

## THE INVERSE SCATTERING PROBLEM FOR ANISOTROPIC MEDIA

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We give a survey of recent results on the inverse scattering problem for anisotropic media including both uniqueness theorems and reconstruction algorithms.

### 1. Introduction

The inverse electromagnetic scattering problem for anisotropic media is significantly different than the corresponding problem for isotropic media. Further problems arise when an anisotropic dielectric medium is coated by a thin layer of a highly conducting material. Since such problems increasingly occur in practice, it seems appropriate to us to provide a short survey of the salient features of inverse scattering problems of this type. The present paper is our attempt to do so. For the sake of simplicity we will restrict ourselves to the special case when the direct scattering problem can be reduced to a coupled set of scalar equations. In particular we will assume that the scattering object is an infinitely long cylinder with cross section  $D$  and that the incident time-harmonic plane wave is such that the electric field is polarized perpendicular to the axis of the cylinder. We further assume that the scatterer is *orthotropic*, i.e. the index of refraction is given by

$$N = \frac{1}{\epsilon_0} \left( \epsilon(x) + i \frac{\sigma(x)}{\omega} \right) = \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{21} & n_{22} & 0 \\ 0 & 0 & n_{33} \end{pmatrix}$$

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where  $\epsilon_0$  is the permittivity of the free space,  $\omega$  is the frequency, and  $\epsilon(x)$  and  $\sigma(x)$  are the permittivity and conductivity respectively, of the scattering object. In addition we allow for the possibility that the scatterer may be coated on a portion of the boundary with a thin layer of highly conductive material. The real valued function  $\eta > 0$  defined on the coated portion of the boundary describe the physical properties of the coating. We assume that the surface conductivity  $\eta$  is constant along the axis of the cylinder. All the results that we shall present for this special case have analogous results for the general case of the scattering of electromagnetic waves by a bounded obstacle in  $\mathbb{R}^3$  and we refer the reader to <sup>3, 7</sup> and <sup>16</sup> for details in this case. For other work on the scattering for an anisotropic medium see <sup>10, 13, 17, 18</sup>.

## 2. The Model Problem

Let the bounded domain  $D \subset \mathbb{R}^2$  with Lipschitz boundary  $\Gamma$  be the cross section of the cylinder such that the exterior domain  $D_e := \mathbb{R}^2 \setminus \overline{D}$  is connected. We denote by  $\nu$  the outward unit normal to  $\Gamma$  defined almost everywhere on  $\Gamma$ . The boundary  $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2}$  is split into two open disjoint parts  $\Gamma_1$  and  $\Gamma_2$ . Here  $\Gamma_1$  corresponds to the uncoated part and  $\Gamma_2$  corresponds to the coated part. Under the assumptions given above, the incident, scattered and interior magnetic fields have the form

$$H^i = (0, 0, u^i), \quad H_0 = (0, 0, v), \quad H^s = (0, 0, u^s)$$

respectively. Assuming that  $N^{-1}$  exists and expressing the electric fields in terms of magnetic fields lead to the following transmission problem for  $v$  and  $u$ :

$$\begin{aligned} (i) \quad & \nabla \cdot A \nabla v + k^2 v = 0 && \text{in } D \\ (ii) \quad & \Delta u + k^2 u = 0 && \text{in } D_e \\ (iii) \quad & v - u = 0 && \text{on } \Gamma_1 \\ (iv) \quad & v - u = -i\eta \frac{\partial u}{\partial \nu} && \text{on } \Gamma_2 \\ (v) \quad & \frac{\partial v}{\partial \nu_A} - \frac{\partial u}{\partial \nu} = 0 && \text{on } \Gamma \\ (vi) \quad & u = u^s + u^i \\ (vii) \quad & \lim_{r \rightarrow \infty} \sqrt{r} \left( \frac{\partial u^s}{\partial r} - iku^s \right) = 0, \end{aligned}$$

where  $r = |x|$ ,  $u^s$  is the scattered field and  $u^i$  is the given incident field. In the case of incident plane waves the incident field is given by  $u^i := e^{ikx \cdot d}$ ,

$d \in \Omega := \{x : |x| = 1\}$ . Moreover

$$\frac{\partial v}{\partial \nu_A}(x) := \nu(x) \cdot A(x) \nabla v(x), \quad x \in \Gamma,$$

$$A = \frac{1}{n_{11}n_{22} - n_{12}n_{21}} \begin{pmatrix} n_{11} & n_{21} \\ n_{12} & n_{22} \end{pmatrix}.$$

and the radiation condition holds uniformly with respect to  $\hat{x} = x/|x|$ .

In the following we assume that  $A$  is a  $2 \times 2$  matrix-valued function whose entries are continuously differentiable functions in  $\overline{D}$  such that  $A$  is symmetric,  $\mathcal{R}e(\bar{\xi} \cdot A \xi) \geq \gamma |\xi|^2$ ,  $\mathcal{R}e(\bar{\xi} \cdot A^{-1} \xi) \geq \gamma |\xi|^2$  and  $\mathcal{I}m(\bar{\xi} \cdot A \xi) \leq 0$  for all  $\xi \in \mathbb{C}^2$  and  $x \in \overline{D}$  where  $\gamma$  is a positive constant. Note that due to the symmetry of  $A$  we have  $\mathcal{R}e(\bar{\xi} \cdot A \xi) = \bar{\xi} \cdot \mathcal{R}e(A) \xi$  and  $\mathcal{I}m(\bar{\xi} \cdot A \xi) = \bar{\xi} \cdot \mathcal{I}m(A) \xi$ . Moreover we require that  $\eta \in L_\infty(\Gamma_2)$  and  $\eta(x) \geq \eta_0 > 0$  almost everywhere on  $\Gamma_2$ .

Under the above assumptions the direct scattering problem has a unique solution  $v \in H^1(D)$  and  $u^s \in H_{loc}^1(D_e)$  where the boundary conditions are assumed in the sense of the trace operator<sup>5</sup>. It is known that the scattered field  $u^s$  has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O(r^{-3/2}) \quad (1)$$

as  $r \rightarrow \infty$  where  $r = |x|$ ,  $\hat{x} = x/|x|$ ,  $k$  is fixed and  $u_\infty(\hat{x}, d)$  is the *far field pattern* of the scattered field  $u^s$ . The *inverse scattering problem* that we shall consider in this paper is to determine the shape  $D$ , the surface conductivity  $\eta$  and the index of refraction  $A$  from a knowledge of the far field pattern  $u_\infty(\hat{x}, d)$  for  $\hat{x}, d \in \Omega$  (and possibly a range of frequencies  $k$ ). It in fact suffices in the following to only know  $u_\infty(\hat{x}, d)$  in a limited aperture, i.e. for  $d \in \Omega_1 \subset \Omega$  and  $\hat{x} \in \Omega_2 \subset \Omega$  where  $\Omega_1$  and  $\Omega_2$  are open subsets of the unit circle  $\Omega$ .

### 3. Uniqueness Theorems

We begin our analysis of the above inverse scattering problem by considering conditions under which  $D$  and  $\eta$  are uniquely determined.

**Theorem 3.1.**<sup>17, 3</sup> *Assume that either  $\bar{\xi} \cdot \mathcal{R}e(A - I) \xi \geq \delta \|\xi\|^2$  or  $\bar{\xi} \cdot \mathcal{R}e(I - A) \xi \geq \delta \|\xi\|^2$  in  $D$  for some  $\delta > 0$ . Then  $D$  is uniquely determined by  $u_\infty(\hat{x}, d)$  for  $\hat{x}, d \in \Omega$  and a fixed value of the wave number  $k$ .*

The proof of the above theorem is due to Hähner<sup>17</sup> for the case  $\eta = 0$  and to Cakoni and Colton<sup>3</sup> for the case  $\eta > 0$ . In each case the proof is based on the well-posedness of the *interior transmission problem* which we will discuss later on in the paper.

**Theorem 3.2.**<sup>6</sup> *Assume that  $\mathcal{I}m(A) = 0$  but is otherwise arbitrary. Then  $\eta \in L_\infty(\Gamma_2)$  is uniquely determined from  $u_\infty(\hat{x}, d)$  for  $\hat{x}, d \in \Omega$  and a fixed value of the wave number  $k$  (except for possibly a discrete set of  $k$  values).*

In the contrast to the above positive results,  $u_\infty(\hat{x}, d)$  does not in general uniquely determine  $A$  even if  $u_\infty(\hat{x}, d)$  is known for an interval of values of  $k$ <sup>15</sup> !

#### 4. The Interior Transmission Problem

Let  $H_0^{(1)}$  denote a Hankel function of the first kind of order zero and let

$$\Phi(x, z) := \frac{i}{4} H_0^{(1)}(k|x - z|)$$

be the radiating fundamental solution to the Helmholtz equation with source at  $x = z$ . Then  $\Phi(x, z)$  has the far field pattern

$$\Phi_\infty(\hat{x}, z) = \frac{e^{1\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot z}.$$

We now define the *far field equation*

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z) \quad (2)$$

for  $g \in L^2(\Omega)$  and  $z \in \mathbb{R}^2$  where  $F : L^2(\Omega) \rightarrow L^2(\Omega)$  is *far field operator*

$$Fg(\hat{x}) := \int_{\Omega} u_\infty(\hat{x}, d)g(d) ds(d). \quad (3)$$

The far field equation will play a basic role in our forthcoming analysis. We began by noting<sup>2</sup> that for  $z \in D$  the far field equation has a solution  $g$  if and only if there exists a solution  $(v_z, w_z)$  of the *interior transmission problem*

$$\begin{aligned} \nabla \cdot A \nabla w_z + k^2 w_z &= 0 && \text{in } D \\ \Delta v_z + k^2 v_z &= 0 && \text{in } D \\ w_z - v_z &= \Phi(\cdot, z) && \text{on } \Gamma_1 \\ w_z - v_z &= \Phi(\cdot, z) - i\eta \frac{\partial}{\partial \nu} (v_z + \Phi(\cdot, z)) && \text{on } \Gamma_2 \\ \frac{\partial w_z}{\partial \nu_A} - \frac{\partial v_z}{\partial \nu} &= \frac{\partial \Phi(\cdot, z)}{\partial \nu} && \text{on } \Gamma \end{aligned}$$

such that  $v_z$  is a *Herglotz wave function*

$$v_g(x) = \int_{\Omega} e^{ikx \cdot d} g(d) ds(d), \quad x \in \mathbb{R}^2, \quad g \in L^2(\Omega). \quad (4)$$

**Definition 4.1.** *Transmission eigenvalues* are the values of  $k > 0$  for which the homogeneous interior transmission problem (i.e.  $\Phi(\cdot, z)$  is replaced by zero in the above formulation of the interior transmission problem) has a nontrivial solution.

Due to their importance in the study of the far field operator, the interior transmission problem and transmission eigenvalues have come to play an important role in inverse scattering theory<sup>11</sup>. Here we summarize what is known about the interior transmission problem in the case of orthotropic media. We assume that in  $D$  we have either  $\|\mathcal{R}e(A^{-1})\|_2 \geq \delta > 1$  or  $0 < \beta \leq \|\mathcal{R}e(A^{-1})\|_2 \leq \delta < 1$  for positive constants  $\delta$  and  $\beta$  where  $\|\cdot\|_2$  denotes the Euclidean norm.

- (1) The interior transmission problem satisfies the Fredholm alternative in  $\mathbb{H}^1(D, \Gamma_2) \times H^1(D)$ , where

$$\mathbb{H}^1(D, \Gamma_2) := \left\{ u \in H^1(D) \quad \text{such that} \quad \frac{\partial u}{\partial \nu} \in L^2(\Gamma_2) \right\}.$$

In the case where  $\eta = 0$  the interior transmission problem satisfies the Fredholm alternative in  $H^1(D) \times H^1(D)$ <sup>5</sup>.

- (2) If  $\mathcal{I}m(A) < 0$  in  $D$  then there are no transmission eigenvalues<sup>2</sup>.  
(3) If  $\mathcal{I}m(A) = 0$  then the set of transmission eigenvalues is either empty or a discrete set<sup>4</sup>.  
(4) If  $\mathcal{I}m(A) = 0$  and  $\Gamma_2 \neq \emptyset$ , the transmission eigenvalues form a subset of the transmission eigenvalues corresponding to the same interior transmission problem for  $\eta = 0$  (or  $\Gamma_2 = \emptyset$ )<sup>5</sup>.  
(5) Any transmission eigenvalue  $k > 0$  must satisfies

$$k^2 \geq \frac{\lambda(D)}{\sup_D \|A^{-1}\|_2}$$

if  $\|A^{-1}\|_2 \geq \delta > 1$  where  $\lambda(D)$  is the first Dirichlet eigenvalue of  $-\Delta$  in  $D$ <sup>4</sup>. If  $0 < \beta \leq \|A^{-1}\|_2 \leq \delta < 1$  then all that can be said is that

$$k^2 \geq \lambda(D).$$

It is an open question whether transmission eigenvalues exist for anisotropic media. The first result related to the existence of the transmission eigenvalues in the case of isotropic media is given in <sup>19</sup>.

The following two remarks play an important role in our method for solving the inverse scattering problem.

**Remark 4.1.** Let  $k$  be a transmission eigenvalue and  $(v, w)$  the non-zero solution of the homogeneous interior transmission problem. Then, if  $v$  is a Herglotz wave function, the far field operator  $F : L^2(\Omega) \rightarrow L^2(\Omega)$  is not injective and the range is not dense <sup>2</sup>.

**Remark 4.2.** If  $(v_z, w_z)$  is the solution to the interior transmission problem then for every  $\epsilon > 0$  there is a Herglotz wave function  $v_{g_\epsilon}$  <sup>5</sup> such that

$$\|v_{g_\epsilon} - v_z\|_{\mathbb{H}^1(D, \Gamma_2)} < \epsilon.$$

In particular,

$$\|v_{g_\epsilon} - v_z\|_{H^1(D)} < \epsilon$$

which is proven independently in <sup>14</sup>.

## 5. The Determination of $D$

We now return to the far

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z).$$

Assuming that  $k > 0$  is not a transmission eigenvalue, we have the following theorem:

**Theorem 5.1.** <sup>5</sup> For  $z \in D$  and a given  $\epsilon > 0$  there exists a  $g_z^\epsilon \in L^2(\Omega)$  such that

$$\|Fg_z^\epsilon - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} < \epsilon$$

and the Herglotz wave function  $v_{g_z^\epsilon}$  converges in  $\mathbb{H}^1(D, \Gamma_2)$  to  $v_z$  where  $v_z, w_z$  is the solution of the interior transmission problem. Furthermore,

$$\lim_{z \rightarrow \partial D} \|v_{g_z^\epsilon}\|_{\mathbb{H}^1(D, \Gamma_2)} = \infty \text{ and } \lim_{z \rightarrow \partial D} \|g_z^\epsilon\|_{L^2(\Omega)} = \infty.$$

On the other hand for  $z \in \mathbb{R}^2 \setminus \overline{D}$  and a given  $\epsilon > 0$ , every  $g_z^\epsilon \in L^2(\Omega)$  that satisfies

$$\|Fg_z^\epsilon - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} < \epsilon$$

is such that

$$\lim_{\epsilon \rightarrow 0} \|v_{g_z^\epsilon}\|_{\mathbb{H}^1(D, \Gamma_2)} = \infty.$$

The above theorem can be used to numerically determine  $D$  <sup>5</sup>. This is done by 1) constructing a rectangular grid of "sampling points"  $\mathcal{G}$  in a region known a priori to contain  $D$ , 2) for  $z_i \in \mathcal{G}$  solve the regularized far field equation

$$(\alpha I + F^* F)g_{z_i} = \Phi_\infty(\hat{x}, z)$$

and 3) choose a cut-off value  $C$  and assert that  $z \in D$  if and only if  $\|g_z\| \leq C$ . The above algorithm for determining  $D$  is called the *linear sampling method* and was first introduced for the case of isotropic media by Colton and Kirsch <sup>9</sup> and Colton, Piana and Potthast <sup>12</sup>. An open question is whether or not the regularized solution  $g_z$  behaves in the same way as the  $g_z^\epsilon$  of Theorem 5.1. A positive answer to this question in the case when the far field operator is normal has been given by Arens and Lechleiter in <sup>1</sup>, <sup>17</sup>. Examples of the reconstruction of  $D$  are shown in the figure below which is taken from <sup>8</sup>.

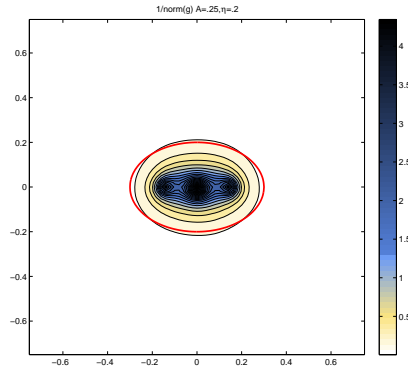


Figure 1. Here it is shown the reconstruction of the partially coated (on the lower half) ellipse with  $A = 0.25I$  and  $\eta = 0.2$ .

## 6. The Determination of $\eta$

We now assume that  $D$  has been determined by the linear sampling method described in the previous section and furthermore assume that  $\mathcal{I}m(A) = 0$ , i.e. the scattering obstacle is a dielectric. Let  $\tilde{\eta}$  denote the extension by

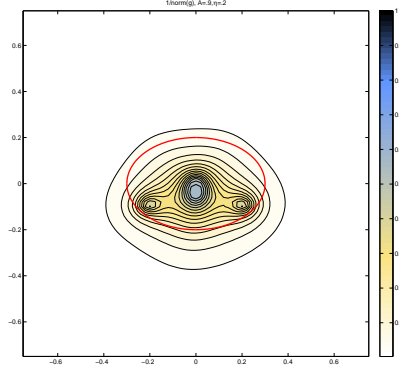


Figure 2. Here it is shown the reconstruction of the partially coated (on the lower half) ellipse with  $A = 0.9I$  and  $\eta = 0.2$ . Note that the scatterer is very weak and the presence of the metallic coating makes it more visible.

zero of  $\eta$  to the whole boundary  $\partial D$  and let  $(v_z, w_z)$  be the solution of the interior transmission problem. Then for  $z \in D$  we have the integral equation <sup>5</sup>

$$\int_{\partial D_2} \tilde{\eta}(x) |v_z(x) + \Phi(x, z)|^2 ds(x) = -1/4 - \text{Im}(v_z(z)).$$

Recalling that  $v_z(z) \approx v_{g_z^\epsilon}(z)$  where  $g_z^\epsilon$  is the approximate solution of the far field equation given in Theorem 5.1, we now have a method for determining  $\tilde{\eta}$  and hence  $\eta$ . Numerical examples using this method for determining  $\eta$  can be found in <sup>6</sup>. An example from this paper is given below.

## 7. Estimates for $A$

As noted in Section 3,  $A$  is not uniquely determined from the far field pattern  $u_\infty(\hat{x}, d)$  even if the far field pattern is known for an interval of  $k$  values. However from point (5) and Remark 4.1 of Section 4 we have that, first, if  $\|A^{-1}(x)\|_2 \geq \delta > 1$  for all  $x \in D$  and some constant  $\delta$  then

$$\sup_D \|A^{-1}\|_2 \geq \frac{\lambda(D)}{k^2}$$

where  $k$  is the first transmission eigenvalue and  $\lambda(D)$  is the first Dirichlet eigenvalue of  $-\Delta$  in  $D$  and, second, the norm of the (regularized) solution

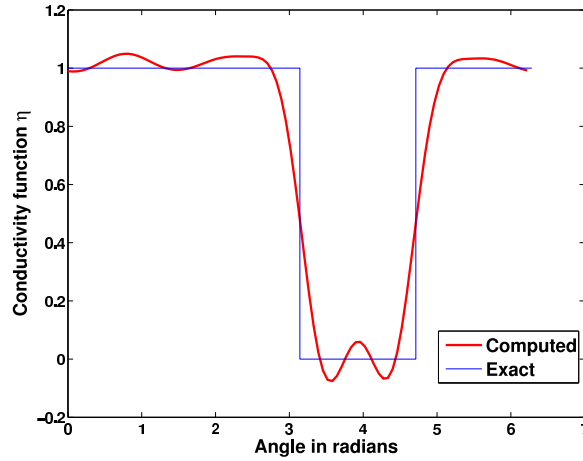


Figure 3. Here it shown the reconstructed  $\eta$  for a partially coated circle. The circle is not coated on the part corresponding to the angular sector  $\pi < \phi < 3\pi/2$ . The blue line show the exact  $\eta$ .

to

$$(Fg)(\hat{x}) = \Phi_{\infty}(\hat{x}, z), \quad z \in D$$

should be large for such values of  $k$ . These two points provide a method for obtaining a lower bound for  $\sup_D \|A^{-1}\|_2$ . The following example is taken from <sup>4</sup>.

Table 1. First transmission eigenvalues ( $k_0$ ) and lower bounds  $n_{\min}$  of the index of refraction  $n$  where  $A^{-1} = nI$  for  $D$  as defined in the caption to Figure 4

$n$	2.	3.	4.	6.	9.	12.	16.
$k_0$	15.5	8.1	6.3	4.5	3.3	2.8	2.3
$n_{\min}$	0.2	0.6	1.	1.9	3.5	4.9	7.2

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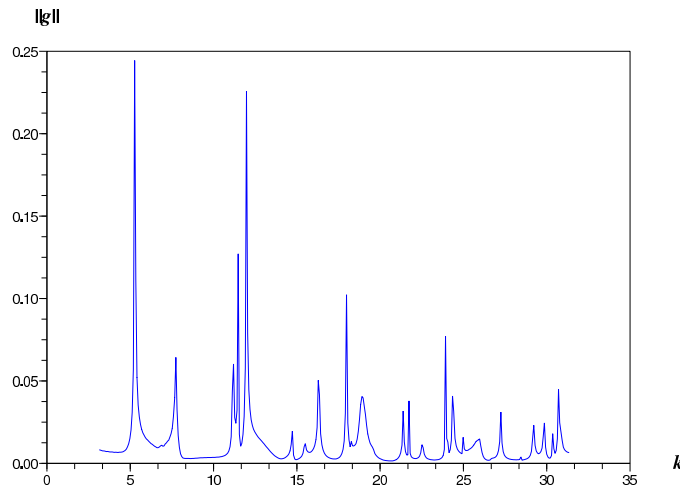


Figure 4. Here we show the transmission eigenvalues corresponding to the scatterer  $D$  being the L-shape  $= \{[-0.5, 0.5] \times [-0.5, 0.5]\} \setminus \{[0, 0.5] \times [0, 0.5]\}$ ,  $A^{-1} = nI$ ,  $n = 4$  and  $\eta = 0$ .

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