n_0})\, dx = 0. \tag{2.22}$$

In the following we denote by $n_* = \inf_D(n)$ and $n^* = \sup_D(n)$.

THEOREM 2.5. *Let $n \in L^\infty(D)$ satisfy either one of the following assumptions*

1) $1+\alpha \le n_* \le n(x) \le n^* < \infty$,

2) $0 < n_* \le n(x) \le n^* < 1-\beta$.

for some positive constant $\alpha > 0$ and $\beta > 0$. Then, there exists an infinite set of transmission eigenvalues with $+\infty$ as the only accumulation point.

Proof. First we notice that based on the analytic Fredholm theory it is shown [11], [13], [20] that, under assumption 1) or 2), the set of transmission eigenvalues is at most discrete with $+\infty$ as the only possible accumulation point. In the following we show that indeed there exists an infinite countable set of transmission eigenvalue. First let us suppose that assumption 1) holds. This assumption also implies that

$$0 < \frac{1}{n^*-1} \le \frac{1}{n(x)-1} \le \frac{1}{n_*-1} < \infty.$$

Therefore, from Lemma 2.3, $\mathbb{A}_\tau$ and $\mathbb{B}$ defined by (2.12) satisfy the requirement of Theorem 2.1 with $U = H_0^2(D)$ and from Lemma 2.4 they also satisfy the assumption 1) of Theorem 2.1 with $\tau_0 \le \lambda_1(D)/n^*$. Next let k_{1,n_*} be the first transmission eigenvalue for the ball B of radius $R=1$ and $n_0 := n_*$. This transmission eigenvalue is the first zero of (2.20) in $\mathbb{R}^2$ or (2.21) in $\mathbb{R}^3$ for $R := 1$ and $n_0 := n_*$ (or possibly similar determinants for higher order Bessel functions). By a scaling argument, it is obvious that $k_{\epsilon,n_*} := k_{1,n_*}/\epsilon$ is the first transmission eigenvalue corresponding to the ball of radius $\epsilon > 0$ with index of refraction n_*. Now, take $\epsilon > 0$ small enough such that D contains $m := m(\epsilon) \ge 1$ disjoint balls $B_\epsilon^1, B_\epsilon^2 \dots B_\epsilon^m$ of radius ϵ, i.e. $\overline{B_\epsilon^j} \subset D$, $j = 1...m$ and $\overline{B_\epsilon^j} \cap \overline{B_\epsilon^i} = \emptyset$ for $j \ne i$. Surely, $k_{\epsilon,n_*} := k_{1,n_*}/\epsilon$ is the first transmission eigenvalue for each of these balls with index of refraction n_* and let $u^{B_\epsilon^j,n_*} \in H_0^2(B_\epsilon^j)$, $j = 1...m$ be the corresponding eigenfunction. The extension by zero $\tilde{u}^j$ of $u^{B_\epsilon^j,n_*}$ to the whole D is obviously in $H_0^2(D)$ due to the zero conditions on $\partial B_{\epsilon,n_*}^j$. Furthermore, the vectors $\{\tilde{u}^1, \tilde{u}^2, \dots \tilde{u}^m\}$ are linearly independent and orthogonal in $H_0^2(D)$ since they have disjoint supports. Furthermore from (2.22) we have that

$$0 = \int\limits_D \frac{1}{n_*-1}(\Delta \tilde{u}^j + k_{\epsilon,n_*}^2 \tilde{u}^j)(\Delta \overline{\tilde{u}}^j + k_{\epsilon,n_*}^2 n_* \overline{\tilde{u}}^j)\, dx \tag{2.23}$$

$$= \int\limits_D \frac{1}{n_*-1}|\Delta \tilde{u}^j + k_{\epsilon,n_*}^2 \tilde{u}^j|^2\, dx + k_{\epsilon,n_*}^4 \int\limits_D |\tilde{u}^j|^2\, dx - k_{\epsilon,n_*}^2 \int\limits_D |\nabla \tilde{u}^j|^2\, dx \tag{2.24}$$

for $j = 1...m$. Let us denote by $\mathcal{U}$ the n-dimensional subspace of $H_0^2(D)$ spanned by $\{\tilde{u}^1, \tilde{u}^2, \dots \tilde{u}^m\}$. Since each $\tilde{u}^j$, $j = 1, ..., m$ satisfies (2.23) and they have disjoint supports, we have that for $\tau_1 := k^2_{\epsilon,n_*}$ and for every $\tilde{u} \in \mathcal{U}$

$$(\mathbb{A}_{\tau_1}\tilde{u} - \tau_1\mathbb{B}\tilde{u},\, \tilde{u})_{H^2_0(D)} = \int\limits_D \frac{1}{n-1}|\Delta\tilde{u} + \tau_1\tilde{u}|^2\,dx + \tau_1^2\int\limits_D |\tilde{u}|^2\,dx - \tau_1\int\limits_D |\nabla\tilde{u}|^2\,dx$$

$$\leq \int\limits_D \frac{1}{n_*-1}|\Delta\tilde{u} + \tau_1\tilde{u}|^2\,dx + \tau_1^2\int\limits_D |\tilde{u}|^2\,dx - \tau_1\int\limits_D |\nabla\tilde{u}|^2\,dx = 0. \tag{2.25}$$

This means that the assumption 2) of Theorem 2.1 is also satisfied and therefore we can conclude that there are $m(\epsilon)$ transmission eigenvalues (counting multiplicity) inside $[\tau_0,\, k_{\epsilon,n_*}]$. Note that $m(\epsilon)$ and k_{ϵ,n_*} both go to $+\infty$ as $\epsilon \to 0$. Since the multiplicity of each eigenvalue is finite we have shown, by letting $\epsilon \to 0$ that there exists a infinite countable set of transmission eigenvalues that accumulate to ∞.

Now, if the index of refraction is such that the assumption 2) holds, then we have that

$$0 < \frac{n_*}{1-n_*} \leq \frac{n(x)}{1-n(x)} \leq \frac{n^*}{1-n^*} < \infty,$$

and therefore according to Lemma 2.3 and Lemma 2.4, $\tilde{\mathbb{A}}_\tau$ and $\mathbb{B}$, $\tau > 0$ satisfy the requirements and assumption 1) of Theorem 2.1 with $U = H^2_0(D)$ for $\tau_0 \leq \lambda_1(D)$. In this case we can estimate

$$\left(\tilde{\mathbb{A}}_\tau u - \tau\mathbb{B}u,\, u\right)_{H^2_0(D)} = \int\limits_D \frac{n}{1-n}|\Delta u + \tau u|^2\,dx + \int\limits_D |\Delta u|^2\,dx - \tau\int\limits_D |\nabla u|^2\,dx$$

$$\leq \int\limits_D \frac{n^*}{1-n^*}|\Delta u + \tau u|^2\,dx + \int\limits_D |\Delta u|^2\,dx - \tau\int\limits_D |\nabla u|^2\,dx \tag{2.26}$$

Hence, the rest of the proof for checking the validity of assumption 2) of Theorem 2.1 goes exactly in the same way as for the previous case if one replaces n_* by n^*. □

Remark 2.2. From the proof of Theorem 2.5 it follows that for every $j \in \mathbb{N}$ the equation $\lambda_j(\tau) - \tau = 0$ has at least one solution, where $\lambda_j(\tau)$ is the j-th eigenvalue of the auxiliary eigenvalue problem (2.14) or (2.15). Combining Theorem 2.5 with the results of Corollary 3.1 in [7] we have the following estimates for transmission eigenvalues. We call B_{r_1} the largest ball of radius r_1 such that $B_{r_1} \subset D$ and B_{r_2} the smallest ball of radius r_2 such that $D \subset B_2$. For a given $0 < \epsilon \leq r_2$, let $m(\epsilon) \in \mathbb{N}$ be the number of balls B_ϵ of radius ϵ that are contained in D. We denote by k_{1,n_*} and k_{1,n^*} the first transmission eigenvalue corresponding to the ball B_1 of radius one with the index of refraction n_* and n^*, respectively.

Corollary 2.6. *Assume that* $n \in L^\infty(D)$.

1) *If* $1 + \alpha \leq n_* \leq n(x) \leq n^* < \infty$, *then*

$$0 < \frac{k_{1,n^*}}{r_2} \leq k_{1,D,n(x)} \leq \frac{k_{1,n_*}}{r_1}. \tag{2.27}$$

There are at least $m(\epsilon)$ *transmission eigenvalues in the interval* $\left[\frac{k_{1,n^*}}{r_2},\, \frac{k_{1,n_*}}{\epsilon}\right]$.

2) *If* $0 < n_* \leq n(x) \leq n^* < 1 - \beta$, *then*

$$0 < \frac{k_{1,n_*}}{r_2} \leq k_{1,D,n(x)} \leq \frac{k_{1,n^*}}{r_1}. \tag{2.28}$$

There are at least $m(\epsilon)$ *transmission eigenvalues in the interval* $\left[\frac{k_{1,n_*}}{r_2},\, \frac{k_{1,n^*}}{\epsilon}\right]$.

where, $k_{1,D,n(x)}$ is the first transmission eigenvalue corresponding to D and the given index of refraction $n(x)$. Note that in the case of the assumption 1) in the above corollary we have that $\frac{\lambda_1(D)}{n^*} < \frac{k_{1,n^*}}{r_2}$ where recall that $\lambda_1(D)$ is the first Dirichlet eigenvalue for $-\Delta$ in D, whence we obtained an improved lower bound for the first transmission eigenvalue. Indeed, from Lemma 2.4 applied to the ball of radius r_2 with index of refraction n^* we have that $k_{1,n^*}/r_2 > \lambda_1(B_{r_2})/n^* > \lambda_1(D)/n^*$, where the latter inequality is true by the monotonicity of the Dirichlet eigenvalue with respect to the domain, i.e. $\lambda_1(B_{r_2}) > \lambda_1(D)$. For similar reasons, in the case of assumption 2) of the above corollary, we provide an improved lower bound, i.e. $k_{1,n_*}/r_2 > \lambda_1(D)$.

2.3. The anisotropic inhomogeneous media. We now turn our attention to the interior transmission eigenvalue problem corresponding to the scattering problem for anisotropic media. We consider two problems, namely the interior transmission problem for the anisotropic Maxwell's equations, and the interior transmission problem for the anisotropic scalar equation. In the following we show that these eigenvalue problems can be analyzed in the same way as the problem discussed in Section 2.2.

DEFINITION 2.7. *A real valued $d \times d$, $d = 2, 3$ matrix function $K \in L^\infty(D, \mathbb{R}^{d\times d})$ is said to be bounded positive definite on D if there exists a constant $\gamma > 0$ such that $\overline{\boldsymbol{\xi}} \cdot K\boldsymbol{\xi} \geq \gamma\,|\boldsymbol{\xi}|^2$, for all $\boldsymbol{\xi} \in \mathbb{C}^d$ and a.e. in D.*

Problem 1: Anisotropic Maxwell's equations: Let $D \subset \mathbb{R}^3$ satisfy the assumptions stated at the beginning of Section 2. In terms of electric fields the interior transmission eigenvalue problem for anisotropic Maxwell's equations (where the magnetic permeability is assumed to be scalar and constant) is formulated as the problem of finding two vector valued functions $\mathbf{E}$ and $\mathbf{E}_0$ satisfying

$$\operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\, N\,\mathbf{E} = 0 \quad \text{in} \quad D \tag{2.29}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{E}_0 - k^2\,\mathbf{E}_0 = 0 \quad \text{in} \quad D \tag{2.30}$$

$$\mathbf{E}\times\nu = \mathbf{E}_0\times\nu \quad \text{on} \quad \partial D \tag{2.31}$$

$$\operatorname{curl}\mathbf{E}\times\nu = \operatorname{curl}\mathbf{E}_0\times\nu \quad \text{on} \quad \partial D \tag{2.32}$$

where N is a 3×3 matrix valued function defined on D with $L^\infty(D)$ real valued entries, i.e. $N \in L^\infty(D, \mathbb{R}^{3\times3})$. To properly formulate this eigenvalue problem, we consider the Hilbert spaces

$$H(\operatorname{curl}, D) := \{\mathbf{u} \in (L^2(D))^3 :\ \operatorname{curl}\mathbf{u} \in (L^2(D))^3\},$$
$$H_0(\operatorname{curl}, D) := \{\mathbf{u} \in H(\operatorname{curl}, D) :\ \mathbf{u}\times\nu = 0 \text{ on } \partial D\},$$

equipped with the scalar product $(\mathbf{u}, \mathbf{v})_{\operatorname{curl}} = (\mathbf{u}, \mathbf{v})_D + (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v})_D$ where $(\cdot,\cdot)_D$ denotes the $(L^2(D))^3$ scalar product and the corresponding norm $\|\cdot\|_{\operatorname{curl}}$. Next we define

$$\mathcal{U}(D) := \{\mathbf{u} \in H(\operatorname{curl}, D) :\ \operatorname{curl}\mathbf{u} \in H(\operatorname{curl}, D)\},$$
$$\mathcal{U}_0(D) := \{\mathbf{u} \in H_0(\operatorname{curl}, D) :\ \operatorname{curl}\mathbf{u} \in H_0(\operatorname{curl}, D)\},$$

equipped with the scalar product $(\mathbf{u}, \mathbf{v})_{\mathcal{U}} = (\mathbf{u}, \mathbf{v})_{\operatorname{curl}} + (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v})_{\operatorname{curl}}$ and the corresponding norm $\|\cdot\|_{\mathcal{U}}$. We further require that N, N^{-1} and either $(N-I)^{-1}$ or $(I-N)^{-1}$ are bounded positive definite real matrix valued functions on D. Hence a solution of (2.29)-(2.32) is such that $\mathbf{E} \in (L^2(D))^3$, $\mathbf{E}_0 \in (L^2(D))^3$ and $\mathbf{E} - \mathbf{E}_0 \in \mathcal{U}_0(D)$. As it is shown in [9] and [14], (2.29)-(2.32) is equivalent to finding $\mathbf{u} = \mathbf{E} - \mathbf{E}_0 \in \mathcal{U}_0(D)$ such that

$$(\operatorname{curl}\operatorname{curl} - k^2 N)(N-I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u}) = 0. \tag{2.33}$$

which in a variational form can be written as

$$\int_D (N-I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u})\cdot(\operatorname{curl}\operatorname{curl}\mathbf{v} - k^2N\mathbf{v})\,dx = 0 \qquad \text{for all } \mathbf{v}\in\mathcal{U}_0(D). \tag{2.34}$$

Problem 2: Anisotropic scalar equation: This problem can be stated in $\mathbb{R}^2$ as well as in $\mathbb{R}^3$. Hence the bounded region $D \in \mathbb{R}^d$, $d = 2, 3$ satisfies the assumptions stated at the beginning of Section 2. The interior transmission eigenvalue problem for anisotropic scalar equations reads

$$\nabla\cdot A\nabla w + k^2\, w = 0 \qquad \text{in} \quad D \tag{2.35}$$
$$\Delta v + k^2\, v = 0 \qquad \text{in} \quad D \tag{2.36}$$
$$w = v \qquad \text{on} \quad \partial D \tag{2.37}$$
$$\frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \qquad \text{on} \quad \partial D \tag{2.38}$$

where

$$\frac{\partial w}{\partial \nu_A}(x) := \nu(x)\cdot A(x)\nabla v(x), \qquad x\in\partial D. \tag{2.39}$$

Letting $N := A^{-1}$, in terms of new vector valued functions

$$\mathbf{w} = A\nabla w, \quad \text{and} \quad \mathbf{v} = \nabla v$$

the above problem can be written as (see [6] and [9] for details)

$$\nabla(\nabla\cdot\mathbf{w}) + k^2N\mathbf{w} = 0 \quad \text{in} \quad D \tag{2.40}$$
$$\nabla(\nabla\cdot\mathbf{v}) + k^2\mathbf{v} = 0 \quad \text{in} \quad D \tag{2.41}$$
$$\nu\cdot\mathbf{w} = \nu\cdot\mathbf{v} \quad \text{on} \quad \partial D \tag{2.42}$$
$$\nabla\cdot\mathbf{w} = \nabla\cdot\mathbf{v} \quad \text{on} \quad \partial D. \tag{2.43}$$

Here the $d\times d$, $d = 2, 3$ matrix valued function N satisfies the same assumptions as the 3×3 matrix N in Problem 1. The suitable spaces to analyze this problem are

$$H(\operatorname{div}, D) := \left\{\mathbf{u}\in(L^2(D))^d :\ \nabla\cdot\mathbf{u}\in L^2(D)\right\}, \qquad d = 2, 3$$
$$H_0(\operatorname{div}, D) := \{\mathbf{u}\in H(\operatorname{div}, D) :\ \nu\cdot\mathbf{u} = 0 \text{ on } \partial D\}$$

and

$$\mathcal{H}(D) := \left\{\mathbf{u}\in H(\operatorname{div}, D) :\ \nabla\cdot\mathbf{u}\in H^1(D)\right\}$$
$$\mathcal{H}_0(D) := \left\{\mathbf{u}\in H_0(\operatorname{div}, D) :\ \nabla\cdot\mathbf{u}\in H^1_0(D)\right\}$$

equipped with the scalar product $(\mathbf{u},\mathbf{v})_{\mathcal{H}(D)} := (\mathbf{u},\mathbf{v})_{L^2(D)} + (\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v})_{H^1(D)}$ and corresponding norm $\|\cdot\|_{\mathcal{H}}$. Hence, a solution $\mathbf{u}$, $\mathbf{v}$ of the interior transmission eigenvalue problem (2.40)-(2.43) is such that $\mathbf{u}\in(L^2(D))^d$, $\mathbf{v}\in(L^2(D))^d$ and $\mathbf{w}-\mathbf{v}\in\mathcal{H}_0(D)$. Similarly, to the case of Problem 1, (2.40)-(2.43) has an equivalent formulation as a fourth order differential equation for $\mathbf{u} := \mathbf{w}-\mathbf{v}\in\mathcal{H}_0(D)$

$$\left(\nabla\nabla\cdot + k^2N\right)(N-I)^{-1}\left(\nabla\nabla\cdot\mathbf{u} + k^2\mathbf{u}\right) = 0 \qquad \text{in} \quad D \tag{2.44}$$

which can be written in the following variational form

$$\int_D (N-I)^{-1}\left(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u}\right)\cdot\left(\nabla\nabla\cdot\overline{\mathbf{v}}+k^2N\overline{\mathbf{v}}\right)\,dx=0 \qquad \text{for all } \mathbf{v}\in\mathcal{H}_0(D). \quad (2.45)$$

We note that (2.34) and (2.45) have the same structure, where the operators (curl curl) and ($\nabla\nabla\cdot$) together with corresponding traces are swapped.

Definition 2.8. Transmission eigenvalues corresponding to the Problem 1 (resp. Problem 2) are the values of $k>0$ for which the homogeneous interior transmission problem (2.29)-(2.32) (resp. (2.40)-(2.43)) has nonzero solutions $\mathbf{w}\in L^2(D)$ and $\mathbf{v}\in L^2(D)$ such that $\mathbf{w}-\mathbf{v}$ is in $\mathcal{U}_0(D)$ (resp. $\mathcal{H}_0(D)$). This solution $\mathbf{u}:=\mathbf{w}-\mathbf{v}$ is called a corresponding eigenfunction. Both eigenvalue problems (2.34) and (2.45) can be written as an operator equation

$$\mathbb{A}_\tau\mathbf{u}-\tau\mathbb{B}\mathbf{u}=0 \qquad \text{and} \qquad \tilde{\mathbb{A}}_\tau\mathbf{u}-\tau\mathbb{B}\mathbf{u}=0, \qquad \text{for} \quad \mathbf{u}\in\mathcal{S}, \qquad (2.46)$$

where $\mathcal{S}$ stands for $\mathcal{U}_0(D)$ if Problem 1 is considered and for $\mathcal{H}_0(D)$ if Problem 2 is considered. Here, the bounded linear operators operators $\mathbb{A}_\tau:\mathcal{S}\to\mathcal{S}$, $\tilde{\mathbb{A}}_\tau:\mathcal{S}\to\mathcal{S}$ and $\mathbb{B}:\mathcal{S}\to\mathcal{S}$ are the operators defined using Riesz representation theorem (i.e. defined by (2.12) where $H_0^2(D)$ is replaced by $\mathcal{S}$) associated with the sesquilinear forms $\mathcal{A}_\tau$, $\tilde{\mathcal{A}}$ and $\mathcal{B}$ which in the case of Problem 1 are defined by (see [14] for more details)

$$\mathcal{A}_\tau(\mathbf{u},\mathbf{v}):=\left((N-I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{u}-\tau\mathbf{u}),\,(\operatorname{curl}\operatorname{curl}\mathbf{v}-\tau\mathbf{v})\right)_D+\tau^2\,(\mathbf{u},\,\mathbf{v})_D \quad (2.47)$$

$$\begin{aligned}\tilde{\mathcal{A}}_\tau(\mathbf{u},\mathbf{v}):=&\left(N(I-N)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{u}-\tau\mathbf{u}),\,(\operatorname{curl}\operatorname{curl}\mathbf{v}-\tau\mathbf{v})\right)_D \\ &+(\operatorname{curl}\operatorname{curl}\mathbf{u},\,\operatorname{curl}\operatorname{curl}\mathbf{v})_D\end{aligned} \quad (2.48)$$

and

$$\mathcal{B}(\mathbf{u},\mathbf{v}):=(\operatorname{curl}\mathbf{u},\,\operatorname{curl}\mathbf{v})_D\,, \quad (2.49)$$

respectively, where $(\cdot,\,\cdot)_D$ denotes the $L^2(D)$-inner product, whereas in the case of Problem 2

$$\mathcal{A}_\tau(\mathbf{u},\mathbf{v}):=\left((N-I)^{-1}\left(\nabla\nabla\cdot\mathbf{u}+\tau\mathbf{u}\right),\left(\nabla\nabla\cdot\mathbf{v}+\tau\mathbf{v}\right)\right)_D+\tau^2\,(\mathbf{u},\mathbf{v})_D \quad (2.50)$$

$$\begin{aligned}\tilde{\mathcal{A}}_\tau(\mathbf{u},\mathbf{v}):=&\left(N(I-N)^{-1}\left(\nabla\nabla\cdot\mathbf{u}+\tau\mathbf{u}\right),\left(\nabla\nabla\cdot\mathbf{v}+\tau\mathbf{v}\right)\right)_D \\ &+(\nabla\nabla\cdot\mathbf{u},\nabla\nabla\cdot\mathbf{v})_D\end{aligned} \quad (2.51)$$

and

$$\mathcal{B}(\mathbf{u},\mathbf{v}):=(\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v})_D\,, \quad (2.52)$$

respectively.

The properties of these operators are studied in [6], [9] and [14]. Let $\sigma_*(x)>0$ and $\sigma^*(x)>0$ be the smallest and the largest eigenvalue, respectively, of the positive definite symmetric $d\times d$, $d=2,3$ matrix N. Recall that the largest eigenvalue $\sigma^*(x)$ which coincides with the Euclidean norm $\|N(x)\|_2$ is given by by $\sigma^*(x)=\sup_{\|\xi\|=1}(\bar{\xi}\cdot N(x)\,\xi)$ and the smallest eigenvalue $\sigma_*(x)$ is given by $\sigma_*(x)=\inf_{\|\xi\|=1}(\bar{\xi}\cdot N(x)\,\xi)$. In the following we denote by $n^* = \sup_D \sigma^*(x)$ and $n_* = \inf_D \sigma_*(x)$. Again, let $\lambda_1(D)$ be the Dirichlet eigenvalue for $-\Delta$ in D. The following lemma is proven in [6],[9] and [14].

LEMMA 2.9. *Let $\mathcal{S}$ stands for $\mathcal{U}_0(D)$ if Problem 1 is considered and for $\mathcal{H}_0(D)$ if Problem 2 is considered. The operators $\mathbb{A}_\tau : \mathcal{S} \to \mathcal{S}$, $\tilde{\mathbb{A}}_\tau : \mathcal{S} \to \mathcal{S}$, $\tau > 0$ and $\mathbb{B} : \mathcal{S} \to \mathcal{S}$ are self-adjoint. Furthermore, $\mathbb{B}$ is a positive compact operator. If $(N - I)^{-1}$ is a bounded positive definite matrix function on D (Definition 2.7), then $\mathbb{A}_\tau$ is a positive definite operator and*

$$(\mathbb{A}_\tau u - \tau \mathbb{B} u, u)_{\mathcal{S}} \geq \alpha \|u\|^2_{\mathcal{S}} > 0 \quad \textit{for all} \qquad 0 < \tau < \frac{\lambda_1(D)}{n^*} \quad \textit{and} \quad u \in \mathcal{S}.$$

If $N(I - N)^{-1}$ is a bounded positive definite matrix function on D, then $\tilde{\mathbb{A}}_\tau$ is a positive definite operator and

$$\left(\tilde{\mathbb{A}}_\tau u - \tau \mathbb{B} u, u\right)_{\mathcal{S}} \geq \alpha \|u\|^2_{\mathcal{S}} > 0 \quad \textit{for all} \qquad 0 < \tau < \lambda_1(D) \quad \textit{and} \quad u \in \mathcal{S}.$$

Note that the kernel of $\mathbb{B} : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ is given by

$$\text{Kernel}(\mathbb{B}) = \left\{\mathbf{u} \in \mathcal{U}_0(D) \quad \text{such that } \mathbf{u} := \nabla \varphi, \ \varphi \in H^1(D)\right\},$$

whereas the kernel of $\mathbb{B} : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ is given by

$$\text{Kernel}(\mathbb{B}) = \{\mathbf{u} \in \mathcal{H}_0(D) \quad \text{such that } \mathbf{u} := \text{curl}\, \varphi, \ \varphi \in H(\text{curl}\,, D)\}.$$

To carry over the approach of Section 2.2 to the eigenvalue problems for anisotropic medium, namely Problem 1 and Problem 2, we also need to consider the corresponding interior transmission eigenvalue problems for a ball with constant index of refraction. To this end, let $B_R \in \mathbb{R}^3$ be a ball of radius R centered at the origin and $n_0 > 0$ a constant different from one. In [11] it is shown, by using separation of variables, that

$$\text{curl}\,\text{curl}\, \mathbf{w} - k^2 n_0 \mathbf{w} = 0 \quad \text{in} \quad B_R \tag{2.53}$$
$$\text{curl}\,\text{curl}\, \mathbf{v} - k^2 \mathbf{v} = 0 \quad \text{in} \quad B_R \tag{2.54}$$
$$\mathbf{w} \times \nu = \mathbf{v} \times \nu \quad \text{on} \quad \partial B_R \tag{2.55}$$
$$\text{curl}\, \mathbf{w} \times \nu = \text{curl}\, \mathbf{v} \times \nu \quad \text{on} \quad \partial B_R \tag{2.56}$$

has a countable discrete set of eigenvalues. Let us denote by k_{R,n_0} the first transmission eigenvalue which as for (2.16)-(2.19) is zero of a determinant similar to (2.21) involving spherical Bessel functions of the variable kR and of order greater or equal to one (see e.g. [11] page 263). Let $\mathbf{u}^{B_R,n_0} = \mathbf{w}^{B_R,n_0} - \mathbf{v}^{B_R,n_0}$ be the eigenfunction corresponding to k_{R,n_0}. We have that $\mathbf{u}^{B_R,n_0} \in \mathcal{U}_0(B_R)$ and

$$\int_{B_R} \frac{1}{n_0 - 1} (\text{curl}\,\text{curl}\, \mathbf{u}^{B_R,n_0} - k^2_{R,n_0} \mathbf{u}^{B_R,n_0}) \cdot (\text{curl}\,\text{curl}\, \overline{\mathbf{u}}^{B_R,n_0} - k^2_{R,n_0} n_0 \overline{\mathbf{u}}^{B_R,n_0})\, dx = 0. \tag{2.57}$$

Similarly, for Problem 2, we denote by k_{R,n_0} the first transmission eigenvalue of

$$\Delta w + k^2 n_0 w = 0 \quad \text{in} \quad B_R \tag{2.58}$$
$$\Delta v + k^2 v = 0 \quad \text{in} \quad B_R \tag{2.59}$$
$$w = v \quad \text{on} \quad \partial B_R \tag{2.60}$$
$$\frac{1}{n_0} \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial B_R \tag{2.61}$$

where here B_R is a two or three dimensional ball of radius R [6] (this problem also have infinitely many eigenvalues). The corresponding eigenfunction $\mathbf{u}^{B_R,n_0} = n_0 \nabla w^{B_R,n_0} - \nabla v^{B_R,n_0}$ is in $\mathcal{H}_0(B_R)$, where w^{B_R,n_0}, v^{B_R,n_0} is a nonzero solution to (2.58)-(2.61). Furthermore we have that

$$\int_{B_R} \frac{1}{n_0 - 1} (\nabla\nabla\cdot\mathbf{u}^{B_R,n_0} + k_{R,n_0}^2 \mathbf{u}^{B_R,n_0}) \cdot (\nabla\nabla\cdot\overline{\mathbf{u}}^{B_R,n_0} + k_{R,n_0}^2 n_0 \overline{\mathbf{u}}^{B_R,n_0})\, dx = 0. \quad (2.62)$$

By definition, the eigenvectors $\mathbf{u}^{B_R,n_0}$ for (2.53)-(2.56) and (2.53)-(2.56) are not in the kernel of $\mathbb{B} : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ and $\mathbb{B} : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$, respectively. Finally, if $\overline{B}_R \subset D$ then the extension by zero $\tilde{\mathbf{u}}$ of eigenvectors $\mathbf{u}^{B_R,n_0}$ for (2.53)-(2.56) and (2.53)-(2.56) to the whole D is in $\mathcal{U}_0(D)$ and $\mathcal{H}_0(D)$, respectively.

The above discussion provides all the necessary ingredients to apply Theorem 2.1 to (2.46) in order to prove the existence of an infinite discrete set of transmission eigenvalues for both, Problem 1 and Problem 2. The following theorem can now be proven exactly in the same way as Theorem 2.5.

THEOREM 2.10. *Assume that $N \in L^\infty(D, \mathbb{R}^{d\times d})$, $d = 2, 3$, satisfies either one of the following assumptions*

1) $1 + \alpha \le n_* \le (\bar{\xi} \cdot N(x)\, \xi) \le n^* < \infty$,

2) $0 < n_* \le (\bar{\xi} \cdot N(x)\, \xi) \le n^* < 1 - \beta$.

for every $\xi \in \mathbb{C}^d$ such that $\|\xi\| = 1$, and some positive constants $\alpha > 0$ and $\beta > 0$. Then both, Problem 1 and Problem 2 have an infinite countable set of transmission eigenvalues with $+\infty$ as the only accumulation point.

The tools developed in this section, also enable us to generalize the result of Corollary 3.1 in [7] in a straightforward manner to both interior transmission eigenvalue problems for anisotropic media. Then the obtained result combined with Theorem 2.10 yield the following estimates for transmission eigenvalues.

Let again B_{r_1} be the largest ball of radius r_1 such that $B_{r_1} \subset D$ and B_{r_2} be the smallest ball of radius r_2 such that $D \subset B_2$. For a given $0 < \epsilon \le r_2$, let $m(\epsilon) \in \mathbb{N}$ be the number of balls B_ϵ of radius ϵ that are contained in D. We denote by k_{1,n_*} and k_{1,n^*} the first transmission eigenvalue of either (2.53)-(2.56) or (2.58)-(2.61) for the ball B_1 of radius one with index of refraction n_* and n^*, respectively.

COROLLARY 2.11. *Assume that $N \in L^\infty(D, \mathbb{R}^{d\times d})$, $d = 2, 3$ and let $k_{1,D,N(x)}$ be the first transmission eigenvalue for either Problem 1 (2.29)-(2.32) or Problem 2 (2.29)-(2.32).*

1) If $1 + \alpha \le n_* \le (\bar{\xi} \cdot N(x)\, \xi) \le n^* < \infty$ for every $\xi \in \mathbb{C}^d$ such that $\|\xi\| = 1$, and some constant $\alpha > 0$, then

$$0 < \frac{k_{1,n^*}}{r_2} \le k_{1,D,N(x)} \le \frac{k_{1,n_*}}{r_1}. \quad (2.63)$$

Both, Problem 1 and Problem 2 have at least $m(\epsilon)$ transmission eigenvalues in the interval $\left[\frac{k_{1,n^*}}{r_2}, \frac{k_{1,n_*}}{\epsilon}\right]$.

2) If $0 < n_* \le (\bar{\xi} \cdot N(x)\, \xi) \le n^* < 1 - \beta$ for every $\xi \in \mathbb{C}^d$ such that $\|\xi\|$, $d = 2, 3$ and some constant $\beta > 0$, then

$$0 < \frac{k_{1,n_*}}{r_2} \le k_{1,D,N(x)} \le \frac{k_{1,n^*}}{r_1}. \quad (2.64)$$

Both, Problem 1 and Problem 2 have at least $m(\epsilon)$ transmission eigenvalues in the interval $\left[\frac{k_{1,n_*}}{r_2}, \frac{k_{1,n^*}}{\epsilon}\right]$.

3. The case of inhomogeneous media with cavities. Motivated by recent application of transmission eigenvalues in detecting presence of cavities inside dielectric materials [1], we now want to show that there are infinitely many transmission eigenvalues for the case of inhomogeneous dielectric media with cavities, i.e. inhomogeneous media D with regions $D_0 \subset D$ where the index of refraction is the same as of the background medium. More precisely, inside D we consider a region $D_0 \subset D$ which can possibly be multiply connected such that $\mathbb{R}^d \setminus \overline{D}_0$, $d = 2, 3$ is connected and assume that its boundary ∂D_0 is piece-wise smooth. We denote by ν the unit outward normal to ∂D and ∂D_0. Now we consider the interior transmission eigenvalue problem (2.3)-(2.6) with $n \in L^\infty(D)$ a real valued function such that $n \geq c > 0$, $n = 1$ in D_0 and $|n-1| \geq c > 0$ almost everywhere in $D \setminus \overline{D}_0$. Hence now we have $1/|n-1| \in L^\infty(D \setminus \overline{D}_0)$. The interior transmission problem for inhomogeneous medium with cavities is investigated in [5]. To prove our result we need to recall the analytical framework developed in [5]. To this end, we introduce the following Hilbert space

$$V(D, D_0, k) := \{u \in H^2(D) \text{ such that } \Delta u + k^2 u = 0 \text{ in } D_0\}$$

equipped with the $H^2(D)$ scalar product and look for the solution v and w both in $L^2(D)$ such that $u = w - v$ in $V(D, D_0, k)$. It is shown in [5] that (2.3)-(2.6) with n satisfying the above assumption, can be written in the variation form as

$$\int_{D\setminus\overline{D}_0} \frac{1}{n-1} \left(\Delta + k^2\right) u \, \left(\Delta + k^2\right) \bar{\psi} \, dx + k^2 \int_{D\setminus\overline{D}_0} (\Delta u + k^2 u) \, \bar{\psi} \, dx = 0 \qquad (3.1)$$

which is required to be valid for all $\psi \in V_0(D, D_0, k)$.

Definition 3.1. *Values of $k > 0$ for which (3.1) has nontrivial solutions $u \in V_0(D, D_0, k)$ are called transmission eigenvalues. These nontrivial solutions are called corresponding eigenfunctions.*

It is important to note that, as it is discussed in [5], if k^2 is not both a Dirichlet and a Neumann eigenvalue for $-\Delta$ in D_0, then Definition 3.1 of transmission eigenvalues is equivalent to the existence of a nontrivial weak solution to (2.3)-(2.6). However, if for a k there exists a nontrivial solution to (2.3)-(2.6) then k is a transmission eigenvalue according to Definition 3.1.

Next, let us define the following bounded sesquilinear forms on $V_0(D, D_0, k) \times V_0(D, D_0, k)$:

$$\mathcal{A}(u, \psi) = \pm \int_{D\setminus\overline{D}_0} \frac{1}{n-1} \left(\Delta u \, \Delta\bar{\psi} + \nabla u \cdot \nabla\bar{\psi} + u \, \bar{\psi}\right) dx + \int_{D_0} \left(\nabla u \cdot \nabla\bar{\psi} + u \, \bar{\psi}\right) dx \qquad (3.2)$$

and

$$\mathcal{B}_k(u, \psi) = \pm k^2 \int_{D\setminus\overline{D}_0} \frac{1}{n-1} \left(u(\Delta\bar{\psi} + k^2\bar{\psi}) + (\Delta u + k^2 n u)\bar{\psi}\right) dx \mp \int_{D\setminus\overline{D}_0} \frac{1}{n-1} \left(\nabla u \cdot \nabla\bar{\psi} + u \, \bar{\psi}\right) dx - \int_{D_0} \left(\nabla u \cdot \nabla\bar{\psi} + u \, \bar{\psi}\right) dx \qquad (3.3)$$

where the upper sign corresponds to the case when $n - 1 \geq c > 0$ almost everywhere in $D \setminus \overline{D}_0$ whereas the lower sign corresponds to the case when $1 - n \geq c > 0$

almost everywhere in $D \setminus \overline{D}_0$. Hence k is a transmission eigenvalue if and only if the homogeneous problem

$$\mathcal{A}(u_0, \psi) + \mathcal{B}_k(u_0, \psi) = 0 \text{ for all } \psi \in V_0(D, D_0, k) \tag{3.4}$$

has nonzero solution. Let $A_k : V(D, D_0, k) \to V(D, D_0, k)$ and B_k be the self-adjoint operators associated with $\mathcal{A}$ and $\mathcal{B}_k$, respectively, by using the Riesz representation theorem. In [5] it is shown that the operator $A_k : V(D, D_0, k) \to V(D, D_0, k)$ is positive definite, therefore $A_k^{-1} : V(D, D_0, k) \to V(D, D_0, k)$ exists and the operator $B_k : V(D, D_0, k) \to V(D, D_0, k)$ is compact. Hence we can define the operator $A_k^{-1/2}$ by $A_k^{-1/2} = \int_0^\infty \lambda^{-1/2} dE_\lambda$ where dE_λ is the spectral measure associated with the positive operator A_k^{-1}. In particular, $A_k^{-1/2}$ is also bounded, positive definite and self-adjoint. Thus we have that (2.44) is equivalent to finding $u \in V(D, D_0, k)$ such that

$$u + A_k^{-1/2} B_k A_k^{-1/2} u = 0. \tag{3.5}$$

In particular, it is obvious that k is a transmission eigenvalue if and only if the operator

$$I_k + A_k^{-1/2} B_k A_k^{-1/2} : V(D, D_0, k) \to V(D, D_0, k) \tag{3.6}$$

has a nontrivial kernel where I_k is the identity operator on $V(D, D_0, k)$. To avoid dealing with function spaces depending on k we introduce the orthogonal projection oprator P_k from $H_0^2(D)$ onto $V(D, D_0, k)$ and the corresponding injection $R_k : V(D, D_0, k) \to H_0^2(D)$. Then one easily sees that $A_k^{-1/2} B_k A_k^{-1/2}$ is injective on $V(D, D_0, k)$ if and only if

$$I + R_k A_k^{-1/2} B_k A_k^{-1/2} P_k : H_0^2(D) \to H_0^2(D) \tag{3.7}$$

is injective. Indeed, if $u + R_k A_k^{-1/2} B_k A_k^{-1/2} P_k u = 0$ then by taking the inner product of the latter with the component $w = u - P_k u$ which is orthogonal to $P_k u$, we have that

$$\begin{aligned} 0 &= (u,\, w)_{H^2} + \left(R_k A_k^{-1/2} B_k A_k^{-1/2} P_k u,\, w \right)_{H^2} \\ &= (w,\, w)_{H^2} + \left(A_k^{-1/2} B_k A_k^{-1/2} P_k u,\, P_k w \right)_{H^2} = \|w\|_{H^2}^2, \end{aligned} \tag{3.8}$$

whence $w = 0$. The injectivity of $A_k^{-1/2} B_k A_k^{-1/2}$ now implies the injectivity of (3.7) since the component $P_k u$ is in $V(D, D_0, k)$. The converse is obvious. Furthermore as it is discussed in [5], $T_k := R_k A_k^{-1/2} B_k A_k^{-1/2} P_k : H_0^2(D) \to H_0^2(D)$ is a compact operator and the mapping $k \to R_k A_k^{-1/2} B_k A_k^{-1/2} P_k$ is continuous. Therefore, from the max-min principle for the eigenvalues $\lambda(k)$ of the compact and self-adjoint operator $R_k A_k^{-1/2} B_k A_k^{-1/2} P_k$ we can conclude that $\lambda(k)$ is a continuous function of k. Finally, it is clear that the multiplicity of a transmission eigenvalue is finite since it corresponds to the multiplicity of the eigenvalue $\lambda(k) = -1$ [19]. The proof of the existence of transmission eigenvalues is based on the following theorem which is proven in [5] (see also [18]). This theorem is a modified version of Theorem 2.1 and the proof is based on the max-min principle for $\lambda(k)$ and the continuity of $\lambda(k)$ on k.

THEOREM 3.2. *Let* $T_k := R_k A_k^{-1/2} B_k A_k^{-1/2} P_k$. *Assume that*

1) There is a κ_0 such that $I + T_{\kappa_0}$ is positive on $H_0^2(D)$.
2) There is a $\kappa_1 > \kappa_0$ such that $I + T_{\kappa_1}$ is non positive on a p-dimensional subspace W_k of $H_0^2(D)$.

Then there are p transmission eigenvalues in $[\kappa_0, \kappa_1]$ counting their multiplicity.

In the following we set $n_* = \inf_{D\setminus\overline{D}_0}(n)$ and $n^* = \sup_{D\setminus\overline{D}_0}(n)$. Recall that we denote by $\lambda_1(D)$ the first Dirichlet eigenvalue for $-\Delta$ on D.

Theorem 3.3. *Let $n \in L^\infty(D)$, $n = 1$ in D_0 satisfy either one of the following assumptions*
1) $1 + \alpha \le n_* \le n(x) \le n^* < \infty$,
2) $0 < n_* \le n(x) \le n^* < 1 - \beta$.

on $D \setminus \overline{D}_0$ for some positive constant $\alpha > 0$ and $\beta > 0$. Then, there exists an infinite set of transmission eigenvalues with $+\infty$ as the only accumulation point.

Proof. First we assume that the assumption 1) holds in which case we have

$$0 < \frac{1}{n^*-1} \le \frac{1}{n(x)-1} \le \frac{1}{n_*-1} < \infty \qquad \text{in} \qquad D \setminus \overline{D}_0.$$

We note that $T_k := I + R_k A_k^{-1/2} B_k A_k^{-1/2} P_k$ is positive on $H_0^2(D)$ if and only if $A_k + B_k$ is positive on $V(D, D_0, k)$. Hence, combining the terms in (3.1) in a different way we have that for $u \in V(D, D_0, k)$,

$$\begin{aligned}(A_k u + B_k u, u)_{H_0^2(D)} = &\int_{D\setminus\overline{D}_0} \frac{1}{n-1}|\Delta u + k^2 nu|^2\,dx - k^4\int_{D\setminus\overline{D}_0} n|u|^2\,dx \\ &+ k^2\int_{D\setminus\overline{D}_0}|\nabla u|^2\,dx - k^4\int_{D_0}|u|^2\,dx + k^2\int_{D_0}|\nabla u|^2\,dx.\end{aligned} \tag{3.9}$$

For $n^* = \sup_{D\setminus\overline{D}_0} n > 1$, if the sum of the last four terms in (3.9) is nonnegative then we have $A_k + B_k$ is positive. Hence we have

$$\begin{aligned}-k^2\int_{D\setminus\overline{D}_0} n|u_0|^2\,dx + \int_{D\setminus\overline{D}_0}|\nabla u_0|^2\,dx - k^2\int_{D_0}|u_0|^2\,dx + \int_{D_0}|\nabla u_0|^2\,dx \\ \ge \int_D |\nabla u_0|^2\,dx - k^2 n^*\int_D |u_0|^2\,dx \ge (\lambda_1(D) - k^2 n^*)\|u_0\|^2_{L^2(D)}.\end{aligned} \tag{3.10}$$

Therefore, all $\kappa_0 > 0$ such that $\kappa_0^2 \le \dfrac{\lambda_1(D)}{n^*}$ satisfy the assumption 1) of Theorem 3.2. Next, we proceed in the same way as in Theorem 2.5. To this end, take $\epsilon > 0$ small enough such that $D \setminus \overline{D}_0$ contains $m := m(\epsilon) \ge 1$ disjoint balls $B_\epsilon^1, B_\epsilon^2 \dots B_\epsilon^m$ of radius ϵ, i.e. $\overline{B_\epsilon^j} \subset D \setminus \overline{D}_0$, $j = 1...m$ and $\overline{B_\epsilon^j} \cap \overline{B_\epsilon^i} = \emptyset$ for $j \ne i$. Let $k_{\epsilon,n_*} := k_{1,n_*}/\epsilon$ is the first transmission eigenvalue for each of these balls with index of refraction n^*, where k_{1,n_*} is the first transmission eigenvalue for the ball B_1 of radius one with index of refraction n_* (i.e. the firsts eigenvalue corresponding to (2.16-(2.19) with $R := 1$ and $n_0 := n_*$). Denote by $u^{B_\epsilon^j, n_*} \in H_0^2(B_\epsilon^j)$, $j = 1...m$ be the eigenfunction corresponding to k_{ϵ,n_*}. The extension by zero $\tilde{u}^j$ of $u^{B_\epsilon^j, n_*}$ to the whole D is obviously in $V(D, D_0, k)$ and the vectors $\{\tilde{u}^1, \tilde{u}^2, \dots \tilde{u}^m\}$ are linearly independent and orthogonal since they have disjoint supports included in $D \setminus \overline{D}_0$. Let us denote by $\mathcal{U}$ the n-dimensional subspace of $V(D, D_0, k)$ spanned by $\{\tilde{u}^1, \tilde{u}^2, \dots \tilde{u}^m\}$. Since each $\tilde{u}^j$, $j = 1...m$ satisfies (2.23) and they have disjoint supports. we have that for $\kappa_1 := k_{\epsilon,n_*}$ and for every

$\tilde{u}^j \in \mathcal{U}$ (note that $\tilde{u}^j = 0$ in a neighborhood of D_0)

$$\begin{aligned}
&(A_{\kappa_1}\tilde{u} + B_{\kappa_1}\tilde{u}, \tilde{u})_{H_0^2(D)} \qquad (3.11)\\
&= \int\limits_{D\setminus\overline{D}_0} \frac{1}{n-1}|\Delta\tilde{u} + \kappa_1\tilde{u}|^2\,dx + \kappa_1^4 \int\limits_{D\setminus\overline{D}_0} |\tilde{u}|^2\,dx - \kappa_1^2 \int\limits_{D\setminus\overline{D}_0} |\nabla\tilde{u}|^2\,dx\\
&\le \int\limits_{D\setminus\overline{D}_0} \frac{1}{n_*-1}|\Delta\tilde{u} + \kappa_1^2\tilde{u}|^2\,dx + \kappa_1^4 \int\limits_{D\setminus\overline{D}_0} |\tilde{u}|^2\,dx - \tau_1 \int\limits_{D\setminus\overline{D}_0} |\nabla\tilde{u}|^2\,dx = 0.
\end{aligned}$$

This means that the assumption 2) of Theorem 3.2 is also satisfied and therefore there are $m(\epsilon)$ transmission eigenvalues (counting multiplicity) inside $[\kappa_0, k_{\epsilon,n_*}]$. Note that $m(\epsilon)$ and k_{ϵ,n_*} both goes to $+\infty$ as $\epsilon \to 0$. Since the multiplicity of each eigenvalue is finite we have shown that there exists a infinite countable set of transmission eigenvalues that accumulate to $+\infty$.

Now assume that the assumption 2) holds. Similarly to the previous case, from the definition (3.2) and (3.3) of A_k and B_k we have that

$$\begin{aligned}
(A_k u + B_k u, u)_{H_0^2(D)} &= \int_{D\setminus\overline{D}_0} \frac{1}{1-n}|\Delta u + k^2 u|^2\,dx - k^4 \int_{D\setminus\overline{D}_0} |u|^2\,dx\\
&+k^2 \int_{D\setminus\overline{D}_0} |\nabla u|^2\,dx - k^4 \int_{D_0} |u|^2\,dx + k^2 \int_{D_0} |\nabla u|^2\,dx. \qquad (3.12)
\end{aligned}$$

Hence we have that $A_k + B_k$ is positive as long as

$$\begin{aligned}
&-k^2 \int_{D\setminus\overline{D}_0} n|u|^2\,dx + \int_{D\setminus\overline{D}_0} |\nabla u|^2\,dx - k^2 \int_{D_0} |u|^2\,dx + \int_{D_0} |\nabla u|^2\,dx \qquad (3.13)\\
&\qquad \ge \int_D |\nabla u|^2\,dx - k^2 \int_D |u|^2\,dx \ge (\lambda_1(D) - k^2)\|u_0\|^2_{L^2(D)} \ge 0.
\end{aligned}$$

Therefore, $\kappa_0 > 0$ such that $\kappa_0^2 \le \lambda_1(D)$ satisfy the assumption 1) of Theorem 3.2. The rest of the proof can be done exactly in the same way as for the first part where n_* is replaced by n^*. □

Finally, we rewrite the main result of Theorem 3.3 in the following corollary. To this end, we call B_{r_1} the largest ball of radius r_1 such that $B_{r_1} \subset D \setminus \overline{D}_0$. For a given $0 < \epsilon \le r_2$, let $m(\epsilon) \in \mathbb{N}$ be the number of disjoint balls B_ϵ of radius ϵ that are contained in $D \setminus \overline{D}_0$. We denote by k_{1,n_*} and k_{1,n^*} the first transmission eigenvalue for the ball B_1 of radius one with index of refraction n_* and n^* respectively (i.e. the first eigenvalue corresponding to (2.16)-(2.19) with $R := 1$ and $n_0 := n_*$ and $n_0 := n^*$, respectively). Finally let $\lambda_1(D)$ be the first Dirichlet eigenvalue for $-\Delta$ in D.

Corollary 3.4. *Assume that $n \in L^\infty(D \setminus \overline{D}_0)$, $n = 1$ in D_0 and let $k_{1,D,D_0,n(x)}$ be the first transmission eigenvalue corresponding to (2.3)-(2.6) for these D, D_0 and $n(x)$.*

1) If $1 + \alpha \le n_* \le n(x) \le n^* < \infty \quad$ on $\quad D \setminus \overline{D}_0$, then

$$0 < \sqrt{\frac{\lambda_1(D)}{n^*}} \le k_{1,D,D_0,n(x)} \le \frac{k_{1,n_*}}{r_1}. \qquad (3.14)$$

There are at least $m(\epsilon)$ transmission eigenvalues in the interval $\left[\sqrt{\frac{\lambda_1(D)}{n^*}},\ \frac{k_{1,n_*}}{\epsilon}\right]$.

2) If $0 < n_* \le n(x) \le n^* < 1-\beta$ on $D \setminus \overline{D}_0$, then

$$0 < \sqrt{\lambda_1(D)} \le k_{1,D,D_0,n(x)} \le \frac{k_{1,n^*}}{r_1}. \tag{3.15}$$

There are at least $m(\epsilon)$ transmission eigenvalues in the interval $\left[\sqrt{\lambda_1(D)},\ \frac{k_{1,n^*}}{\epsilon}\right]$.

Alternative lower bounds for the first transmission eigenvalue that involve the geometry of D_0 can be found in [5].

Acknowledgments. This research was initiated while F.C. was visiting Ecole Polytechnique supported by the associate team ISIP of INRIA-UDEL. The hospitality of this institution is gratefully acknowledged. The research of D.G. was supported by the grant PEBE2007 of the National Technical University of Athens.

REFERENCES

[1] F. Cakoni, M. Cayoren and D. Colton, *Transmission eigenvalues and the nondestructive testing of dielectrics*, Inverse Problems, 26 (2008) paper 065016.

[2] F. Cakoni and D. Colton, *Qualitative Methods in Inverse Scattering Theory*, Springer, Berlin, 2006.

[3] F. Cakoni, D. Colton and H. Haddar, *On the determination of Dirichlet and transmission eigenvalues from far field data*, submitted for publication.

[4] F. Cakoni, D. Colton and P. Monk, *On the use of transmission eigenvalues to estimate the index of refraction from far field data*, Inverse Problems, 23(2007), pp. 507-522 .

[5] F. Cakoni, D. Colton and H. Haddar, *The interior transmission problem for region with cavities*, submitted for publication.

[6] F. Cakoni, D. Colton and H. Haddar, *The computation of lower bounds for the norm of the index of refraction in an anisotropic media*, J. Integral Equations and Applications, 21, No 2 (2009), pp. 203-227.

[7] F. Cakoni and D. Gintides, *New results on transmission eigenvalues*, submitted for publication.

[8] F. Cakoni and H. Haddar, *A variational approach for the solution of electromagnetic interior transmission problem for anisotropic media*, Inverse Problems and Imaging, 1 no 3 (2007), pp. 443-456.

[9] F. Cakoni and H. Haddar, *On the existence of transmission eigenvalues in an inhomogeneous medium*, Applicable Analysis, 88 (2009), pp. 475-493.

[10] D. Colton, A. Kirsch and L. Päivärinta, *Far field patterns for acoustic waves in an inhomogeneous medium*, SIAM Jour. Math. Anal., 20 (1989), pp. 1472-1483.

[11] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 2nd edition, Vol. 93 in Applied Mathematical Sciences, Springer, New York, 1998.

[12] D. Colton and P. Monk, *The inverse scattering problem for acoustic waves in an inhomogeneous medium*, Quart. Jour. Mech. Applied Math, 41 (1988), pp. 97-125.

[13] D. Colton, L. Päivärinta and J. Sylvester, *The interior transmission problem*, Inverse Problems and Imaging 1 (2007), pp. 13-28.

[14] H. Haddar, *The interior transmission problem for anisotropic Maxwell's equations and its applications to the inverse problem.* Math. Methods Applied Sciences, 27, Issue 18 (2004), pp. 2111-2129.

[15] A. Kirsch, *An integral equation approach and the interior transmission problem for Maxwell's equations*, Inverse Problems and Imaging, 1 (2007), pp. 159-179.

[16] A. Kirsch, *On the existence of transmission eigenvalues*, Inverse Problems and Imaging 3 (2009), pp. 155-172.

[17] A. Kirsch and N. Grinberg, *The Factorization Method for Inverse Problems*, Oxford University Press, Oxford, 2008

[18] L. Päivärinta and J. Sylvester, *Transmission eigenvalues*, SIAM J. Math. Anal., **40** (2008), 738-753.

[19] M. Reed and B Simon, *Functional Analysis*, Academic Press, 1980.

[20] B. P. Rynne and B.D. Sleeman, *The interior transmission problem and inverse scattering from inhomogeneous media*, SIAM J. Math. Anal., 22 (1992), pp. 1755-1762 (1992).