Solutions of Exam 1. #1 and #4 will do in class.

#2 a) if $A \cap B \neq \emptyset$ then

$0 \in \{ d(a,b) \geq 0 : a \in A,\ b \in B \}$ pick $\begin{matrix} a = c \\ b = c \end{matrix}$ $c \in A \cap B$

$\Rightarrow d(A,B) = 0$

The converse is not true: example $\begin{matrix} A = (-1,0) \\ B = (0,1) \end{matrix}$

$A \cap B = \emptyset$ but $d(A,B) = 0$ since we take

$a_n \in A$ where $a_n = -\frac{1}{n}$
$b_n \in B$ where $b_n = \frac{1}{n}$

$d(a_n, b_n) = \frac{2}{n}$

that is smaller then any positive number for n large enough.

(*) does not define a metric on $\mathcal{P}(X)$ since the first property $d(x,y) = 0 \Leftrightarrow x = y$ is violates.

b) $x \in \overline{A} \Leftrightarrow d(\{x\}, A) = 0$

1) $\Rightarrow$ if $x \in A$ then $d(\{x\}, A) = 0$

x limit point of A. Then $\forall \varepsilon\ \exists\ y_\varepsilon \in A$ such that

$d(x, y_\varepsilon) < \varepsilon \Rightarrow$ ~~d(A,B) ≤ d(x~~

$0 \leq d(\{x\}, A) \leq d(x, y_\varepsilon) < \varepsilon$ for every $\varepsilon > 0$

$\Rightarrow d(\{x\}, A) = 0$

2) $\Leftarrow$ $d(\{x\}, A) = 0 = \inf \{ d(x,y) \mid y \in A \}$

By the property of inf $\forall \varepsilon > 0 \quad \exists \; y_\varepsilon \in A$ s.th.

$$0 \leq d(x, y_\varepsilon) < \varepsilon \quad \Rightarrow \quad x \in \overline{A}$$

c) This is Problem 1 (a) in Solutions of HW5

Now if $d(A,B) = 0$ we have $d(A, \{b\}) = 0$ which from (b) implies $b \in \overline{A}$ but since A is close $b \in A$ and $b \in B$ which contradicts the fact that $A \cap B = \emptyset$

#3 a) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

$x \in \overline{A \cap B} \Rightarrow \forall \; B_r(x)$ contains $y \in A \cap B \Rightarrow$

$\forall \; B_r(x)$ contains $y \in A$ and $y \in B \Rightarrow x \in \overline{A}$ and $x \in \overline{B}$

$\Rightarrow x \in \overline{A} \cap \overline{B}$

Take $\underset{\text{rational}}{A = \mathbb{Q}}$ $\quad \underset{\text{irrational}}{B = \mathbb{I}}$ $\qquad \overline{A \cap B} = \emptyset$

$\qquad\qquad\qquad\qquad\qquad\qquad \overline{A} \cap \overline{B} = \mathbb{R}$

or $A = (0, 1]$ $\qquad \overline{A \cap B} = [\frac{1}{2}, 1]$

$B = \{0\} \cup [\frac{1}{2}, 1]$ $\quad \overline{A} \cap \overline{B} = [\frac{1}{2}, 1] \cup \{0\}$

(b) A open $\Rightarrow A \cap \overline{B} \subseteq \overline{A \cap B}$

$x \in A \cap \overline{B} \Rightarrow x \in A$ and $x \in \overline{B}$

a) if $x \in B$ then $x \in A \cap B \subset \overline{A \cap B}$

b) $x \in A$ and x limit point of B.

Since A is open there exist $B_r(x) \subset A$

On the other hand every ball $B_\varepsilon(x)$ contains a ball of radius less or equal to r. So we assume that $\varepsilon \leq r$. Since x l.p of B then $\exists\, y \in B_\varepsilon(x)$

$y \neq x$ and $y \in B$

But since $\varepsilon \leq r$ $y \in A$ thus

Every ball $B_\varepsilon(x)$ of radius $\leq r$ (and therefore any ball) contains $y \in A \cap B \Rightarrow x \in \overline{A \cap B}$

(c) We have $\overline{A} = X$ and $\overline{B} = X$ but since

A is open $A \cap \overline{B} \subseteq \overline{A \cap B} \Rightarrow \overline{A \cap \overline{B}} \subseteq \overline{A \cap B}$

$\Rightarrow \overline{A \cap X} \subseteq \overline{A \cap B} \Rightarrow \overline{A} \subseteq \overline{A \cap B} \Rightarrow X \subseteq \overline{A \cap B}$

$\Rightarrow X = \overline{A \cap B}$

Example $A = \mathbb{Q}$ dense in $\mathbb{R}$

$B = \mathbb{I}$ " - "

$A \cap B = \emptyset$ not dense in $\mathbb{R}$