THE TRANSMISSION EIGENVALUE PROBLEM

F. CAKONI AND D. COLTON *

Department of Mathematical Sciences
University of Delaware
Newark, Delaware 19716-2553, USA
E-mail: cakoni@math.udel.edu, colton@math.udel.edu

D. GINTIDES

Department of Mathematics
National Technical University of Athens
Zografou Campus, 15780 Athens, Greece
E-mail: dgindi@math.ntua.gr

This paper provides a survey of recent developments on the transmission eigenvalue problem.

1. Introduction

In recent years the transmission eigenvalue problem has come to play an important role in inverse scattering theory [14]. This is due to the fact that transmission eigenvalues can be determined from the far field data of the scattered wave and used to obtain estimates for the material properties of the scattering object [3], [5], [8]. For the case of scattering of acoustic waves by a bounded simply connected inhomogeneous medium $D \subset \mathbb{R}^3$, the *transmission eigenvalue problem* is to find $k \neq 0$, $w \in L^2(D)$, $v \in L^2(D)$, $w - v \in H^2(D)$ such that

$$\Delta w + k^2 n(x) w = 0 \quad \text{in} \quad D \tag{1}$$

$$\Delta v + k^2 v = 0 \quad \text{in} \quad D \tag{2}$$

$$w = v \quad \text{on} \quad \partial D \tag{3}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial D \tag{4}$$

*The research of F.C. and D.C. was supported in part by the Air Force of Scientific Research under Grant FA-9550-08-1-0138.

where ν is the unit outward normal to the smooth boundary ∂D and the index of refraction $n(x)$ is positive. Values of $k \neq 0$ such that there exists a nontrivial solution to (1)-(4) are called *transmission eigenvalues*. Research on transmission eigenvalues has focused on two main themes: 1) conditions on $n \in L^\infty(D)$ such that transmission eigenvalues exist and form a discrete set [9], [11], [16], [17], [18] and 2) the determination of lower and upper bounds for $n(x)$ from a knowledge of the transmission eigenvalues corresponding to (1)-(4) [6], [7], [8] [9]. The purpose of this paper is to provide a brief survey of both of these research directions.

2. Scattering by an Inhomogeneous Medium

We begin by considering the scattering problem for a non-absorbing inhomogeneous medium and show how the transmission eigenvalues corresponding to this medium can be determined from the far field pattern of the scattered wave. The scattering problem under consideration is to find a function $u \in H^1_{loc}(\mathbb{R}^3)$ such that

$$\Delta u + k^2 n(x) u = 0 \quad \text{in} \quad \mathbb{R}^3 \tag{5}$$

$$u = u^i + u^s \tag{6}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{7}$$

where $x \in \mathbb{R}^3$, $r = |x|$, $k > 0$ is the wave number, $u^i(x) = e^{ikx\cdot d}$ with $|d| = 1$ is the incident field, u^s is the scattered field satisfying the Sommerfeld radiation condition (7) uniformly in $\hat{x} = x/|x|$ and $n \in L^\infty(D)$ such that $n(x) > 0$ for $x \in \overline{D}$ and $n(x) = 1$ for $x \in \mathbb{R}^3 \setminus D$. Here D is defined as in the introduction and under our assumptions on D and n the scattering problem (5)-(7) is uniquely solvable in $H^1_{loc}(\mathbb{R}^3)$. It can be shown [12] that u^s has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{r} u_\infty(\hat{x}; d, k) + O(r^{-2}) \tag{8}$$

as $r \to \infty$ uniformly in $\hat{x}$ where u_∞ is the *far field pattern* of the scattered field u^s and we can define the *far field operator* $F : L^2(\Omega) \to L^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}; d, k) g(d) ds(d) \tag{9}$$

where Ω is the unit sphere in $\mathbb{R}^3$. We note that by linearity $(Fg)(\hat{x})$ is the far field pattern corresponding to (5)-(7) where the incident field is replaced by the *Herglotz wave function*

$$v_g(x) := \int_\Omega e^{ikx\cdot d} g(d) ds(d). \tag{10}$$

Theorem 2.1. [12] *For $k > 0$ the far field operator is injective with dense range if and only if there does not exist a nontrivial solution $v, w \in L^2(D)$, $v - w \in H^2(D)$ of the transmission eigenvalue problem (1)-(4) such that v is a Herglotz wave function.*

Theorem (2.1) suggests the possiblity of being able to determine real transmission eigenvalues from measured far field data. In particular, let F^δ denote the noisy far field operator corresponding to noisy measurements $u^\delta_\infty(\hat{x}, d, k)$ os the far filed data and let $\Phi_\infty(\hat{x}, z, k)$ be the far field pattern of the fundamental solution

$$\Phi(x, z, k) = \frac{e^{ik|x-z|}}{4\pi|x - z|}. \tag{11}$$

We are interested in the Tikhonov regularized solution $g^\delta_{z,\epsilon}$ of the far field equation defined as the unique minimizer of the *Tikhonov functional* [12]

$$\|F^\delta g - \Phi(\cdot, z)\|^2_{L^2(\Omega)} + \epsilon\|g\|^2_{L^2(\Omega)} \tag{12}$$

where the positive number ϵ is known as the *Tikhonov regularization parameter*. Let $\epsilon(\delta)$ be a sequence of regularization parameter such that $\epsilon(\delta)$ be a sequence of regularization parameters such that $\epsilon(\delta) \to 0$ as $\delta \to 0$ and let $g_{z,\delta} := g^\delta_{z,\epsilon(\delta)}$ be the minimizer of (12) with $\epsilon = \epsilon(\delta)$. Then it follows from the results of Arens [1], [2] that if $z \in D$ and $k > 0$ is not a transmission eigenvalue then the Herglotz wave function $v_{g_{z,\delta}}$ converges in the $L^2(D)$ norm as $\delta \to 0$. The following theorem shows that this is not in general true if k is a transmission eigenvalue.

Theorem 2.2. [5] *Let $k > 0$ be a transmission eigenvalue and assume that the far field operator F has dense range. Assume further that either $n(x) > 1$ for $x \in \overline{D}$ or $0 < n(x) < 1$ for $x \in \overline{D}$. Then for almost every $z \in D$, $\|v_{g_{z,\delta}}\|_{L^2(D)}$ can not be bounded as $\delta \to 0$.*

Note that by Theorem 2.1 F has dense range as long as the solution v of the interior transmission problem is not a Herglotz wave function, i.e. v can not be expanded to all of $\mathbb{R}^3$ such that v has the representation (10). Thus in practice real transmission eigenvalues are determined by solving the *far field equation*

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z, k) \tag{13}$$

for an interval of k values and then locating those values of k for which $\|g\|_{L^2(\Omega)}$ is relatively large [3].

3. The Transmission Eigenvalue Problem: The Case of Spherically Stratified Index of Refraction

In the previous section many questions about transmission eigenvalues were left open. In particular, do transmission eigenvalues exist? Is the set of transmission eigenvalues discrete? What is the relationship between transmission eigenvalues and the index of refraction n? The issue of discreteness is of particular importance in the linear sampling method and the factorization method for solving the inverse scattering problem [4], [15] since in these methods one needs to avoid those frequencies that correspond to transmission eigenvalues and hence is important to know that the set of transmission eigenvalues is discrete. On the other hand, transmission eigenvalues are clearly related to the physical properties of the scattering object and hence it is important to know if they exist and what their connection is to the index of refraction.

In this section we will begin to answer these questions for the case when $n(x) = n(r)$ is spherically stratified, D is the ball $\{x : |x| < a\}$ and $n \in C^2[0, a]$. In this case we can expand v and w in a series of spherical harmonics

$$v(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l^m j_l(kr) Y_l^m(\hat{x}) \tag{14}$$

$$w(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} b_l^m y_l(kr) Y_l^m(\hat{x}) \tag{15}$$

$$\tag{16}$$

where $j_l$is a spherical Bessel function of order l and y_l is a real valued solution of

$$y'' + \frac{2}{r} y' + \left(k^2 n(r) - \frac{l(l+1)}{r^2} \right) y = 0 \tag{17}$$

normalized such that $y_l(r)$ behaves like $j_j(kr)$ as $r \to 0$. From Section 2.3 of [10] we can represent this solution in the form

$$y_l(r) = j_l(kr) + k^2 \int_0^r G(r, s, k) j_l(ks) ds \tag{18}$$

where G is real valued, twice continuously differentiable for $0 \leq s \leq r$ and is an even entire function of k of finite exponential type. It can furthermore be shown that y_l is bounded for k on the positive real axis (c.f. Section 9.4 of [12]).

Theorem 3.1. [13] *Assume that $n(x) = n(r)$ is spherically stratified, D is the ball $\{x : |x| < a\}$ and $n \in C^2[0, a]$. Then if $n(r)$ is not identically equal to one there exits a countably infinite number of transmission eigenvalues for (1)-(4) with infinity as the only possible accumulation point. If in addition*

$$\frac{1}{a}\int_0^a \sqrt{n(r)}dr \neq 1 \tag{19}$$

then there exist a countably infinite number of positive transmission eigenvalues *for (1)-(4) with infinity as the only possible accumulation point.*

Proof. Assume $n = n(r)$ satisfies the hypothesis of the theorem. A necessary and sufficient condition for k to be a transmission eigenvalue is that k is a zero of the determinant

$$d_l(k) := \det \begin{pmatrix} y_l(a) & -j_l(ka) \\ \\ y_l'(a) & -kj_l'(ka) \end{pmatrix} \tag{20}$$

for some non-negative integer l (c.f. Section 9.4 of [12]). Since the spherical Bessel functions are entire functions of k of finite exponential type and bounded for k on the positive real axis, by the above discussion we see that $d_l(k)$ also has this property. Furthermore, by the series expansion of j_l we see that $d_l(k)$ is an even function of k and $d_l(0) = 0$. Hence if $d_l(k)$ does not have a countably infinite number of zeros, by the Hadamard factorization theorem [19] $d_l(k)$ must be identically zero. We will now show that $d_l(k)$ is not identically zero for every l unless $n(r)$ is identically equal to one.

Assume that $d_l(k)$ is identically zero for every non-negative integer l. Noting that $j_l(kr)Y_l^m(\hat{x})$ is a Herglotz wave function, it follows from the proof of Theorem 8.16 in [12] that

$$\int_0^a j_l(k\rho)y_l(\rho)\rho^2 m(\rho)\, d\rho = 0 \tag{21}$$

for all k where $m(r) := 1 - n(r)$. Hence, using the Taylor series expansion of $j_l(k\rho)$ and (18) we see that

$$\int_0^a \rho^{2l+2} m(\rho)\, d\rho = 0 \tag{22}$$

for all non-negative integers l. By Muntz's theorem [19] we now have that $m(r) = 0$, i.e. $n(r) = 1$.

If (19) is valid then it follows from Theorem 2 of [14] that there exist a countably infinite number of *positive* transmission eigenvalues. □

4. The Transmission Eigenvalue Problem: The Case of General Index of Refraction

We now consider the transmission eigenvalue problem (1)-(4) for general $n \in L^\infty$. We will assume that n satisfies one of the following assumptions: 1) $1+\alpha \leq n_* \leq n(x) \leq n^* < \infty$, 2) $0 < n_* \leq n(x) \leq n^* \leq < 1-\beta$ for some $\alpha > 0$ and $\beta > 0$ where $n_* = \inf_D(n)$ and $n^* = \sup_D(n)$ Analogous to Theorem 3.1, we have the following theorem:

Theorem 4.1. [8] *Assume that $n \in L^\infty(D)$ and $k > 0$. Then under the above assumptions on n there exist infinitely many positive transmission eigenvalues with infinity as the only accumulation point. Furthermore, if $1+\alpha \leq n_* \leq n(x) \leq n^*$ and $\lambda(D)$ is the first transmission eigenvalue for $-\Delta$ in D then the first positive transmission eigenvalue $k_{1,D,n(x)}$ satisfies*

$$k^2_{1,D,n(x)} \geq \frac{\lambda(x)}{n^*}. \tag{23}$$

Outline of Proof: The transmission eigenvalue problem can be written for $u := w - v \in H^2_0(D)$ as an eigenvalue problem for the fourth order equation

$$\left(\Delta + k^2 n\right) \frac{1}{n-1} \left(\Delta + k^2\right) u = 0 \tag{24}$$

which in variational form is formulated as finding a function $u \in H^2_0(D)$ such that

$$\int_D \frac{1}{n-1} (\Delta u + k^2 u)(\Delta \overline{\varphi} + k^2 n \overline{\varphi})\, dx = 0 \tag{25}$$

for all $\varphi \in H^2_0(D)$. To fix our ideas, we assume from now on that $1+\alpha \leq n_* \leq n(x) \leq n^* < \infty$. Letting $\tau := k^2$ we have that

$$\int_D \frac{1}{n-1} \Delta u\, \Delta \overline{\varphi}\, dx - \tau \int_D \frac{n+1}{n-1} \nabla u \cdot \nabla \overline{\varphi}\, dx + \tau^2 \int_D \frac{n}{n-1} u\, \overline{\varphi}\, dx = 0 \tag{26}$$

which by the Riesz representation theorem takes the form

$$\mathbb{K}_1 u - \tau \mathbb{K}_2 u + \tau^2 \mathbb{K}_3 u = 0 \qquad \text{in } H^2_0(D)$$

where $\mathbb{K}_1 : H_0^2(D) \to H_0^2(D)$ is self adjoint and coercive and $\mathbb{K}_2, \mathbb{K}_3 : H_0^2(D) \to H_0^2(D)$ are self adjoint, compact and positive. Hence we can write

$$u - \tau \mathbb{K}_1^{-1}\mathbb{K}_2 u + \tau^2 \mathbb{K}_1^{-1}\mathbb{K}_3 u = 0 \qquad \text{in } H_0^2(D)$$

and setting $\phi = \tau \mathbb{K}_1^{-1/2}\mathbb{K}_3^{1/2}$ we obtain

$$\begin{pmatrix} u \\ \phi \end{pmatrix} - \tau \begin{pmatrix} \mathbb{K}_1^{-1}\mathbb{K}_2 & -\mathbb{K}_1^{-1/2}\mathbb{K}_3^{1/2} \\ \mathbb{K}_1^{-1/2}\mathbb{K}_3^{1/2} & 0 \end{pmatrix} \begin{pmatrix} u \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{27}$$

This is clearly not an eigenvalue problem for a compact self adjoint operator. However from (27) we see that the transmission eigenvalues (if they exist) form a discrete set with infinity as the only possible accumulation point and that each transmission eigenvalue has finite multiplicity.

We now define the following bounded sesquilinear forms on $H_0^2(D) \times H_0^2(D)$:

$$\mathcal{A}_\tau(u,v) = \int\limits_D \frac{1}{n-1}(\Delta u + \tau u)\,(\Delta \varphi + \tau \varphi)\,ds + \tau^2 \int\limits_D u\,\varphi\,ds, \tag{28}$$

and

$$\mathcal{B}(u,v) = \int\limits_D \nabla u\,\nabla \varphi\,ds \tag{29}$$

and let $\mathbb{A}_\tau : H_0^2(D) \to H_0^2(D)$ and $\mathbb{B} : H_0^2(D) \to H_0^2(D)$ be the corresponding operators defined by the Riesz representation theorem. The interior transmission problem now reads

$$(\mathbb{A}_\tau - \tau\mathbb{B})\,u = 0 \qquad \text{in } H_0^2(D). \tag{30}$$

where $\mathbb{A}_\tau$ is self adjoint, coercive and depends continuously on $\tau \in (0, \infty)$ and $\mathbb{B}$ is a self adjoint compact and non-negative operator.

We now consider the generalized eigenvalue problem

$$(\mathbb{A}_\tau - \lambda(\tau)\mathbb{B})\,u = 0 \qquad \text{in } H_0^2(D) \tag{31}$$

noting that $k^2 = \tau$ defines a transmission eigenvalue $k = \sqrt{\tau}$ if and only if $\tau > 0$ satisfies $\lambda(\tau) = \tau$. Since the generalized eigenvalue problem (31) can be written in the form

$$\left(I - \lambda(\tau)\mathbb{A}_\tau^{-1/2}\mathbb{B}\mathbb{A}_\tau^{-1/2}\right) u = 0 \qquad \text{in } H_0^2(D)$$

we conclude that for a fixed $\tau > 0$ there exists an increasing sequence of eigenvalues $\lambda_j(\tau)$, such that $\lambda_j(\tau) \to \infty$ as $j \to \infty$. These eigenvalues satisfy

$$\lambda_j(\tau) = \min_{W \subset \mathcal{U}_j} \left(\max_{u \in W \setminus \{0\}} \frac{(\mathbb{A}_\tau u, u)}{(\mathbb{B}u, u)} \right) \tag{32}$$

where $\mathcal{U}_j$ denotes the set of all j dimensional subspaces W of $H_0^2(D)$ such that $W \cap \ker(\mathbb{B}) = \{0\}$. We now recall the following theorem from [9]:

Theorem 4.2. *Assume that there exist two positive constants $\tau_0 > 0$ and $\tau_1 > 0$ such that*

(1) $\mathbb{A}_{\tau_0} - \tau_0 \mathbb{B}$ is positive on U,
(2) $\mathbb{A}_{\tau_1} - \tau_1 \mathbb{B}$ is non positive on a m dimensional subspace of U.

Then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1, \ldots, m$, has at least one solution in $[\tau_0, \tau_1]$ where $\lambda_j(\tau)$ is the j^{th} eigenvalue (counting multiplicity) of $\mathbb{A}_\tau$ with respect to $\mathbb{B}$.

To establish the existence of the constant τ_0 we note that

$$\begin{aligned} (\mathbb{A}_\tau u - \tau \mathbb{B} u, u)_{H^2} = & \int_D \frac{1}{n-1} |\Delta u + n\tau u|^2 \, ds - \tau^2 \int_D |u|^2 \, ds \\ & + \tau \int_D |\nabla u|^2 \, ds. \end{aligned} \tag{33}$$

Since

$$\lambda(D) \le \inf_{u \in H_0^2(D)} \frac{\int_D |\nabla u|^2 \, ds}{\int_D |u|^2 \, ds}$$

$\mathbb{A}_{\tau_0} - \tau_0 \mathbb{B}$ is a positive operator for all τ_0 such that $0 < \tau_0 < \lambda(D)/n^*$ where $\lambda(D)$ is the first Dirichlet eigenvalue for $-\Delta$ in D (In particular (23) is valid).

To establish the existence of the constant τ_1, let B be a ball contained in D and let $k_{n_*}^B$ and $u_{n_*}^B \in H_0^2(B)$ be a transmission eigenvalue and eigenvector, respectively, corresponding to the ball B and index of refraction n_*. Set $\tau_1 := (k_{n_*}^B)^2$ and let $\tilde{u} \in H_0^2(D)$ be the extension by zero of $u_{n_*}^B$ to the whole of D. Then

$$\begin{aligned}(\mathbb{A}_{\tau_1}\tilde{u} - \tau_1\mathbb{B}\tilde{u}, \tilde{u})_{H^2} &= \int\limits_D \frac{1}{n-1}|\Delta\tilde{u} + \tau_1\tilde{u}|^2\, ds + \tau_1^2 \int\limits_D |\tilde{u}|^2\, ds \\ &- \tau_1 \int\limits_D |\nabla\tilde{u}|^2\, ds \\ &\leq \int\limits_B \frac{1}{n_*-1}|\Delta u^B_{n_*} + \tau_1 u^B_*|^2\, ds + \tau_1^2 \int\limits_B |u^B_{n_*}|^2\, ds \\ &- \tau_1 \int\limits_B |\nabla u^B_{n_*}|^2\, ds = 0.\end{aligned} \tag{34}$$

and hence $\mathbb{A}_{\tau_1} - \tau_1\mathbb{B}$ is a non-positive operator in the subspace spanned by $\tilde{u}$.

The above argument shows that there exists at least one transmission eigenvalue between $\sqrt{\lambda(D)/n^*}$ and $k^B_{n_*}$. To establish the existence of infinitely many transmission eigenvalues take $\epsilon > 0$ small enough such that D contains $m(\epsilon) \geq 1$ disjoint balls $B^1_\epsilon, B^2_\epsilon, \ldots B^{m(\epsilon)}_\epsilon$ of radius ϵ. Let k_{ϵ,n_*} be the first transmission eigenvalue for each of these balls with index of refraction n_* and let $u^{B_j}_{n_*} \in H^2_0(B_j)$, $j = 1, \ldots, m(\epsilon)$ be the corresponding eigenfunctions and $\tilde{u}^j \in H^2_0(D)$ their extension by zero to the whole of D. Note that $\{\tilde{u}^1, \tilde{u}^2, \ldots \tilde{u}^{m(\epsilon)}\}$ are linearly independent in $H^2_0(D)$ since they have disjoint supports. By the above $\mathbb{A}_{\tau_1} - \tau_1\mathbb{B}$ is a non-positive operator in the subspace spanned by $\{\tilde{u}^1, \tilde{u}^2, \ldots \tilde{u}^{m(\epsilon)}\}$ where $\tau_1 := (k_{\epsilon,n_*})^2$. Hence there exists $m(\epsilon)$ transmission eigenvalues less then k_{ϵ,n_*}. Letting $\epsilon \to 0$ and noting the finite multiplicity of transmission eigenvalues proves the results.

This completes the outline of the proof of Theorem 4.1.

The inequality (23) can be further improved [7], [8]. To this end let $B_{r_1} \subset D \subset B_{r_2}$ where B_{r_1} is the largest such ball and B_{r_2} is the smallest such ball. For a given $0 < \epsilon \leq r_2$, let $m(\epsilon) \in \mathbb{N}$ be the number of balls B_ϵ of radius ϵ that are contained in D. We denote by k_{1,n_*} and k_{1,n^*} the first transmission eigenvalue for the ball of radius one with index of refraction n_* and n^*, respectively. Finally let $k_{1,D,n(x)}$ denote the first transmission eigenvalue corresponding to D and the given index of refraction $n(x)$.

Corollary 4.1. *Assume that $n \in L^\infty(D)$. Then*

(1) If $1+\alpha \le n_* \le n(x) \le n^* < \infty$, *then*

$$0 < \frac{k_{1,n^*}}{r_2} \le k_{1,D,n(x)} \le \frac{k_{1,n_*}}{r_1}. \tag{35}$$

(2) If $0 < n_* \le n(x) \le n^* < 1-\beta$, *then*

$$0 < \frac{k_{1,n_*}}{r_2} \le k_{1,D,n(x)} \le \frac{k_{1,n^*}}{r_1}. \tag{36}$$

Proof. First we note that from the scaling properties of the spherical Bessel functions, the first transmission eigenvalue for B_{r_2} with index of refraction n^* equals $k_{1,n^*}/r_2$. Similar scaling properties hold for the first transmission eigenvalue for B_{r_2} and index of refraction n_* as well as for B_{r_1} with index of refraction n^* or n_*. The right inequality in (35) is shown in the proof of Theorem 4.1. In the following we show the validity of the left inequality in (35) for the case when $1+\alpha \le n_* \le n(x) \le n^* < \infty$. Obviously for any $u \in H_0^2(D)$ we have

$$\begin{aligned}\frac{\frac{1}{n^*-1}\|\Delta u + \tau u\|_D^2 + \tau^2\|u\|_D^2}{\|\nabla u\|_D^2} &\le \frac{\int_D \frac{1}{n(x)-1}|\Delta u+\tau u|^2\,dx + \tau^2\|u\|_D^2}{\|\nabla u\|_D^2}\\ &\le \frac{\frac{1}{n_*-1}\|\Delta u + \tau u\|_D^2 + \tau^2\|u\|_D^2}{\|\nabla u\|_D^2}.\end{aligned} \tag{37}$$

Therefore from (32) we have that for an arbitrary $\tau > 0$

$$\begin{aligned}\lambda_1(\tau, B_{r_2}, n^*) \le \lambda_1(\tau, D, n^*) \le \lambda_1(\tau, D, n(x))\\ \le \lambda_1(\tau, D, n_*) \le \lambda_1(\tau, B_{r_1}, n_*)\end{aligned}$$

where $\lambda_1(\tau, D, n^*)$, $\lambda_1(\tau, D, n(x))$ and $\lambda_1(\tau, D, n_*)$ are the first eigenvalues of the auxiliary problem for D and n^*, $n(x)$ and n_*, respectively, and $\lambda_1(\tau, B_{r_2}, n^*)$ and $\lambda_1(\tau, B_{r_1}, n_*)$ are the first eigenvalues of the auxiliary problem for B_{r_2}, n^* and B_{r_1}, n_*, respectively. Now for $\tau_1 := (k_{1,n_*}/r_1)^2$ we have that $\lambda_1(\tau_1, D, n(x)) - \tau_1 \le 0$ since in the subspace spanned by the extension by zero to the whole of D of the eigenfunction $u_{1,n_*}^{B_1}$ the Rayleigh quotient minus τ for $\tau = \tau_1$ becomes negative. On the other hand, for $\tau_0 := (k_{1,n^*}/r_2)^2$, we have $\lambda_1(\tau_0, B_{r_2}, n^*) - \tau_0 = 0$ and hence $\lambda_1(\tau_0, D, n(x)) - \tau_0 \ge 0$. Therefore the first eigenvalue $k_{1,D,n(x)}$ corresponding to D and $n(x)$ is between $k_{1,n^*}/r_2$ and $k_{1,n_*}/r_1$. Note that there is no transmission eigenvalue for D and $n(x)$ that is less then $k_{1,n^*}/r_2$. Indeed if there is a transmission eigenvalue strictly less then $k_{1,n^*}/r_2$ then by the monotonocity of the eigenvalues of the auxiliary problem with respect to the domain and the fact that for τ small enough there are no transmission eigenvalues we would have found an eigenvalue of the ball B_{r_2} and n^* that is strictly smaller then the first eigenvalue. □

Remark 4.1. The inequality (23) was first obtained in [14].

Remark 4.2. The existence of finitely many transmission eigenvalues (under more restrictive assumptions then in Theorem 4.1) was first given in [17].

Remark 4.3. Theorem 4.1 can be generalized to include the case where D has cavities, i.e. there exit regions in D where $n(x) = 1$ [6].

References

1. T. Arens, *Why linear sampling method works*, *Inverse Problems* **20**, 163-173 (2004).
2. T. Arens and A. Lechleiter, *The linear sampling method revisited*, *J. Int. Eqns. Appl.* **21**, 179-202 (2009)
3. F. Cakoni, M. Cayoren and D. Colton, *Transmission eigenvalues and the non-destructive testing of dielectrics*, *Inverse Problems* **24**, 066016 (2008).
4. F. Cakoni and D Colton and H Haddar, *Qualitative Methods in Inverse Scattering Theory*, Springer, Berlin, 2006.
5. F. Cakoni, D Colton and H Haddar, *On the determination of Dirichlet and transmission eigenvalues from far field data*, to appear.
6. F. Cakoni, D. Colton and H. Haddar, *The interior transmission problem for regions with cavities*, to appear.
7. F. Cakoni and D. Gintides, *New results on transmission eigenvalues*, to appear.
8. F. Cakoni, D. Gintides and H. Haddar, *The existence of an infinite discrete set of transmission eigenvalues*, to appear.
9. F. Cakoni and H Haddar, *On the existence of transmission eigenvalues in an inhomogeneous medium*, *Applicable Analysis*, **88**, 475-493 (2009).
10. D. Colton, *Analytic Theory of Partial Differential Equations*, Pitman, Boston, 1980.
11. D. Colton, A. Kirsch and L. Päivärinta, *Far field patterns for acoustic waves in an inhomogeneous medium*, *SIAM Jour. Math. Anal.*, **20**, 1472-1483 (1989).
12. D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 2nd edition, Springer, New York, 1998.
13. D. Colton, P. Monk and J. Sun, *Analytic and computational methods for transmission eigenvalues*, to appear.
14. D. Colton, L. Päivärinta and J. Sylvester, *The interior transmission problem*, *Inverse Problems and Imaging* **1**, 13-28 (2007).
15. A. Kirsch and N. Grinberg, *Thd Factorization Method for Inverse Problems*, Oxford University Press, Oxford, 2008.

16. A. Kirsch, *On the existence of transmission eigenvalues*, *Inverse Problems and Imaging* **3**, 155-172 (2009).
17. L. Päivärinta and J. Sylvester, *Transmission Eigenvalues*, *SIAM J. Math. Anal.* **40**, 738-753 (2008).
18. B. P. Rynne and B.D. Sleeman, *The interior transmission problem and inverse scattering from inhomogeneous media*, *SIAM J. Math. Anal.* **22**, 1755-1762 (1992).
19. R.M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, San Diego, 2001.