Title: Inhomogeneous Media Identification

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Inhomogeneous Media Identification

Definition

Inhomogeneous media identification is the problem of determining the physical properties of an unknown inhomogeneity from its response to various interrogating modalities. This response, recorded in measured data, comes as a result of the interaction of the inhomogeneity with an exciting physical field. Inhomogeneous media identification is mathematically modeled as the problem of determining the coefficients of some partial differential equations with initial or boundary data from a knowledge of the solution on the measurement domain.

Formulation of the Problem

This survey discusses only the problem of *inhomogeneous media identification in inverse scattering therory.* Scattering theory is concerned with the effects that inhomogeneities have on the propagation of waves and in particular time-harmonic waves. In the context of this presentation, scattering theory provides the mathematical tools for imaging of inhomogeneous media via acoustic, electromagnetic or elastic waves with applications to such fields as radar, sonar, geophysics, medical imaging and nondestructive testing. For reasons of brevity, we focus our attention on the case of acoustic waves and refer the reader to Cakoni-Colton-Monk [3] for a comprehensive reading on media identification using electromagnetic waves. Since the literature in the area is enormous, we have only referenced a limited number of papers and monographs and hope that the reader can use these as starting point for further investigations.

We begin by considering the propagation of sound waves of small amplitude in R^3 viewed as a problem in fluid dynamics. Let $p(x,t)$ denote the pressure of the fluid which is a small perturbation of the static case, i.e. $p(x,t) = p_0 + \epsilon P_1(x,t) + \cdots$ where $p_0 > 0$ is a constant. Assuming that $p_1(x,t)$ is time harmonic, $p_1(x,t) = \Re\left\{u(x)e^{-i\omega t}\right\}$, we have that u satisfies (Colton-Kress 1998 [8])

$$\Delta u + \frac{\omega^2}{c^2(x)} u = 0 \tag{1}$$

where ω is the frequency and $c(x)$ is the sound speed. Equation (1) governs the propagation of time harmonic acoustic waves of small amplitude in a slowly varying inhomogeneous medium. We still must prescribe how the wave motion is initiated and what is the boundary of the region contained in the fluid. We shall only consider the simplest case when the inhomogeneity is of compact support denoted by D, the region of consideration is all of R^3 and the wave motion is caused by an incident field u^i satisfying the unperturbed linearized equations being scattered by the inhomogeneous medium. Assuming that $c(x) = c_0 = \text{constant}$ for $x \in R^3 \setminus \overline{D}$, the total field $u = u^i + u^s$ satisfies

$$\Delta u + k^2 n(x) u = 0 \qquad \text{in } R^3 \tag{2}$$

and the scattered field u^s fulfills the Sommerfeld radiation condition

$$\lim_{|x|\to\infty} |x| \left(\frac{\partial u^s}{\partial |x|} - iku^s \right) = 0 \tag{3}$$

which holds uniformly in all directions $x/|x|$ where, $k = \omega/c_0$ is the wave number and $n = c_0^2/c^2$ is the refractive index in the case of non absorbing media. An absorbing medium is modeled by adding an absorption term which leads to a refractive index with a positive imaginary part of the form

$$n(x) = \frac{c_0^2}{c^2(x)} + i\frac{\gamma(x)}{k}$$

in terms of an absorption coefficient $\gamma > 0$ in $\overline{D}$. In the sequel, the *refractive index* n is assumed to be a piecewise continuous complex valued fuction such that $n(x) = 1$ for $x \notin D$ and $\Re(n) > 0$ and $\Im(n) \geq 0$. For a vector $d \in R^3$, with $|d| = 1$, the function $e^{ikx\cdot d}$ satisfies the Helmholtz equations in R^3 and it is called a *plane wave*, since $e^{i(kx\cdot d - \omega t)}$ is constant on the planes $kx \cdot d - \omega t$ =const. Summarizing, given the incident field u^i and the physical properties of the inhomogeneity, the *direct scattering problem* is to find the scattered wave and in particular its behavior at large distances from the scattering object, i.e. its far field behavior. The *inverse scattering problem* takes this answer to the direct scattering problem as its starting point and asks what is the nature of the scatterer that gave rise to such far field behavior?

Identification of Inhomogeneities from Far Field Data

It can be shown that radiating solutions u^s to the Helmholtz equation (i.e. solutions that satisfy the Sommerfeld radiation condition (3) assume the asymptotic behavior

$$u^s(x) = \frac{e^{ik|x|}}{|x|}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \qquad |x| \to +\infty \tag{4}$$

uniformly for all directions $\hat{x}$ where the function u_∞ defined on the unit sphere S^2 is known as the far field pattern of the scattered wave. For plane wave incidence $u^i(x, d) = e^{ikx\cdot d}$ we indicate the dependence of the far field pattern on the incident direction d and the observation direction $\hat{x}$ by writing $u_\infty = u_\infty(\hat{x}, d)$. The *inverse scattering problem* or in other words *inhomogeneous media identification problem* can now be formulated as the problem of determining the index of refraction n (and hence also its support D)

from a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for $\hat{x}$ and d on the unit sphere S^2 (or a subset of S^2). All the results presented here are valid in R^2 as well. Also, it is possible to extend our discussion to the case of point source incidence and near field measurements (see [4]).

Uniqueness

A first question to approach the problem is whether the inhomogeneous media is identifiable from the exact data, which in mathematical terms is known as the uniqueness problem. The uniqueness problem for inverse scattering by an inhomogeneous medium in R^3 was solved by Nachman (1988), Novikov (1988) and Ramm (1988) who based their analysis on the fundamental work of Sylvester and Uhlmann citeUS. Their uniqueness proof was considerably simplified by Hähner (1996) (see [12], [14], [2] and the references in [8] and [9]). The uniqueness problem for an inhomogeneous media in R^2, which is a formerly determined problem, was recently solved by Bukhgeim (2008) [2]. In particular, under the assumptions on the refractive index stated in Introduction, the following uniqueness result holds.

Theorem 1. *The refractive index n in (2) is uniquely determined from $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in S^2$ and a fixed value of the wave number k.*

It is important to notice that owning to fact that u_∞ is real analytic in $S^2 \times S^2$, for the uniqueness problem it suffices to know $u_\infty(\hat{x}, d)$ for $\hat{x}, d$ on subsets of S^2 having an accumulation point.

The identifiability problem for matrix index of refraction of an anisotropic media is more complicated. In the mathematical model of the scattering by anisotropic media the equation (2) is replaced by

$$\nabla \cdot A\nabla u + k^2 n(x) u = 0 \qquad \text{in } R^3 \tag{5}$$

where n satisfies the same assumptions as in Introduction, and A is a 3×3 piece-wise continuous matrix-valued function with positive definite real part i.e. $\xi \cdot \Re(A)\xi > \alpha|\xi|^2$, $\alpha > 0$ in $\overline{D}$, non-positive imaginary part, i.e. $\xi \cdot \Im(A)\xi \leq 0$ in $\overline{D}$ and $A = I$ in $R^3 \setminus \overline{D}$. In general, it is known that $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in S^2$ does not uniquely determine the matrix A even it is known for all wave numbers $k > 0$, and hence without further a priori assumptions the determination of D is the most that can be hoped. To this end, Hähner (2000) proved that the support D of an anisotropic inhomogeneity is uniquely determined from $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in S^2$ and a fixed value of the wave number k provided that either $\xi \cdot \Re(A)\xi > \beta|\xi|^2$ or $\xi \cdot \Re(A^{-1}))\xi > \beta|\xi|^2$ for some constant $\beta > 1$.

Reconstruction Methods

Recall the scattering problem described by (2)-(3) for the total field $u = u^i + u^s$ with plane wave incident field $u^i := e^{ikx \cdot d}$. The total field satisfies the *Lippmann-Schwinger equation* (see [8])

$$u(x) = e^{ikx \cdot d} - \frac{k^2}{4\pi} \int_{R^3} \frac{e^{ik|x-y|}}{|x-y|} m(y)u(y)\, dy, \qquad x \in R^3, \tag{6}$$

and the corresponding far field pattern is given by

$$u_\infty(\hat{x}, d) = -\frac{k^2}{4\pi} \int_{R^3} e^{-ik\hat{x} \cdot y} m(y)u(y)\, dy, \qquad \hat{x}, d \in S^2. \tag{7}$$

Since the function $m := 1 - n$ has support D, the integrals in (6) and (7) can in fact be written over a bounded domain containing D. The goal is to reconstruct $m(x)$ from a knowledge of (the measured) far field pattern $u_\infty(\hat{x}, d)$ based on (7). The dependence of (7) on the unknown m is in a nonlinear fashion, thus the inverse medium problem is genuinely a nonlinear problem. The reconstruction methods can, roughly speaking, be classified into three groups, Born or weak scattering approximation, nonlinear optimization techniques and qualitative methods (we remark that this classification is not inclusive).

Born Approximation

Born approximation, known otherwise as weak scattering approximation, turns the inverse medium scattering problem into e linear problem and therefore is often employed in practical applications. This process is justified under restrictive assumption that the scattered field due to the inhomogeneous media is only a small perturbation of incident field, which at a given frequency, is valid if either the corresponding contrast $n-1$ is small or the support D is small. Hence, assuming that $k^2\|m\|_\infty$ is sufficiently small, one can replace u in (7) by the plane wave incident field $e^{ikx\cdot d}$, thus obtaining the linear integral equation for m

$$u_\infty(\hat{x}, d) = -\frac{k^2}{4\pi}\int_{R^3} e^{-ik(\hat{x}-d)\cdot y} m(y)\, dy, \qquad \hat{x}, d \in S^2. \tag{8}$$

Solving (8) for the unknown m corresponds to inverting the Fourier transform of m restricted to the ball of radius $2k$ centered at the origin, i.e., only incomplete data is available. This causes uniqueness ambiguities and leads to severe ill-posedness of the inversion. For details we refer the reader to Langenberg [11].

Nonlinear Optimization Techniques

These methods avoid incorrect model assumptions inherent in weak scattering approximation and consider the full nonlinear inverse medium problem. To write a nonlinear optimization setup, note that the inverse medium problem is equivalent to solving the system of equations composed by (6) and (7) for u and m where u_∞ is in practice the (noisy) measured data u_∞^δ with $\delta > 0$ being the noise level. Thus, a simple least square approach looks for minimizing the cost functional

$$\mu(u, m) := \frac{\|u^i + Tmu - u\|^2_{L^2(B\times S^2)}}{\|u^i\|^2_{L^2(B\times S^2)}} + \frac{\|u_\infty^\delta - Fmu - u\|^2_{L^2(S^2\times S^2)}}{\|u_\infty^\delta\|^2_{L^2(B\times S^2)}}$$

for u and m over admissible sets, where Tmu denotes the integral in (6) and Fmu denotes the integral in (7). The discrete versions of this optimization problem suffer from a large number of unknowns and thus is expensive. Regularization techniques are needed to handle instability due to ill-posedness.

Newton Iterative Method

A more rigorous mathematical approach to deal with nonlinearity in (6) and (7) is the Newton type iterative method. To this end, it is possible to reformulate the inverse medium problem as a nonlinear operator equation by introducing the operator $\mathcal{F}$: $m \to u_\infty$ that maps $m := 1 - n$ to the far field pattern $u_\infty(\cdot, d)$ for plane incidence $u^i(x) = e^{ikx\cdot d}$. In view of uniqueness theorem, $\mathcal{F}$ can be interpreted as an injective operator from $\mathcal{B}(B)$ (the space of bounded functions defined on a ball B containing the support D of m) into $L^2(S^2 \times S^2)$ (the space of square integrable function on $S^2 \times S^2$). From (7) we can write

$$(\mathcal{F}(m))(\hat{x}, d) = -\frac{k^2}{4\pi}\int_B e^{-ik\hat{x}\cdot y} m(y)u(y)\, dy, \qquad \hat{x}, d \in S^2 \tag{9}$$

where $u(\cdot, d)$ is the unique solution of (6). Note that $\mathcal{F}$ is a compact operator, owning this to its analytic kernel, thus (9) is severely ill-posed. From the latter it can be seen that the Freéchet derivative v_q of u with respect to m (in direction q) satisfies the Lippmann-Schwinger equation

$$v_q(x, d) + \frac{k^2}{4\pi}\int_B \frac{e^{ik|x-y|}}{|x-y|}\left[m(y)v_q(y, d) + q(y)u(y, d)\right]\, dy, \qquad x \in B$$

which implies the following expression for the Fréchet derivative of $\mathcal{F}$

$$(\mathcal{F}'(m)q)(\hat{x}, d) = -\frac{k^2}{4\pi}\int_B e^{-ik(\hat{x}-d)\cdot y}\left[m(y)v_q(y, d) + q(y)u(y, d)\right]\, dy, \qquad \hat{x}, d \in S^2.$$

Observe that $\mathcal{F}'(m)q = v_{q,\infty}$ where $v_{q,\infty}$ is the far field pattern of the radiating solution to $\Delta v + k^2 nv = -k^2 uq$. It can be shown that $\mathcal{F}'(m)$ is injective (see[8], [9]). With help of Fréchet derivative, it is now possible to replace (7) by its linearized version

$$\mathcal{F}(m) + \mathcal{F}'(m)q = u_\infty \tag{10}$$

which, given an initial guess m, it is solved for q to obtain an update $m + q$. Then as in the classical Newton iterations, this linearization procedure is iterated until some stopping criteria is satisfied. Of course the linearized equation inherits the ill-posedness of the nonlinear equation and therefore regularization is required. If u_∞^δ is again the noisy far field measurements, Tikhonov regularization replaces (10) by

$$\alpha q + [\mathcal{F}'(m)]^* \mathcal{F}'(m) q = [\mathcal{F}'(m)]^* \left\{ u_\infty^\delta - \mathcal{F}(m) \right\}$$

with some positive regularization parameter α and the L^2 adjoint $[\mathcal{F}'(m)]^*$ of $\mathcal{F}'(m)$. Of course for the Newton method to work, one needs to start with a good initial guess incorporating available a priori information, but in principle the method con be formulated for one or few incident directions.

Qualitative Methods

In recent years alternative methods for imaging of inhomogeneous media have emerged which avoid incorrect model assumptions of weak approximations but, as opposed to nonlinear optimization techniques, require essentially no a priori information on the scattering media. Nevertheless, they seek limited information about scattering object and need multistatic data, i.e several incident fields each measured at several observation directions. Such methods come under the general title of qualitative methods in inverse scattering theory. Most popular examples of such approaches are linear sampling method (Cakoni-Colton [4]), factorization method (Kirsch-Grinberg [10]) and singular sources method (Potthast [13]). Typically, these methods seek to determine an approximation to the support of the inhomogeneity by constructing a support indicator function, and in some cases provide limited information on material properties of inhomogeneous media. We provide here a brief exposé of the linear sampling method. To this end let us define the *far field operator* $F : L^2(S^2) \to L^2(S^2)$ by

$$(Fg)(\hat{x}) := \int_{S^2} u_\infty(\hat{x}; d, k) g(d) ds(d) \tag{11}$$

We note that by linearity $(Fg)(\hat{x})$ is the far field pattern corresponding to (1) where the incident field u^i is a *Herglotz wave function* $v_g(x) := \int_{S^2} e^{ikx\cdot d} g(d) ds(d)$. For given $k > 0$ the far field operator is injective with dense range if and only if there does not exist a nontrivial solution $v, w \in L^2(D)$, $v - w \in H^2(D)$ of the transmission eigenvalue problem

$$\Delta w + k^2 n(x) w = 0 \quad \text{and} \quad \Delta v + k^2 v = 0 \quad \text{in} \quad D \tag{12}$$

$$w = v \quad \text{and} \quad \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial D \tag{13}$$

such that v is a Herglotz wave function. Values of $k > 0$ for which (12)-(13) has nontrivial solutions are called *transmission eigenvalues.* If $\Im(n) = 0$ there exits an infinite discrete set of transmission eigenvalues accumulating only at $+\infty$, [7]. Consider now the *far field equation* $(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z, k)$ where $\Phi_\infty(x, z, k) := \frac{1}{4\pi} e^{-ik\hat{x}\cdot z}$ (is the far field pattern of the fundamental solution $\frac{e^{ik|x-y|}}{4\pi|x-y|}$ to the Helmholtz equation). The far field equation is severely ill-posed owing to the compactness of the far field operator which is an integral operator with analytic kernel.

Theorem 2. *Assume that k is not a transmission eigenvalue. Then: 1) If $z \in D$ for given $\epsilon > 0$ there exists $g_{z,\epsilon,k} \in L^2(S^2)$ such that $\|Fg_{z,\epsilon,k} - \Phi_\infty(\cdot, z, k)\|_{L^2(S^2)} < \epsilon$ and the corresponding Herglotz function satisfies $\lim_{\epsilon \longrightarrow 0} \|v_{g_{z,\epsilon,k}}\|_{L^2(D)}$ exists finitely, and for a fixed $\epsilon > 0$, $\lim_{z \longrightarrow \partial D} \|v_{g_{z,\epsilon,k}}\|_{L^2(D)} = +\infty$. 2) If $z \in R^3 \setminus \overline{D}$ and $\epsilon > 0$ every $g_{z,\epsilon,k} \in L^2(S^2)$ satisfying $\|Fg_{z,\epsilon,k} - \Phi_\infty(\cdot, z, k)\|_{L^2(S^2)} < \epsilon$ is such that $\lim_{\epsilon \longrightarrow 0} \|v_{g_{z,\epsilon,k}}\|_{L^2(D)} = +\infty$.*

The *linear sampling method* is based on attempting to compute the function $g_{z,\epsilon,k}$ in the above theorem by using Tiknonov regularization as the unique minimizer of the *Tikhonov functional* (see [8])

$$\|F^\delta g - \Phi(\cdot, z)\|^2_{L^2(\Omega)} + \alpha \|g\|^2_{L^2(S^2)} \tag{14}$$

where the positive number $\alpha := \alpha(\delta)$ is known as the *Tikhonov regularization parameter* and $F^\delta g$ is the noisy far field operator where u_∞ in (7) is replaced by the noisy far field data u^δ_∞ with $\delta > 0$ being the noise level (note that $\alpha_\delta \to 0$ as $\delta \to 0$). In particular, one expects that this regularized solution will be relatively smaller for $z \in D$ than $z \in R^3\backslash\overline{D}$ and this behavior can be visualized by color coding the values of the regularized solution on a grid over some domain containing the support D of the inhomogeneity and thus providing a reconstruction of D. A precise mathematical statement on the described behavior of the regularized solution to the far field equation is based on factorization method which instead of the far field operator F considers $(F^*F)^{1/4}$ where F^* is the L^2 adjoint of F (see [10]). For numerical examples using linear sampling method we refer the reader to [4].

Having reconstructed the support of the inhomogeneity D, we then obtain information on $n(x)$ for non-absorbing media, i.e. if $\Im(n) = 0$. Assume to this end that $n(x) > 1$ (similar results holds for $0 < n(x) < 1$), fix a $z \in D$ and consider a range of wave number $k > 0$. If $g_{\delta,z,k}$ is now the Tikhonov regularized solution of the far field equation (14) then we have that: 1) for $k > 0$ *not* a transmission eigenvalue $\lim\limits_{\delta\to 0} \|v_{g_{\delta,z,k}}\|_{L^2(D)}$ exists finitely [1], 2) for $k > 0$ a transmission eigenvalue $\lim\limits_{\delta\to 0} \|v_{g_{\delta,z,k}}\|_{L^2(D)} = +\infty$ (for almost all $z \in D$) [5]. In practice, this means if $\|g_{\delta,z,k}\|_{L^2(S^2)}$ is plotted against k, the transmission eigenvalues will appear as sharp picks, and thus providing a way to compute transmission eigenvalues from far field measured data. A detailed study of transmission eigenvalue problem [7] reveals that the first transmission is related to the index of refraction n. More specifically, letting $n_* = \inf_D n(x)$ and $n^* = \sup_D n(x)$ the following Faber-Krahn type inequalities holds

$$k^2_{1,n(x),D} \geq \frac{\lambda_1(D)}{n^*} \tag{15}$$

where $k_{1,n(x),D}$ is the first transmission eigenvalue corresponding to d and $n(x)$ and $\lambda_1(D)$ is the first Dirichlet eigenvalue for $-\Delta$ in D, and

$$0 < k_{1,D,n^*} \leq k_{1,D,n(x)} \leq k_{1,D,n_*} \tag{16}$$

which is clearly seen to be isoperimetric for $n(x)$ equal to a constant. In particular, (16) shows that for n constant the first transmission eigenvalue is monotonic decreasing function of n and moreover, this dependence can be shown to be continuous and strictly monotonic. Using (16), for a measured first transmission eigenvalue $k_{1,D,n(x)}$ we can determine a unique constant n_0 that satisfies $0 < n_* \leq n_0 \leq n^*$, where this constant is such that $k_{1,D,n_0} = k_{1,D,n(x)}$. This n_0 is an integrated average of $n(x)$ over D.

A more interesting question is what does the first transmission eigenvalue say about the matrix index of refraction A for the scattering problem for anisotropic media (5). Assuming $n = 1$, $\overline{\xi} \cdot \Re(A)\xi > |\xi|^2$ and $\overline{\xi} \cdot \Im(A)\xi = 0$ in (5), similar analysis for the corresponding transmission eigenvalue problem leads to the isoperimetric inequality $0 < k_{1,D,a^*} \leq k_{1,D,A(x)} \leq k_{1,D,a_*}$ [6]. Hence, it is possible to compute a constant a_0 such that k_{1,D,a_0} equals the (measured) first transmission eigenvalue $k_{1,D,A(x)}$ and this constant satisfies $0 < a_* \leq a_0 \leq a^*$, where $a_* = \inf\limits_D a_1(x)$, $a^* = \sup a_3(x)$ and $a_1(x)$ and $a_3(x)$ are the smallest and the largest eigenvalues of the matrix $A^{-1}(x)$, respectively. The latter inequality is of particular interest since $A(x)$ is not uniquely determined from the far field and to our knowledge this is the only information obtainable to date about $A(x)$ that can be determined from far field data (see [6]) for numerical examples).

Cross-references

Inverse medium problem, inverse scattering, impedance tomography, optical tomography, thermo-acoustic imaging, seismic tomography..

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