Mixed Boundary Value Problems in Inverse Electromagnetic Scattering

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Introduction

Mixed boundary value problems in electromagnetic scattering theory arise when the scattering object is a composite material such that parts of the scatterer have different electrical properties.

Such scattering objects can be:

- *Partially coated perfect conductors.*
- *Thin objects with one side a perfect conductor and the other side an imperfect conductor or dielectric.*
- *Partially coated dielectrics.*

Introduction

In general the physical properties of the scattering object are not known a priori, e.g. it is not known if an object is coated or not and if so what the extent and the composition of the coating is.

The associated inverse scattering problem is to determine the shape and physical properties of the obstacle from a knowledge of the asymptotic behavior of the scattered field due to the scattering of an incident time-harmonic electromagnetic plane wave at fixed frequency.

Introduction

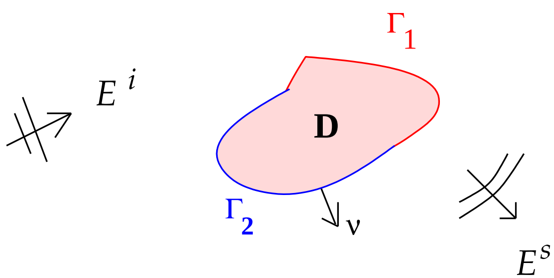
The direct scattering problem

- *The mathematical analysis of mixed boundary value problems is particularly difficult due to the non-standard solution space.*
- *No matter how smooth the boundary data is, the change of boundary conditions causes the scattered field to be singular at the interface. This give rise to numerical and computational* difficulties.

The inverse scattering problem

- *Since the physical structure of the composite medium is not known a priori, the use of weak scattering approximations and/or nonlinear optimization techniques are problematic.*

Partially Coated Dielectrics: TE Polarization Case



Assume

- *The scatterer is an infinitely long cylinder.*
- *The incident wave is such that the electric field is polarized $\perp$ to the cylinder axis.*

- *The dielectric is orthotropic, i.e. the index of refraction is given by*

$$N(x) = \frac{1}{\epsilon_0}\left(\epsilon(x) + i\frac{\sigma(x)}{\omega}\right) = \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{21} & n_{22} & 0 \\ 0 & 0 & n_{33} \end{pmatrix}.$$

Thus the incident, scattered and interior magnetic fields have the form

$$H^i(0,\ 0,\ u^i), \quad H^{int} = (0,\ 0,\ v), \quad H^s = (0,\ 0,\ u^s)$$

The Forward Problem

$$\nabla \cdot A\nabla v + k^2 v = 0 \quad \text{in } D \quad \text{and} \quad \Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}$$

$$v - (u^s + u^i) = 0 \quad \text{on } \Gamma_1$$

$$v - (u^s + u^i) = -i\eta(x)\frac{\partial(u^s + u^i)}{\partial\nu} \quad \text{on } \Gamma_2$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial(u^s + u^i)}{\partial\nu} = 0 \quad \text{on} \quad \Gamma = \Gamma_1 + \Gamma_2$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

$u^i(x) = e^{ikx\cdot d}$, A *is symmetric*, $\mathcal{R}e\left(\overline{\xi} \cdot A\xi\right) \geq \gamma|\xi|^2 \; \gamma > 0$, $\mathcal{I}m\left(\overline{\xi} \cdot A\xi\right) \geq 0$, $\eta \in L_\infty(\Gamma_2)$, $\eta(x) \geq \eta_0 > 0$, $\frac{\partial v}{\partial \nu_A} := \nu \cdot A\nabla v$.

Far Field Pattern

Theorem: There exists a unique solution (v, u) in $H^1(D) \times H^1_{loc}(\mathbb{R}^2 \setminus \overline{D})$ to the mixed transmission problem.

The scattered field u^s has the asymptotic behaviour

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O\left(r^{-3/2}\right)$$

as $r \to \infty$ where $r = |x|$, $\hat{x} = x/r$, k is fixed and u_∞ is the far field pattern of the scattered field u^s.

The inverse scattering problem is to determine D and η from a knowledge of $u_\infty(\hat{x}, d)$ for $\hat{x}, -d \in \Omega_0 \subset \Omega$ where $\Omega := \{x : \ |x| = 1\}$

Uniqueness Theorems

Uniqueness of D

Theorem: Assume that either $\mathcal{R}e\left(\overline{\xi}\cdot A\xi\right) \geq \gamma|\xi|^2$ or $\mathcal{R}e\left(\overline{\xi}\cdot A^{-1}\xi\right) \geq \gamma|\xi|^2$ for some $\gamma > 1$. Then D is uniquely determined by $u_\infty(\hat{x}, d)$ for $\hat{x}, -d \in \Omega_0 \subset \Omega$.

Open Problem: Remove the assumption that $\mathcal{R}e\left(\overline{\xi}\cdot A\xi\right) \geq \gamma|\xi|^2$ *or* $\mathcal{R}e\left(\overline{\xi}\cdot A^{-1}\xi\right) \geq \gamma|\xi|^2, \gamma > 1$.

Uniqueness Theorems

Uniqueness of D

Theorem: Assume that either $\mathcal{R}e\left(\overline{\xi} \cdot A\xi\right) \geq \gamma|\xi|^2$ or $\mathcal{R}e\left(\overline{\xi} \cdot A^{-1}\xi\right) \geq \gamma|\xi|^2$ for some $\gamma > 1$. Then D is uniquely determined by $u_\infty(\hat{x}, d)$ for $\hat{x}, -d \in \Omega_0 \subset \Omega$.

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Uniqueness of η

Theorem: Given A and D, then $\eta \in C(\overline{\Gamma}_2)$ is uniquely determined from $u_\infty(\hat{x}, d)$ for $\hat{x}, -d \in \Omega_0$.

Open Problem: Remove the assumption that A is fixed.

Solving the Inverse Scattering Problem

We consider the far field equation

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z) \qquad z \in \mathbb{R}^3$$

$g \in L^2(\Omega)$, *the far field operator* $F : L^2(\Omega) \to L^2(\Omega)$ *is defined by*

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d) \, ds$$

and $\Phi_\infty(\hat{x}, z) := \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot z}$ *is the far field pattern of the fundamental solution* $\Phi(x, z) := \frac{i}{4} H_0^{(1)}(k|x - z|)$.

The Herglotz Wave Function

Definition: A Herglotz wave function with kernel g is an entire solution of the Helmholtz equation defined by

$$v_g(x) := \int_\Omega e^{ikx\cdot d} g(d)\, ds(d).$$

Let $H(D) := \{\Delta w + k^2 w = 0,\ w \in H^1(D)\}$. *Then we can write the far field operator as*

$$\boxed{Fg = \mathcal{B} v_g.}$$

where $\mathcal{B} : H(D) \to L^2(\Omega)$ *is the bounded extension of the mapping* $u^i \to u_\infty$.

Solving the Far Field Equation

The interior trasmission problem is to find (v_z, w_z) *such that*

$$
\begin{array}{lll}
\nabla \cdot A\nabla v_z + k^2 v_z = 0 \ \textit{ and } \ \Delta_2 w_z + k^2 w_z = 0 & \textit{in} & D \\
v_z - w_z = \Phi(\cdot, z) & \textit{on} & \Gamma_1 \\
v_z - w_z = \Phi(\cdot, z) - i\eta(x)\dfrac{\partial}{\partial \nu}(w_z + \Phi(\cdot, z)) & \textit{on} & \Gamma_2 \\
\dfrac{\partial v_z}{\partial \nu_A} - \dfrac{\partial w_z}{\partial \nu} = \dfrac{\partial}{\partial \nu}\Phi(\cdot, z) & \textit{on} & \Gamma.
\end{array}
$$

Definition: Values of k *for which there exists a nontrivial solution to the interior transmission problem for* $\Phi = 0$ *are called transmission eigenvalues.*

Transmission Eigenvalues

Remark: If k is a transmission eigenvalue then $\dfrac{\partial w_z}{\partial \nu} = 0$ on Γ_2, i.e. v_z and w_z satisfy

$$\nabla \cdot A\nabla v_z + k^2 v_z = 0 \quad \textit{and} \quad \Delta_2 w_z + k^2 w_z = 0 \qquad \textit{in} \qquad D$$

$$v_z - w_z = 0 \quad \textit{and} \quad \frac{\partial v_z}{\partial \nu_A} - \frac{\partial w_z}{\partial \nu} = 0 \qquad \textit{on} \qquad \Gamma$$

It is unknown whether or not transmission eigenvalues exist!

Partial results showing that if they do exist they form a discrete set can be found in *F. Cakoni, D. Colton and H. Haddar, Jour. Comp. Applied Math. 146 (2002), 285-299.*

Solving the Far Field Equation

Theorem: *Assume that k is not a transmission eigenvalue and that $\Re(\overline{\xi} \cdot A\xi) \geq \gamma|\xi|^2$ or $\Re(\overline{\xi} \cdot A^{-1}\xi) \geq \gamma|\xi|^2$ for some $\gamma > 1$. Then there exists a unique solution $v_z \in H^1(D)$, $w_z \in H^1(D)$ and $\dfrac{\partial w_z}{\partial \nu} \in L^2(\Gamma_2)$ to the interior transmission problem.*

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Theorem: *The set of Herglotz wave functions v_g for all $g \in L^2(\Omega)$ is dense in*

$$\mathbb{H}^1(D) := \left\{ w \in H(D) \quad \text{such that} \quad \frac{\partial w}{\partial \nu} \in L^2(\Gamma_2) \right\}$$

equipped with the graph norm.

Solving the Far Field Equation

- For $z \in D$, $Fg = \mathcal{B}v_g = \Phi_\infty(\hat{x}, z) \Longleftrightarrow$ there exists a solution (w_z, v_z) to the interior transmission problem with $w_z = v_g$.
- The existence and uniqueness of a solution to the interior transmission problem implies
  - $\mathcal{B} : \mathbb{H}^1(D) \to L^2(\Omega)$ is injective with dense range.
  - $\Phi_\infty(\hat{x}, z) \in$ Range $\mathcal{B} \Longleftrightarrow z \in D$.

Solving the Far Field Equation

The above approximation property of Herglotz wave functions now implies the following theorem:

Theorem : *There exists an approximate solution* $g = g_z$ *to the far field equation that behaves as follows:*

- *For* $z \in D$, $\|v_g\|_{\mathbb{H}^1} < \infty \quad and \quad \|g\|_{L^2} < \infty.$
- *As* $z \to \Gamma$, $\|v_g\|_{\mathbb{H}^1} \to \infty \quad and \quad \|g\|_{L^2} \to \infty.$
- *For* $z \in \mathbb{R}^3 \setminus \overline{D}$, $\|v_g\|_{\mathbb{H}^1} \quad and \quad \|g\|_{L^2}$ *are arbitrarily large.*

Linear Sampling Method

The linear sampling method determines g from the far field equation $Fg = \Phi_\infty$.

The support D can be determined by the above behavior of g.

Open Problem: In practice g is obtained by using regularization methods. Does this regularized solution behave in the same way as the approximate solution g?

Determination of η

Assume that k is neither a transmission eigenvalue nor a Neumann eigenvalue for $-\nabla \cdot A\nabla$ and let v_z, w_z be the unique solution of the interior transmission problem. Define

$$\mathcal{V} := \left\{ f \in L^2(\Gamma_2) : \ f = \frac{\partial W_z}{\partial \nu}\bigg|_{\Gamma_2}, \ W_z = w_z + \Phi(\cdot\,, z), \ z \in B_r \right\}$$

where $B_r \subset D$. Two important properties of $W_z = w_z + \Phi(\cdot\,, z)$ are:

- *$\mathcal{V}$ is complete in $L^2(\Gamma_2)$.*

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where $B_r \subset D$. Two important properties of $W_z = w_z + \Phi(\cdot\,, z)$ are:

- *$\mathcal{V}$ is complete in $L^2(\Gamma_2)$.*
- *For any $z_1, z_2 \in D$*

$$2\int_{\Gamma_2} \eta(x) \frac{\partial W_{z_1}}{\partial \nu} \frac{\partial \overline{W}_{z_2}}{\partial \nu}\, ds = -\frac{1}{2} J_0(k|z_1 - z_2|) + i w_{z_1}(z_2) - i \overline{w}_{z_2}(z_1).$$

Determination of η

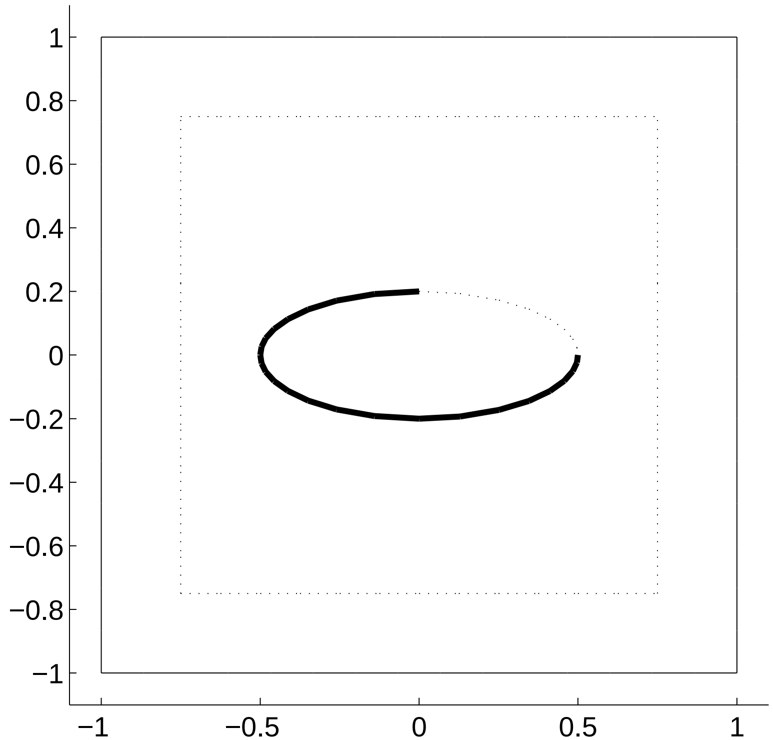
w_z can be approximated in $\mathbb{H}^1(D)$ by the Herglotz wave function v_{g_z} with kernel g_z the (regularized) solution of the far field equation.

- The above properties of w_z provide a method for approximating $\|\eta\|_{L_\infty(\Gamma_2)}$. In particular, if η is a constant we have

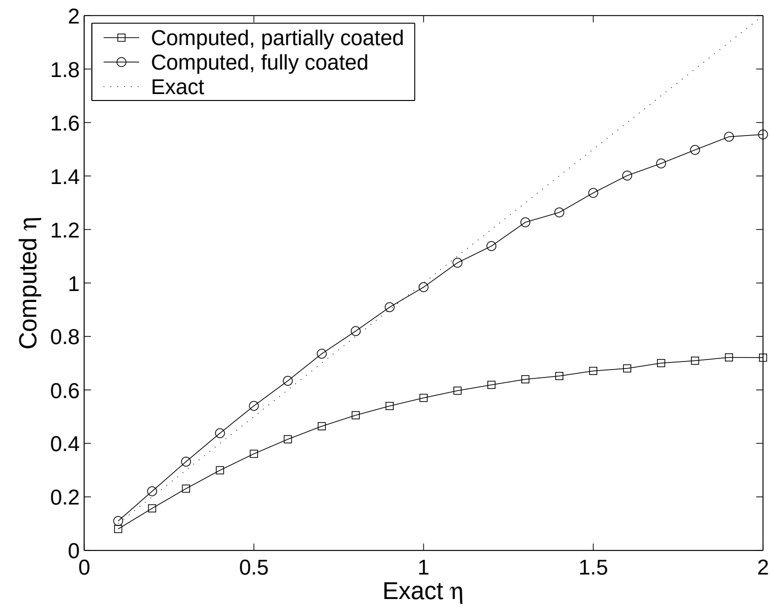
$$\eta = \frac{-\frac{1}{4} - \mathcal{I}\mathrm{m}\,(w_{z_0}(z_0))}{\|\frac{\partial}{\partial\nu}\,(w_{z_0} + \Phi(\cdot\,, z_0))\,\|^2_{L^2(\Gamma_2)}} \qquad z_0 \in D.$$

- Since Γ_2 is unknown, the above expression only provides a lower bound for η.

Determination of η

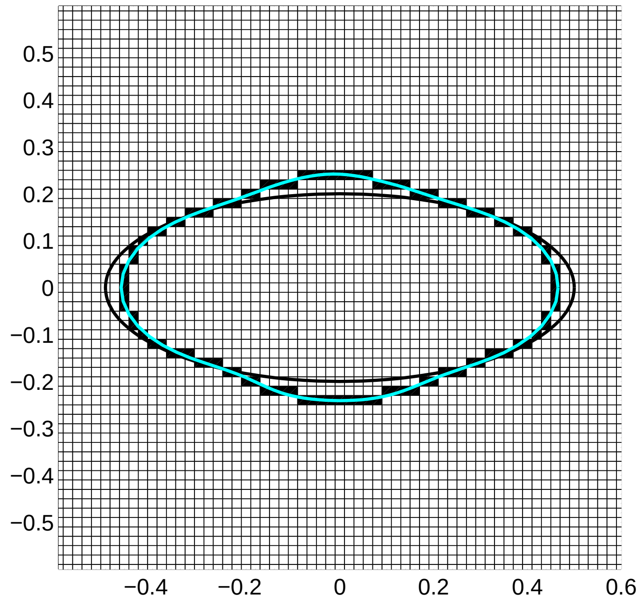


The thick curve is the coated portion

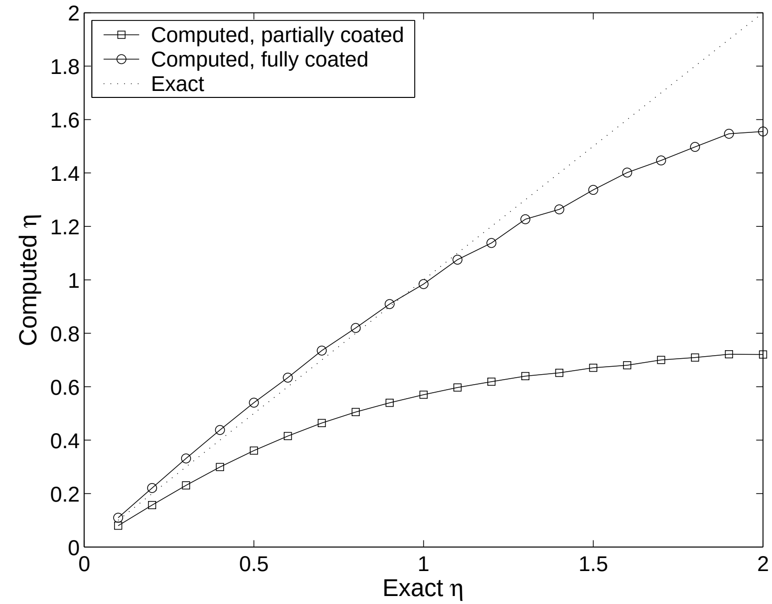


Computation of η using the exact boundary

Determination of η

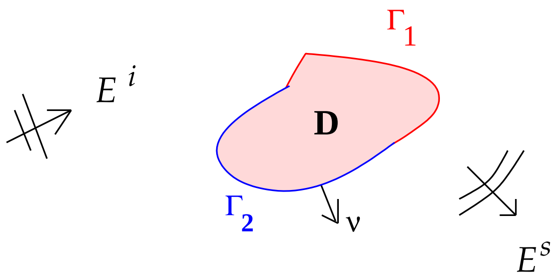


Reconstruction of D (fully coated)



Reconstruction of η (approximate D)

Partially Coated Dielectrics: Maxwell's Equations in $\mathbb{R}^3$



The scattered field E^s, H^s and the interior field E^{int}, H^{int} satisfy the equations

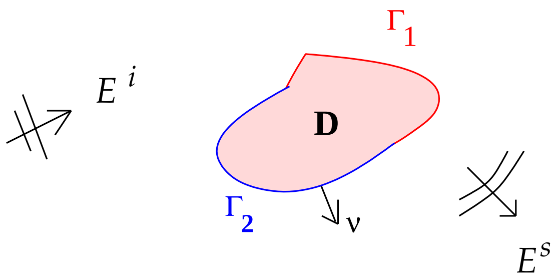
$$\begin{cases} \nabla \times E^s - ikH^s = 0 \\ \nabla \times H^s + ikE^s = 0 \end{cases} \quad \textit{in } \mathbb{R}^3 \setminus \overline{D}$$

$$\begin{cases} \nabla \times E^{int} - ikH^{int} = 0 \\ \nabla \times H^{int} + ikN(x)E^{int} = 0 \end{cases} \quad \textit{in } D$$

$$\lim_{|x|\to\infty} (H^s \times x - |x|E^s) = 0,$$

N *is a symetric matrix,* $\overline{\xi} \cdot \Re(N)\xi \geq \gamma|\xi|^2$, $\gamma > 0$
and $\overline{\xi} \cdot \Im(N)\xi \geq 0$.

Partially Coated Dielectrics



Let $\eta \in L^\infty(\Gamma_2)$ be the surface conductivity and Γ a Lipshitz boundary.

Then

$$\begin{aligned}
\nu \times E^s - \nu \times E^{int} &= -\nu \times E^i \quad \textit{on} \quad \Gamma = \Gamma_1 \cup \Gamma_2 \\
\nu \times H^s - \nu \times H^{int} &= \nu \times H^i \quad \textit{on} \quad \Gamma_1 \\
\nu \times H^s - \nu \times H^{int} &= -\nu \times H^i + \eta(x) \left[\nu \times (E^s + E^i)\right] \times \nu \ \textit{on} \ \Gamma_2
\end{aligned}$$

where $E^i = \frac{i}{k} \nabla \times (\nabla \times p\, e^{ikx\cdot d})$, $H^i = \nabla \times p\, e^{ikx\cdot d}$, $p \in \mathbb{R}^3$.

Inverse Problem

- *The wellposedness of above scattering problem is established by* Cakoni and Colton, Proc. Edinburgh. Math. Soc. 46 (2003), 293-314.

The inverse scattering problem is

> Determine D and η from a knowledge of $E_\infty(\hat{x}, d, p)$ for $\hat{x}$, $-d \in \Omega_0 \subset \Omega := \{x, \ |x| = 1\}$ and three linearly independent polarizations p.

- *The uniqueness of D is proved by* Cakoni and Colton, Proc. Edinburgh. Math. Soc. 46 (2003), 293-314.
- *The uniqueness of η is proved by* Cakoni, Colton and Monk (2004) to appear

Solving the Inverse Problem

We define the far field equation by

$$(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q).$$

where $g \in L^2_t(\Omega)$, $z \in \mathbb{R}^3$, *the far field operator* $F : L^2_t(\Omega) \to L^2_t(\Omega)$ *is given by*

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))\, ds(d)$$

and $E_{e,\infty}(\hat{x}, z, q)$ *is the far field pattern of the electric dipole*

$$E_e(x, z, q) := \frac{i}{k} \nabla \times \nabla \times q\, \Phi(x, z) \text{ with } \Phi(x, z) := \frac{1}{4\pi} \frac{e^{ik|x-z|}}{|x-z|},\ q \in \mathbb{R}^3.$$

Interior Transmission Problem

The analysis of the far field equation leads to the investigation of the interior transmission problem .

$$\nabla \times (\nabla \times E) - k^2 N(x) E = 0 \quad \textit{in} \quad D$$

$$\nabla \times (\nabla \times E_0) - k^2 E_0 = 0 \quad \textit{in} \quad D$$

$$\nu \times (E - E_0) = \nu \times E_e \quad \textit{on} \quad \Gamma = \Gamma_1 \cup \Gamma_2$$

$$\nu \times [\nabla \times (E - E_0)] = \nu \times (\nabla \times E_e) \quad \textit{on} \quad \Gamma_1$$

$$\nu \times [\nabla \times (E - E_0)] = \nu \times (\nabla \times E_e) - ik\eta\nu \times (E_0 + E_e) \times \nu \quad \textit{on} \quad \Gamma_2$$

The well posedness of the interior transmission problem is an open problem!

Interior Transmission Problem

We can prove the following completeness result:

Theorem *There exists a solution E of*

$$\nabla \times (\nabla \times E) - k^2 N(x) E = 0 \text{ in } D$$

and a Herglotz wave function E_g satisfying

$$\nabla \times (\nabla \times E_g) - k^2 E_g = 0 \text{ in } D$$

such that E and E_g satisfy the interior transmission transmission boundary data to arbitrary accuracy.

The Herglotz kernel g is an approximate solution to the far field equation.

Solving the Inverse Scattering Problem

- The support D can be determined by using the linear sampling method.
- If η is a constant on Γ_2 we obtain

$$\eta \approx \frac{\frac{k}{6\pi}\|q\|^2 - \Re(E_{g_z}(z))}{\|\nu \times (E_{g_z} + E_e(\cdot, z, q))\|^2_{L^2_t(\Gamma_2)}} \qquad z \in D \qquad (*)$$

where g_z is the (regularized) solution to the far field equation and E_{g_z} the Herglotz function with kernel g_z.

- Since Γ_2 is unknown, $(*)$ only provides a lower bound for η!
- To treat the case of non constant η the well posedness of the interior transmission problem is needed.

Determine the Support of Coating

Open problem: Determine the support Γ_2 of the coating.

- *We think to use the singularity of the scattered field near the boundary of Γ_2 on Γ. In the scalar case if the boundary is smooth*

$$u^s \in H_{loc}^{\frac{3}{2}-\epsilon}(\mathbb{R}^3 \setminus \overline{D}), \qquad \forall \ \epsilon > 0.$$

  Singularities of this order are observable at the derivative of the far field pattern with respect to the boundary. This suggests a combination of the linear sampling method with Newton type optimization methods to determine Γ_2 .

- *For thick coatings the Herglotz wave function with kernel g the approximate solution of the far field equation is zero on Γ_2.*