

The linear sampling method for the inverse electromagnetic scattering problem for screens

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We consider the inverse electromagnetic scattering problem of determining the shape of a screen from a knowledge the electric far field pattern of the scattered wave at fixed frequency. We adapt the linear sampling method invented by Colton and Kirsch (Inverse Problems 12 (1996) 383-393) for the case of obstacles with nonempty interior.

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1 Introduction

The inverse scattering problem we consider in this paper is to determine the shape of a thin scattering object, referred to as screen, from a knowledge of the electric far field pattern of the scattered wave. For the sake of presentation, we establish the validity of the linear sampling method for solving the inverse scattering problem for the case of perfectly conducting screens. However, the analysis can be carried through for more complicated boundary conditions. We discuss numerical implementation of the linear sampling method and give an example showing the viability of the method.

2 The direct and inverse scattering problem

Let S be a perfectly conducting screen which we suppose to be a subset of the boundary of some Lipschitz domain $D \subset \mathbf{R}^3$. We consider the incident electromagnetic plane waves $E^i(x) = \frac{i}{k} \nabla \times \nabla \times p e^{ikx \cdot d} = ik(d \times p) \times d e^{ikx \cdot d}$. Then the scattered electric field E satisfies

$$\begin{cases} \nabla \times \nabla \times E - k^2 E = 0 & \text{in } \mathbf{R}^3 \setminus \bar{S} \\ \gamma_T E^\pm = c & \text{on } S \\ \lim_{|x| \rightarrow \infty} ((\nabla \times E) \times x - ik|x|E) = 0 \end{cases}$$

where γ_T denotes the tangential trace operator and $c = -\gamma_T E^i$. It is already known that there exists a unique solution of the above problem which is in $H_{loc}(\text{curl}, \mathbf{R}^3 \setminus \bar{S})$ ([3],[1]) where $H_{loc}(\text{curl}, \mathbf{R}^3 \setminus \bar{S}) := \{u \in (L^2_{loc}(\mathbf{R}^3 \setminus \bar{S}))^3 / \text{curl } u \in (L^2_{loc}(\mathbf{R}^3 \setminus \bar{S}))^3\}$. For a function u in $H_{loc}(\text{curl}, \mathbf{R}^3 \setminus \bar{S})$ we have that $\gamma_T \in H_{curl}^{-\frac{1}{2}}(S) := \{u \in (H^{-\frac{1}{2}}(S))^3 / \nu \cdot u = 0, \text{ curl}_S u \in H^{-\frac{1}{2}}(S)\}$. Let us denote by $\tilde{H}_{div}^{-\frac{1}{2}}(S) = (H_{curl}^{-\frac{1}{2}}(S))'$ in the duality pairing with L^2 as a pivot space (a function in $\tilde{H}_{div}^{-\frac{1}{2}}(S)$ can be extended by zero to a function in $H_{div}^{-\frac{1}{2}}(\partial D)$).

The electric scattered field has the asymptotic behavior $E(x) = \frac{e^{ik|x|}}{|x|} \left[E_\infty(\hat{x}; d, p) + O\left(\frac{1}{|x|}\right) \right]$, where $E_\infty(\hat{x}; d, p)$ is known as the electric *far field pattern* of the scattered field and is an infinitely differentiable tangent field in the unit sphere Ω .

The *inverse scattering problem* we are interested in is to determine S from a knowledge of $E_\infty(\hat{x}; d, p)$ for \hat{x} and d in a subset of $\Omega_0 \subset \Omega$ and three linearly independant polarizations $p \in \mathbf{R}^3$. One can show that S is uniquely determined from the electric far field $E_\infty(\hat{x}; d, p)$ given for all $\hat{x}, d \in \Omega$ and $p \in \mathbf{R}^3$.

We introduce the far field operator $F : L^2_t(\Omega_0) \rightarrow L^2_t(\Omega_0)$ by $(Fg)(\hat{x}) := \int_{\Omega_0} E_\infty(\hat{x}; d, g(d)) ds(d)$. which is a linear operator since $E_\infty(\hat{x}; d, p)$ depends linearly on p . By superposition $(Fg) = -\mathcal{B}(\mathcal{H}g)$ where $\mathcal{B} : H_{curl}^{-\frac{1}{2}}(S) \rightarrow L^2_t(\Omega_0)$ maps the boundary data $c \in H_{curl}^{-\frac{1}{2}}(S)$ to the far field pattern of the solution to the corresponding screen problem, and the Herglotz operator $\mathcal{H} : L^2_t(\Omega_0) \rightarrow H_{curl}^{-\frac{1}{2}}(S)$ is defined by $\mathcal{H}g := \gamma_T E_g$ where E_g is the electric Herglotz wave function with kernel g given by $E_g(x) := \int_{\Omega_0} g(d) e^{ikx \cdot d} ds(d)$. It can be shown that $\mathcal{B} : H_{curl}^{-\frac{1}{2}}(S) \rightarrow L^2_t(\Omega_0)$ is compact, injective and has dense range provided that there does not exists a Herglotz function whose tangential trace vanishes on S . Furthermore, the range of \mathcal{B} consists of all functions of the form $\Phi_\infty^L(\hat{x}) := \hat{x} \times \int_L \alpha_L(y) e^{-ik\hat{x} \cdot y} ds_y \times \hat{x}$, where $L \subset S$ and $\alpha_L \in \tilde{H}_{div}^{-\frac{1}{2}}(L)$. in addition, one can also show that the range of $\mathcal{H} : L^2_t(\Omega_0) \rightarrow H_{curl}^{-\frac{1}{2}}(S)$ is dense. From these properties, we can derive the following result about the solution of the *far field equation* $(Fg) = -\mathcal{B}(\mathcal{H}g) = \Phi_\infty^L$ where L is an arbitrary arc and $\alpha_L \in \tilde{H}_{div}^{-\frac{1}{2}}(L)$.

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- Theorem 2.1** 1. If $L \subset S$ then for every $\varepsilon > 0$ there exists a solution $g_\varepsilon^L \in L_t^2(\Omega)$ of $\|Fg_\varepsilon^L - \Phi_\infty^L\|_{L_t^2(\Omega)} \leq \varepsilon$.
2. If $L \not\subset S$ then for every $\varepsilon > 0$ and $\delta > 0$ there exists a solution $g_{\varepsilon,\delta}^L \in L_t^2(\Omega)$ of $\|Fg_{\varepsilon,\delta}^L - \Phi_\infty^L\|_{L_t^2(\Omega)} \leq \varepsilon + \delta$ such that $\lim_{\delta \rightarrow 0} \|g_{\varepsilon,\delta}^L\|_{L_t^2(\Omega)} = \infty$ and $\lim_{\delta \rightarrow 0} \|E_{g_{\varepsilon,\delta}^L}\|_{H(\text{curl}, B_R)} = \infty$, where $E_{g_{\varepsilon,\delta}^L}$ is the electric Herglotz wave function with kernel $g_{\varepsilon,\delta}^L$.

Letting L degenerate to a point z by taking α_L an appropriate delta sequence, we can replace Φ_∞^L by $\frac{ik}{4\pi}[(\hat{x} \times q) \times \hat{x}e^{-ik\hat{x} \cdot z}]$. The above analysis can also be carried out for the case of screens with mixed boundary conditions ([4],[5]).

3 The linear sampling method

The linear sampling determines g from the following equation

$$\int_{\Omega} E_{\infty}(\hat{x}; d, g(d)) ds(d) = \frac{ik}{4\pi} [(\hat{x} \times q) \times \hat{x}e^{-ik\hat{x} \cdot z}] \quad z \in \mathbf{R}^3.$$

Then S is reconstructed as the set of points where $\|g\|_{L^2}$ is bounded. Since the far field equation is ill-posed, g is obtained by using regularization methods such as Tikhonov regularization. It is an open question whether this regularized solution behave in the same way as the approximate solution of the far field equation. Recently, this question is answered positively in ([2]) in certain cases for obstacles with non empty interior.

In practice, we consider the discretized far field equation which reads $A_{\infty}g_{z,q} = b_{z,q}$, where $A_{\infty}(\hat{x}_i, d_j)$ represents the far field data at N incident directions $d_j \in \Omega_0$ and N observation directions $\hat{x}_i = -d_i \in \Omega_0$. We use piecewise continuous linear approximation of g on a triangularization of Ω_0 . We then solve the regularized equation $\alpha g + A_{\infty}^*A_{\infty}g = A_{\infty}^*b$, for three linearly independent polarizations $q = q_1, q_2, q_3$ where α is chosen by Morozov's discrepancy principle. The screen is then visualized by plotting the surface $\mathcal{G}(z) = C \max_{z_i \in \text{mesh}} \frac{1}{3} \sum_{j=1}^3 \|g(\cdot; z_i, q_j)\|^{-1}$ with C chosen by ad hoc methods (see [6]).

In Fig. (1) we show an example of reconstruction for a non-smooth L-shape screen with mixed perfectly conducting - impedance boundary condition. In particular, the screen satisfies perfectly conducting boundary condition on all sides except for the inner side of the vertical square which satisfies the impedance boundary condition with $\lambda = 2$.

Fig. 1 Perfectly conducting screen on the left and a mixed screen on the right

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