The 3D Inverse Electromagnetic Scattering Problem for a Coated Dielectric

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Abstract. We use the linear sampling method to determine the shape and surface conductivity of a partially coated dielectric from a knowledge of the far field pattern of the scattered electromagnetic wave at fixed frequency. A mathematical justification of the method is provided for the full 3D vector case based on the use of a complete family of solutions. Numerical examples are given.

1 Introduction

In a previous paper together with D. Colton [6], we have analyzed the use of the linear sampling method to identify the shape of a coated dielectric in the 2D TM-polarized case. In addition we proposed and tested a heuristic formula for calculating the surface conductivity from far field data. In this paper we extend the techniques of [6] to the full three dimensional electromagnetic scattering problem at a fixed frequency. Using approximation arguments we shall provide a mathematical justification of the linear sampling method of finding the shape of a coated dielectric. Such arguments avoid the need to analyze an appropriate interior problem appearing in the theory (the "interior transmission problem"). Assuming that the interior transmission problem is well-posed (currently an open problem), we then derive a formula for the surface conductivity. Computational results for simple model problems show that the linear sampling method can reconstruct the surface conductivity .

The physical relevance and background for the inverse problem in this paper is discussed in [6] and we direct the reader there for further references.

The plan of our paper is as follows. In Section 2 we formulate the direct and inverse scattering problem for a dielectric that is partially coated by a highly conductive layer. In Section 3, we then use the linear sampling method [10] to determine the shape of the scattering object. We also discuss how to additionally recover the surface conductivity from the scattering data. In Section 4, we conclude by providing some numerical examples.

2 Formulation of the direct and inverse scattering problem

Let $D \subset \mathbb{R}^3$ be a bounded region with boundary Γ such that $D_e := \mathbb{R}^3 \setminus \overline{D}$ is connected. Each simply connected piece of D is assumed to be a Lipschitz curvilinear polyhedron. Moreover we assume that the boundary $\Gamma = \Gamma_1 \cup \Pi \cup \Gamma_2$ is split into two disjoint parts Γ_1 and Γ_2 having Π as their possible common boundary in Γ which is assumed to be a union of Lipschitz curves. The domain D is the support of an anisotropic object that is partially coated on a portion Γ_2 of the boundary by a very thin homogeneous layer of a highly conductive material and the incident field is a time-harmonic electromagnetic plane wave with frequency ω (Γ_1 may be the empty set which corresponds to a fully coated obstacle!). The interior electric and magnetic fields $\tilde{E}^{int}$, $\tilde{H}^{int}$, and the exterior electric and magnetic fields $\tilde{E}^{ext}$, $\tilde{H}^{ext}$, satisfy

$$\begin{cases} \nabla \times \tilde{E}^{ext} - i\omega\mu_0 \tilde{H}^{ext} = 0 \\ \nabla \times \tilde{H}^{ext} + i\omega\epsilon_0 \tilde{E}^{ext} = 0 \end{cases} \quad \text{in } D_e \tag{1}$$

$$\begin{cases} \nabla \times \tilde{E}^{int} - i\omega\mu_0 \tilde{H}^{int} = 0 \\ \nabla \times \tilde{H}^{int} + (i\omega\epsilon(x) - \sigma(x)) \tilde{E}^{int} = 0 \end{cases} \quad \text{in } D \tag{2}$$

and on the boundary Γ

$$\nu \times \tilde{E}^{ext} - \nu \times \tilde{E}^{int} = 0 \quad \text{on } \Gamma \tag{3}$$

$$\nu \times \tilde{H}^{ext} - \nu \times \tilde{H}^{int} = 0 \quad \text{on } \Gamma_1 \tag{4}$$

$$\nu \times \tilde{H}^{ext} - \nu \times \tilde{H}^{int} = \tilde{\eta}\, (\nu \times \tilde{E}^{ext}) \times \nu \quad \text{on } \Gamma_2. \tag{5}$$

The electric permittivity ϵ_0 and magnetic permeability μ_0 of the exterior dielectric medium are positive constants whereas the scatterer has the same magnetic permeability μ_0 as the exterior medium but the electric permittivity ϵ and conductivity σ are real 3×3 matrix valued functions. The constant $\tilde{\eta} > 0$ describes the physical properties of the thin coating layer [1]. If we define $\tilde{E}^{(ext,int)} = \frac{1}{\sqrt{\epsilon_0}} E^{(ext,int)}$, $\tilde{H}^{(ext,int)} = \frac{1}{\sqrt{\mu_0}} H^{(ext,int)}$, $k^2 = \epsilon_0\mu_0\omega^2$, $N(x) = \frac{1}{\epsilon_0}\left(\epsilon(x) + i\frac{\sigma(x)}{\omega}\right)$, and $\tilde{\eta} = \sqrt{\frac{\mu_0}{\epsilon_0}}\eta$ we obtain the transmission problem

$$\begin{cases} \nabla \times E^{ext} - ikH^{ext} = 0 \\ \nabla \times H^{ext} + ikE^{ext} = 0 \end{cases} \quad \text{in } D_e \tag{6}$$

$$\begin{cases} \nabla \times E^{int} - ikH^{int} = 0 \\ \nabla \times H^{int} + ikN(x)E^{int} = 0 \end{cases} \quad \text{in } D \tag{7}$$

$$\nu \times E^{ext} - \nu \times E^{int} = 0 \quad \text{on} \quad \Gamma \tag{8}$$
$$\nu \times H^{ext} - \nu \times H^{int} = 0 \quad \text{on} \quad \Gamma_1 \tag{9}$$
$$\nu \times H^{ext} - \nu \times H^{int} = \eta\,(\nu \times E^{ext}) \times \nu \quad \text{on} \quad \Gamma_2, \tag{10}$$

where the exterior field E^{ext}, H^{ext} is given by

$$E^{ext} = E^i + E^s \tag{11}$$
$$H^{ext} = H^i + H^s, \tag{12}$$

E^s, H^s is the scattered field satisfying the Silver Müller radiation condition

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0 \tag{13}$$

uniformly in $\hat{x} = x/|x|$, $r = |x|$, the incident field E^i,H^i is given by

$$E^i(x) := \frac{i}{k} \nabla \times \nabla \times p e^{ikx\cdot d} = ik(d \times p) \times d e^{ikx\cdot d} \tag{14}$$
$$H^i(x) := \nabla \times p e^{ikx\cdot d} = ikd \times p e^{ikx\cdot d},$$

where the wave number k is a positive constant, $d \in \Omega := \{x \in \mathbb{R}^3 : \ |x| = 1\}$ is a unit vector giving the direction of propagation and p is the polarization vector.
In the following we assume that N is a 3×3 symmetric matrix-valued function whose entries are in $C^1(\overline{D})$, and N satisfies $\bar{\xi} \cdot \Im(N)\,\xi \ \geq 0$ and $\bar{\xi} \cdot \Re(N)\,\xi \geq \gamma|\xi|^2$ for all $\xi \in \mathbb{C}^3$ and all $x \in \overline{D}$ where γ is a positive constant.
In order to formulate precisely the forward problem we need the following spaces. Letting $(H^s(D))^3$, $(H^s_{loc}(D_e))^3$ and $(H^s(\Gamma))^3$, $s \in \mathbb{R}$, denote the product of the standard Sobolev spaces defined on D, D_e and Γ respectively (with the convention $H^0 = L^2$), and

$$H(\text{curl}, D) := \{u \in (L^2(D))^3 \ : \ \nabla \times u \in (L^2(D))^3\}$$
$$L^2_t(\Gamma) := \{u \in (L^2(\Gamma))^3 \ : \ \nu \cdot u = 0 \quad \text{on} \quad \Gamma\}$$
$$L^2_t(\Gamma_2) := \{u|_{\Gamma_2} : u \in L^2_t(\Gamma)\},$$

we introduce the space

$$X(D, \Gamma_2) := \{u \in H(\text{curl}, D) \ : \ \nu \times u|_{\Gamma_2} \in L^2_t(\Gamma_2)\} \tag{15}$$

equipped with the norm

$$\|u\|^2_{X(D,\,\Gamma_2)} = \|u\|^2_{H(curl,\,D)} + \|\nu \times u\|^2_{L^2(\Gamma_2)}. \tag{16}$$

For the exterior domain D_e we define the above spaces in the same way for every $D_e \cap B_R$, with B_R a ball of radius R containing D and denote these spaces by $H_{loc}(\text{curl}, D_e)$ and $X_{loc}(D_e, \Gamma_2)$, respectively. Finally, we introduce the trace space of $X(D, \Gamma_2)$ on Γ by

$$Y(\Gamma) := \left\{ h \in (H^{-1/2}(\Gamma))^3 \ : \ \exists\, u \in H_0(\text{curl}, B_R), \ \begin{array}{l} \nu \times u|_{\Gamma_2} \in L^2_t(\Gamma_2) \\ \text{and} \quad h = \nu \times u|_{\Gamma} \end{array} \right\}$$

where $H_0(\text{curl}, B_R)$ is the space of functions u in $H(\text{curl}, B_R)$ satisfying $\hat{x} \times u = 0$ on the boundary of B_R. As shown in [7] $Y(\Gamma)$ is a Banach space with respect to the norm

$$\|h\|^2_{Y(\Gamma)} := \inf\{\|u\|^2_{H(curl,B_R)} + \|\nu \times u\|^2_{L^2_t(\Gamma_2)}\} \tag{17}$$

where the infimum is taken over all functions $u \in H_0(\text{curl}, B_R)$ such that $\nu \times u|_{\Gamma_2} \in L^2_t(\Gamma_2)$ and $h = \nu \times u|_\Gamma$. In fact $Y(\Gamma)$ coincides with $H^{-\frac{1}{2}}_{div}(\Gamma) \cap L^2_t(\Gamma_2)$ where

$$H^{-\frac{1}{2}}_{div}(\Gamma) := \left(u \in (H^{-\frac{1}{2}}(\Gamma))^3, \quad \nu \cdot u = 0, \quad \text{div}_\Gamma\, u \in H^{-\frac{1}{2}}(\Gamma)\right)$$

is the trace space of $\nu \times u|_\Gamma$ for $u \in H_0(\text{curl}, B_R)$ (see [4] and [2], [3] for the case of Lipshitz boundaries). We also recall that the trace space of $(\nu \times u) \times \nu|_\Gamma$ for $u \in H(\text{curl}, B_R)$ is defined by

$$H^{-\frac{1}{2}}_{curl}(\Gamma) := \left(u \in (H^{-\frac{1}{2}}(\Gamma))^3, \quad \nu \cdot u = 0, \quad \text{curl}_\Gamma\, u \in H^{-\frac{1}{2}}(\Gamma)\right),$$

and a duality relation is defined between $H^{-\frac{1}{2}}_{div}(\Gamma)$ and $H^{-\frac{1}{2}}_{div}(\Gamma)$.
Expressing the magnetic fields in terms of the electric fields, the direct scattering problem becomes a particular case of the following problem: Given $f \in Y(\Gamma)$, $h \in Y(\Gamma)$, $h_2 \in L^2_t(\Gamma_2)$ find $E^s \in X_{loc}(D_e, \Gamma_2)$, $E^{int} \in X(D, \Gamma_2)$ such that

$$\nabla \times \nabla \times E^s - k^2 E^s = 0 \qquad \text{in} \quad D_e \tag{18}$$

$$\nabla \times \nabla \times E^{int} - k^2 N(x) E^{int} = 0 \quad \text{in} \quad D \tag{19}$$

$$\nu \times E^s - \nu \times E^{int} = f \quad \text{on} \quad \Gamma \tag{20}$$

$$\nu \times (\nabla \times E^s) - \nu \times (\nabla \times E^{int}) = h + \begin{cases} 0 & \text{on} \ \ \Gamma_1 \\ ik\eta E^s_T + h_2 & \text{on} \ \ \Gamma_2 \end{cases} \tag{21}$$

$$\lim_{r\to\infty}((\nabla \times E^s) \times x - ikrE^s) = 0 \tag{22}$$

where u_T denotes the tangential component $(\nu \times u) \times \nu$. Note that the direct scattering problem corresponds to $f := -\nu \times E^i|_\Gamma$, $h := -\nu \times (\nabla \times E^i)|_\Gamma$, and $h_2 := ik\eta\, E^i_T|_{\Gamma_2}$.
The following theorem concerning the well-posedness of the above problem was proved in [9].

Theorem 1. *The transmission problem (18)-(22) has a unique solution* $E^{int} \in X(D, \Gamma_2)$, $E^s \in X_{loc}(D_e, \Gamma_2)$ *which satisfies*

$$\|E^{int}\|_{X(D,\Gamma_2)} + \|E^s\|_{X(B_R\setminus\overline{D},\Gamma_2)} \le C\left(\|f\|_{Y(\Gamma)} + \|h\|_{Y(\Gamma)} + \|h_2\|_{L^2_t(\Gamma_2)}\right) \tag{23}$$

for some positive constant C *depending on* R *but not on* f, h *and* h_2.

It is known [11] that the radiating solution E^s to (18)-(22) has the asymptotic behavior

$$E^s(x) = \frac{e^{ik|x|}}{|x|} \left\{ E_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\} \tag{24}$$

as $|x| \to \infty$, where E_∞ is defined on the unit sphere Ω and is known as the *electric far field pattern*. In the case of incident plane waves given by (14) the electric far field pattern depends on the incident direction and polarization which will be indicated by $E_\infty(\hat{x}) = E_\infty(\hat{x}, d, p)$.

The *inverse scattering problem* we are concern with is to determine D and η from a knowledge of the electric far field pattern $E_\infty(\hat{x}, d, p)$ of the scattered field E^s for $\hat{x}$, $-d \in \Omega_0$, where Ω_0 is a subset of the unit sphere Ω, and three linearly independent polarizations p. Note that no a priori knowledge of the amount of coating is required.

3 Analysis of the Inverse Problem

Now we turn to the *inverse problem* for the vector case. Given the incident plane wave $E^i = ik(d \times p) \times d\, e^{ikx\cdot d}$ and the corresponding electric far field pattern $E_\infty(\hat{x}, d, p)$ for $\hat{x}, d$ in the unit sphere Ω and three linearly independent polarizations p, determine D and η. Uniqueness results for the inverse problem can be found in [9]. The aim of this paper is to show how to reconstruct D and η from the given data.

3.1 Shape Reconstruction

The analysis of this inverse problems follows the lines of the analysis of the inverse problem in the scalar case treated in [6]. We define *Maxwell eigenvalues* to be the values of k for which

$$\begin{aligned} \nabla \times \nabla \times E + k^2 N(x) E = 0 \qquad & \text{in} \quad D \\ \nu \times E = 0 \qquad & \text{on} \quad \Gamma, \end{aligned}$$

has a nontrivial solution, and *transmission eigenvalues* the values of k for which

$$\left\{ \begin{aligned} \nabla \times \nabla \times E_0 - k^2 E_0 = 0 \\ \nabla \times \nabla \times E - k^2 N(x) E = 0 \end{aligned} \right. \qquad \text{in} \quad D \tag{25}$$

$$\nu \times E - \nu \times E_0 = 0 \qquad \text{on} \quad \Gamma \tag{26}$$

$$\nu \times (\nabla \times E) - \nu \times (\nabla \times E_0) = 0 \qquad \text{on} \quad \Gamma_1 \tag{27}$$

$$\nu \times (\nabla \times E) - \nu \times (\nabla \times E_0) = -ik\eta(\nu \times E_0) \times \nu \qquad \text{on} \quad \Gamma_2. \tag{28}$$

has a nontrivial solution. Note that if $\bar{\xi} \cdot \Im(N)\,\xi > 0$ at a point $x_0 \in D$ Maxwell eigenvalues and transmission eigenvalues do not exist.

We now define an *electromagnetic Herglotz pair* to be a pair of vector fields of the form

$$E_g(x) = \int_\Omega e^{ikx\cdot d} g(d) ds(d), \qquad H_g(x) = \frac{1}{ik}\nabla \times E_g(x) \tag{29}$$

where $g \in L^2_t(\Omega)$. It is easy to see that $\nabla \times \nabla \times E_g - k^2 E_g = 0$. Next we consider the vector space

$$\begin{aligned} E(D) := \{&E \in X(D, \Gamma_2),\, \nabla \times E \in X(D, \Gamma_2), \\ &\nabla \times \nabla \times E - k^2 N(x)E = 0 \text{ in } D\big\} \end{aligned}$$

and define the subset of $\mathcal{Y}(\Gamma) := H^{-\frac{1}{2}}_{div}(\Gamma) \times Y(\Gamma_1) \times L^2_t(\Gamma_2)$ by

$$\begin{aligned} \mathcal{E} := \{&\nu \times (E_g - E),\, \nu \times \nabla \times (E_g - E)|_{\Gamma_1}, \\ &\nu \times \nabla \times (E_g - E) - ik\eta(\nu \times E_g) \times \nu|_{\Gamma_2}\} \end{aligned}$$

for all $E \in E(D)$, $g \in L^2_t(\Gamma)$, E_g the electric field of the electromagnetic Herglotz pair with kernel g, where $Y(\Gamma_1) := \{h|_{\Gamma_1} :\ h \in Y(\Gamma)\}$.

Theorem 2. *Suppose that k is neither a Maxwell eigenvalue nor a transmission eigenvalue. Then $\mathcal{E}$ is dense in $\mathcal{Y}(\Gamma)$.*

Proof. Let $\varphi \in H^{-\frac{1}{2}}_{curl}(\Gamma)$ and $\psi \in H^{-\frac{1}{2}}_{curl}(\Gamma) \cap L^2_t(\Gamma_2)$ such that

$$\int_\Gamma \nu \times (E_g - E)\cdot\varphi\, ds + \int_\Gamma \nu \times \nabla \times (E_g - E)\psi\, ds - \int_{\Gamma_2} ik\eta (E_g)_T \cdot \psi\, ds = 0 \tag{30}$$

for all $g \in L^2_t(\Omega)$ and $E \in E(D)$. Note that $\varphi \in H^{-\frac{1}{2}}_{curl}(\Gamma)'$ and $\psi|_{\Gamma_1} \in Y(\Gamma_1)'$ (see [7], Section 2.2 for the characterization of the dual space $Y(\Gamma_1)'$). The first and the second integral in (30) is understood in the sense of duality paring between $H^{-\frac{1}{2}}_{div}(\Gamma)$ and $H^{-\frac{1}{2}}_{div}(\Gamma)$ while the third integral containing η is understood in the $L^2_t(\Gamma_2)$ sense. Setting first $E = 0$ in (30) and interchanging the order of integrations we obtain

$$\begin{aligned} 0 = \ &\hat{x} \times \Bigg\{ \int_\Gamma (\varphi \times \nu) e^{-iky\cdot\hat{x}}\, ds + ik\,\hat{x} \times \int_\Gamma (\psi \times \nu) e^{-iky\cdot\hat{x}}\, ds \\ &-ik\eta \int_{\Gamma_2} [(\nu \times \psi) \times \nu] e^{-iky\cdot\hat{x}}\, ds \Bigg\} \times \hat{x} \end{aligned} \tag{31}$$

The right hand side of the above expression is the far field pattern of the following electric and magnetic dipole distribution defined by

$$P(x) = \frac{1}{k^2}\nabla \times \nabla \times \int_\Gamma (\varphi(y) \times \nu)\Phi(x, y)\, ds_y$$

$$-\nabla \times \int_{\Gamma} (\psi(y) \times \nu)\Phi(x,y)\, ds_y$$
$$-i\frac{\eta}{k}\,\nabla \times \nabla \times \int_{\Gamma_2} [(\nu \times \psi(y)) \times \nu]\Phi(x,y)\, ds_y \tag{32}$$

where $\Phi(x,y) := \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}$. Therefore we conclude that $P(x) = 0$ in $D_e := \mathbb{R}^3 \setminus \overline{D}$. Hence taking the limit of $P(x)$ as $x \to \Gamma$ from both sides we obtain

$$\nu \times P^- = -\nu \times \psi \qquad\qquad \nu \times \nabla \times P^- - ik\tilde{\eta}(\nu \times P^-) \times \nu = \nu \times \varphi$$

on Γ, where the superscript - indicates the limit obtained by approaching the boundary Γ from D, and $\tilde{\eta} = 0$ on Γ_1 and $\tilde{\eta} = \eta$ on Γ_2. We remark that $P(x)$ and $\operatorname{curl} P(x)$ are both square integrable in any compact subset of D and D_e. Furthermore, since $\varphi \times \nu$ and $\psi \times \nu$ are in $H_{div}^{-\frac{1}{2}}(\Gamma)$, the potentials over Γ in (32) and the corresponding jump relations are well defined from potential theory for single layer potentials with $H^{-\frac{1}{2}}$ densities [17], while the jump relations for the potential over Γ_2 with L^2 density is interpreted in the sense of the L^2 limit ([11] p.172). Next, setting $E_g = 0$ in (30), using the expressions for φ and ψ and Green's formula together with a parallel surfaces argument (see [14]) we obtain

$$\begin{aligned} 0 &= \int_{\Gamma} \big[(\nu \times P^-) \cdot \nabla \times E - (\nu \times E) \cdot \nabla \times P^- \\ &\qquad -ik\tilde{\eta}(\nu \times E) \cdot (\nu \times P^-)\big]\, ds \\ &= k^2 \int_{D} P^- \cdot (I - N)E\, dx - ik\eta \int_{\Gamma_2} (\nu \times E) \cdot (\nu \times P^-)\, ds. \end{aligned} \tag{33}$$

Note that $P \in L^2(D)$. Now let $F \in H(\operatorname{curl}, D)$ be the unique solution (c.f. [18]) of

$$\begin{aligned} \nabla \times \nabla \times F - k^2 NF &= k^2(I - N)P \qquad &&\text{in} \quad D \\ \nu \times F &= 0 &&\text{on} \quad \Gamma. \end{aligned}$$

Using the vector Green formula for E and F, from (33) we obtain

$$\begin{aligned} \int_{D} (\nu \times E) \cdot \nabla \times F\, ds &= -k^2 \int_{D} P^- \cdot (I - N)E\, dx \\ &= -ik\eta \int_{\Gamma_2} (\nu \times E) \cdot (\nu \times P^-)\, ds. \end{aligned}$$

Hence

$$\int_{\Gamma} (\nu \times E) \cdot \big[\nabla \times F + ik\tilde{\eta}(\nu \times P^-)\big]\, ds = 0$$

for all $E \in E(D)$ whence

$$\nu \times \nabla \times F + ik\tilde{\eta}(\nu \times P^-) \times \nu = 0 \qquad \text{on} \quad \Gamma$$

since k is not a Maxwell eigenvalue. Now we observe that P and $\tilde{E} = P + F$ satisfy

$$\begin{cases} \nabla \times \nabla \times P - k^2 P = 0 \\ \nabla \times \nabla \times \tilde{E} - k^2 N(x)\tilde{E} = 0 \end{cases} \qquad \text{in } D \tag{34}$$

$$\nu \times \tilde{E} - \nu \times P = 0 \qquad \text{on } \Gamma \tag{35}$$

$$\nu \times (\nabla \times \tilde{E}) - \nu \times (\nabla \times P) = 0 \qquad \text{on } \Gamma_1 \tag{36}$$

$$\nu \times (\nabla \times \tilde{E}) - \nu \times (\nabla \times P) = -ik\eta(\nu \times P) \times \nu \qquad \text{on } \Gamma_2 \tag{37}$$

which implies that $P = \tilde{E} = 0$ in D provided k is not a transmission eigenvalue. Therefore $\varphi = \psi = 0$ which proves the theorem. We remark that, in order to conclude that $P = \tilde{E} = 0$ in D, the $H(\text{curl}, D_h)$-regularity of P, where $\overline{D}_h \subset D$ allows us to apply the vector Green's formula in any compact subset D_h of D and then take the limit of the surface integrals since the boundary relations in (34)–(37) hold in the L^2-limit sense (see Lemma 2.1 of [6] for a similar proof in the scalar case). □

Next we define the *far field operator* $F : L^2_t(\Omega) \to L^2_t(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))ds(d), \qquad \hat{x} \in \Omega \quad \text{and} \quad g \in L^2_t(\Omega) \tag{38}$$

and look for solutions $g \in L^2_t(\Omega)$ of the *far field equation*

$$(Fg)(\hat{x}) := E_{e,\infty}(\hat{x}, z, q) \tag{39}$$

where

$$E_{e,\infty}(\hat{x}, z, q) = \frac{ik}{4\pi}(\hat{x} \times q) \times \hat{x}\, e^{-ik\hat{x}\cdot z}$$

is the electric far field pattern of the electric dipole with polarization q given by

$$E_e(x, z, q) := \frac{i}{k}\nabla_x \times \nabla_x \times\, q\,\Phi(x, z) \tag{40}$$

with $\Phi(x, z) := \frac{1}{4\pi}\frac{e^{ik|x-z|}}{|x-z|}$. We can now prove the following theorem.

Theorem 3. *Assume that k is neither a transmission eigenvalue nor a Maxwell eigenvalue and let F be the far field operator corresponding to (6)-(13). Then we have:*

1. *For $z \in D$ and every $\epsilon > 0$ there exists a solution $g^z_\epsilon \in L^2_t(\Omega)$ of the inequality*

$$\|Fg^z_\epsilon - E_{e,\infty}(\cdot,\, z, q)\|_{L^2_t(\Omega)} < \epsilon$$

such that $\|E_{g_\epsilon^z}\|_{X(D,\Gamma_2)} < \infty$ *where* $E_{g_\epsilon^z}$ *is the electric field of the electromagnetic Herglotz pair with kernel* g_ϵ^z*. Moreover, for a fixed* $\epsilon > 0$

$$\lim_{z\to\Gamma} \|g_\epsilon^z\|_{L_t^2(\Omega)} = \infty \quad \textit{and} \quad \lim_{z\to\Gamma} \|E_{g_\epsilon^z}\|_{X(D,\Gamma_2)} = \infty.$$

2. *For* $z \in \mathbb{R}^3 \setminus \overline{D}$ *and every* $\epsilon > 0$ *and* $\delta > 0$ *there exists a solution* $g_{\epsilon,\delta}^z \in L_t^2(\Omega)$ *of the inequality*

$$\|Fg_{\epsilon,\delta}^z - E_{e,\infty}(\cdot,\, z, q)\|_{L^2(\Omega)} < \epsilon + \delta$$

such that

$$\lim_{\delta\to 0} \|g_{\epsilon,\delta}^z\|_{L_t^2(\Omega)} = \infty \quad \textit{and} \quad \lim_{\delta\to 0} \|E_{g_{\epsilon,\delta}^z}\|_{X(D,\Gamma_2)} = \infty,$$

where $E_{g_{\epsilon,\delta}^z}$ *is the electric field of the electromagnetic Herglotz pair with kernel* $g_{\epsilon,\delta}^z$*.*

Remark 1. Note that, in Theorem 3, $E_{g_\epsilon^z}$ for $z \in D$ is such that $\nu \times E_{g_\epsilon^z}$ and $\nu \times \nabla \times E_{g_\epsilon^z}$ converge with respect to the $Y(\Gamma)$ norm as $\epsilon \to 0$.

Proof. The proof follows the lines of the proof of Theorem 2.6 of [6]. Let $\mathcal{B}$ denotes the linear bounded operator which maps $f := \nu \times E|_\Gamma$, $h := \nu \times (\nabla \times E)|_\Gamma$ and $h_2 := ik\eta(\nu \times E) \times \nu|_{\Gamma_2}$, where $E \in X(D, \Gamma_2)$ satisfies $\nabla \times \nabla \times E - k^2 E = 0$, onto the electric far field pattern of the corresponding solution of (18)-(22). Exactly in the same way as in Lemma 2.5 of [6] by making use of the result of Theorem 2 and using the divergence free vector spherical wave functions [11], one can show that $\mathcal{B} : Y(\Gamma) \times Y(\Gamma) \times L_t^2(\Gamma_2) \to L_t^2(\Omega)$ is compact, injective and has dense range provided that k is neither a Maxwell eigenvalue nor a transmission eigenvalue.
Next consider $z \in D$. Given $\epsilon > 0$ from Theorem 2 there exists $E_{g_\epsilon^z}$ with $g_\epsilon^z \in L_t^2(\Omega)$ and $E_\epsilon^z \in E(D)$ such that $\nu \times (E_{g_\epsilon^z} - E_\epsilon^z)$, $\nu \times \nabla \times (E_{g_\epsilon^z} - E_\epsilon^z) - ik\tilde{\eta}(\nu \times E_{g_\epsilon^z}) \times \nu$ approximates $\nu \times E_e(\cdot, z, q)$, $\nu \times \nabla \times E_e(\cdot,\, z, q) - ik\tilde{\eta}(\nu \times E_e(\cdot,\, z, q)) \times \nu$ in the $\mathcal{Y}(\Gamma)$ norm with discrepancy ϵ, where $\tilde{\eta} = \eta$ on Γ_2 and $\tilde{\eta} = 0$ on Γ_1. Noting that $F_{g_\epsilon^z}$ is the far field pattern of the electric scattered field corresponding to $E_{g_\epsilon^z}$ as the incident field, from the estimate (23) and the fact that the far field pattern depends continuously on the scattered field we obtain that

$$\|Fg_{\epsilon,\delta}^z - E_{e,\infty}(\cdot,\, z, q)\|_{L^2(\Omega)} < C\epsilon$$

where C is a positive constant independent of ϵ. As $z \to \Gamma$, using (23) for the solution of the direct scattering problem E_ϵ and $E_e(\cdot\,, z, q)$ together with the fact that $\lim_{z\to\Gamma} \|E_e(\cdot\,, z, q)\|_{H(curl, B_R\setminus\overline{D})} = \infty$ one obtain that

$$\lim_{z\to\Gamma} \|g_\epsilon^z\|_{L_t^2(\Omega)} = \infty \quad \text{and} \quad \lim_{z\to\Gamma} \|E_{g_\epsilon^z}\|_{X(D,\Gamma_2)} = \infty.$$

Now let $z \in \mathbb{R}^3 \setminus \overline{D}$. From the theory of the ill-posed problems applied to the compact operator $\mathcal{B}$, we obtain

$$\mathcal{B}(f_z^\alpha, h_z^\alpha, h_{1z}^\alpha) - E_{e,\infty}(\cdot,\, z, q)\|_{L^2(\Omega)} < \delta$$

for an arbitrary small but fixed δ where $f_z^\alpha = \nu \times E_z^\alpha, h_z^\alpha = \nu \times \nabla \times E_z^\alpha, h_{1z}^\alpha = ik\eta(\nu \times E_z^\alpha \times \nu)$ with $E_z^\alpha \in X(D, \Gamma_2)$ is the regularized solution corresponding to the regularization parameter α chosen by a regular regularization strategy (e.g. the Morozov discrepancy principle). Furthermore, we have that the $Y(\Gamma) \times Y(\Gamma) \times L_t^2(\Gamma_2)$ norm of $(f_z^\alpha, h_z^\alpha, h_{1z}^\alpha)$ goes to infinity as $\alpha \to \infty$. Note that $\alpha \to 0$ as $\delta \to 0$. Now the second part of the theorem follows from the fact that E_z^α can be approximated arbitrarily close with respect to the $X(D, \Gamma_2)$-norm by a Herglotz wave function E_g (see Theorem 2.5 of [7]) and the fact that $Fg = \mathcal{B}(\nu \times E_g,\, \nu \times \nabla \times E_g, ik\eta(\nu \times E_g \times \nu))$. This ends the proof. □

The above result provides a characterization for the boundary Γ of the scattering object D. Unfortunately, since the behavior of $E_{g_\epsilon^z}$ is described in terms of a norm depending on the unknown region D, $E_{g_\epsilon^z}$ can not be used to characterize D. Instead the linear sampling method characterizes the obstacle by the behavior of g_ϵ^z. In particular, given a discrepancy $\epsilon > 0$ and g_ϵ^z the ϵ-approximate solution of the far field equation (39), the boundary of the scatterer is reconstructed as the set of points z where the $L_t^2(\Omega)$ norm of g_ϵ^z becomes large. One can also use $|E_{g_\epsilon^z}|$ as an alternative indicator function of the boundary ∂D of the scattering object D as it will be shown in the numerical examples presented in Section 4.

3.2 Identification of the Surface Conductivity

Assuming now that D is known, we want to determine the surface conductivity η by making use of the approximate solution g to the far field equation (39). In [5] a formula for computing η in the 2D TE-polarized case is derived and the mathematical justification is based on the analysis of a appropriate boundary value problem called the *interior transmission problem*. The interior transmission problem corresponding to our scattering problem reads: Find a solution E, E_0 of the following boundary value problem

$$\begin{cases} \nabla \times \nabla \times E_0^z - k^2 E_0^z = 0 \\ \nabla \times \nabla \times E^z - k^2 N(x) E^z = 0 \end{cases} \quad \text{in } D \tag{41}$$

$$\nu \times E^z - \nu \times (E_0^z + E_e(\cdot, z, q)) = 0 \quad \text{on } \Gamma \tag{42}$$

$$\nu \times (\nabla \times E^z) - \nu \times [\nabla \times (E_0^z + E_e(\cdot, z, q))] = 0 \quad \text{on } \Gamma_1 \tag{43}$$

$$\nu \times (\nabla \times E^z) - \nu \times [\nabla \times (E_0^z + E_e(\cdot, z, q))] = -ik\eta\, [\nu \times (E_0^z - E_e(\cdot, z, q))] \times \nu \quad \text{on } \Gamma_2 \tag{44}$$

where $E_e(\cdot, z, q)$ is the electric dipole given by (40), $z \in D$ and $q \in \mathbb{R}^3$.
As noticed in [6] the completeness result given by Theorem 2 does not suffice to proceed further with the reconstruction of η. It is essential in the following analysis to know that the interior transmission problem has a (weak) solution in appropriate Sobolev spaces. Unfortunately, the well posedness of (41)-(44)

is not yet established. In the case where $\eta = 0$, Haddar in [13] has shown that, provided k is not a transmission eigenvalue and under some assumptions on N, the interior transmission problem has a unique weak solution $E^z \in L^2(D)$ and $E_0^z \in L^2(D)$ such that $E^z - E_0^z \in H(\mathrm{curl}, D)$ and $\nabla \times (E^z - E_0^z) \in H(\mathrm{curl}, D)$.

Conjecture 1. Assume that k is not a transmission eigenvalue and either $\bar{\xi} \cdot \Re(N-I)^{-1}\,\xi \geq \gamma|\xi|^2$ or $\bar{\xi} \cdot \Re(N-I)\,\xi \geq \gamma|\xi|^2$ for all $\xi \in \mathbb{C}^3$, all $x \in \overline{D}$ and some $\gamma > 0$. Then the interior transmission problem (41)-(44) has a unique solution $E^z \in L^2(D)$, $E_0^z \in L^2(D)$ and $E_0^z|_{\Gamma_2} \in L^2(\Gamma_2)$ such that $E^z - E_0^z \in H(\mathrm{curl}, D)$, $\nabla \times (E^z - E_0^z) \in H(\mathrm{curl}, D)$ and $\nu \times \nabla \times (E^z - E_0^z)|_{\Gamma_2} \in L^2(\Gamma_2)$.

Assuming Conjecture 1, we now use the approximate solution g^z for $z \in D$ of the far field equation (39) to give an approximation for the surface conductivity η. To this end we need the following lemma.

Lemma 1. *Assume that k is neither a Maxwell eigenvalue nor a transmission eigenvalue. For any point z in D we have that*

$$\int_D \overline{E}^z \cdot \Im(N) E^z \, dx + k\eta \int_{\Gamma_2} |\nu \times (E_0^z + E_e(\cdot,\, z, q)|^2 \, ds$$
$$= -\frac{k^3}{6\pi}\|q\|^2 + k\Re\left(E_0^z(z)\right) \tag{45}$$

where E^z and E_0^z is a solution to the interior transmission problem (41)-(44).

Proof. From Theorem 2, for given $\epsilon > 0$, there exists a $E_\epsilon^z \in E(D)$ and a electromagnetic Herglotz pair with electric field $E_{g_\epsilon^z}$ and kernel $g_\epsilon^z \in L_t^2(\Omega)$ such that

$$\begin{cases} \nu \times E_\epsilon^z - E_e(\cdot, z, q)) = E_{g_\epsilon^z} + \alpha_\epsilon \\ \\ \nu \times \nabla \times (E_\epsilon^z - E_e(\cdot, z, q)) + ik\tilde{\eta}(\nu \times E_e(\cdot, z, q)) \times \nu \\ = \nu \times \nabla \times E_{g_\epsilon^z} - ik\tilde{\eta}(\nu \times E_{g_\epsilon^z}) \times \nu + \beta_\epsilon \end{cases} \tag{46}$$

on Γ where

$$\|\,(\alpha_\epsilon, \beta_\epsilon)\,\|_{\mathcal{Y}(\Gamma)} < \epsilon. \tag{47}$$

Now, let E^z and E_0^z be the unique solution of the interior transmission problem (41)-(44). Obviously, E_ϵ^z and $E_{g_\epsilon^z}$ and converge to E^z and E_0^z, respectively as $\epsilon \to 0$ with respect to the graph norm $L^2(D) \cap L_t^2(\Gamma_2)$. Hence, E_ϵ^z and $E_{g_\epsilon^z}$ are uniformly bounded together with their curl in the $L^2(D)$ norm. Applying the vector Green's formula to E_ϵ^z and $\overline{E}_\epsilon^z$ in D (see [18] for the case of $H(\mathrm{curl}, D)$ functions) we obtain

$$\int_\Gamma \left(\nu \times E_\epsilon^z \cdot \mathrm{curl}\,\overline{E_\epsilon^z} - \nu \times \overline{E_\epsilon^z} \cdot \mathrm{curl}\,E_\epsilon^z\right) ds = 2i \int_D \overline{E}_\epsilon^z \cdot \Im(N) E_\epsilon^z \, dx. \tag{48}$$

On the other hand, using (46) and defining $W_\epsilon^z := E_{g_\epsilon^z} + E_e(\cdot, z, q)$, we have that

$$\begin{aligned}
&\int\limits_{\Gamma} \left(\nu \times E_\epsilon^z \cdot \operatorname{curl} \overline{E_\epsilon^z} - \nu \times \overline{E_\epsilon^z} \cdot \operatorname{curl} E_\epsilon^z\right) ds \\
=& \int\limits_{\Gamma} \left(\nu \times W_\epsilon^z \cdot \operatorname{curl} \overline{W_\epsilon^z} - \nu \times \overline{W_\epsilon^z} \cdot \operatorname{curl} W_\epsilon^z\right) ds \\
&-2ik\eta \int\limits_{\Gamma_2} |(\nu \times W_\epsilon^z) \times \nu|^2 \, ds + R_\epsilon^z \qquad (49)
\end{aligned}$$

where $|R_\epsilon^z| \leq C\epsilon$ for a positive constant C independent of ϵ. Again using the vector Green's formula, the integral representation formula and connecting the radiating solution $E_e(\cdot,\, z, q)$ to its far field pattern as in [8] Theorem 3.1, we obtain

$$\begin{aligned}
&\int\limits_{\Gamma} \left(\nu \times W_\epsilon^z \cdot \operatorname{curl} \overline{W_\epsilon^z} - \nu \times \overline{W_\epsilon^z} \cdot \operatorname{curl} W_\epsilon^z\right) ds \qquad (50) \\
&= -\frac{ik^3}{3\pi} \|q\|^2 + ikq \cdot \left[E_{g_\epsilon^z}(z) + \overline{E}_{g_\epsilon^z}(z)\right].
\end{aligned}$$

Hence, combining (48), (49) and (50) we have that

$$\begin{aligned}
&2i \int\limits_{D} \overline{E}_\epsilon^z \cdot \Im(N) E_\epsilon^z \, dx + 2ik\eta \int\limits_{\Gamma_2} |(\nu \times W_\epsilon^z) \times \nu|^2 \, ds \qquad (51) \\
&= -\frac{ik^3}{3\pi} \|q\|^2 + ikq \cdot \left[E_{g_\epsilon^z}(z) + \overline{E}_{g_\epsilon^z}(z)\right] - R_\epsilon^z.
\end{aligned}$$

Now letting $\epsilon \to 0$ in (51) we obtain the result. □

Theorem 4. *Let z be a fixed point in D, $\Im(N) = 0$ and assume that k is neither a Maxwell eigenvalue nor a transmission eigenvalue. Then for every $\epsilon > 0$ there exists an electromagnetic Herglotz function $E_{g_\epsilon^z}$ with kernel $g_\epsilon^z \in L_t^2(\Omega)$ an approximate solution of the far field equation (39) such that*

$$\left| \eta + \frac{\frac{k^2}{6\pi}\|q\|^2 - \Re\left(E_{g_\epsilon^z}(z)\right)}{\|\nu \times (E_{g_\epsilon^z} + E_e(\cdot,\, z, q))\|^2_{L_t^2(\Gamma_2)}} \right| \leq \epsilon. \qquad (52)$$

Proof. From the proof of Theorem 3 we have that the kernel g_ϵ^z of the Herglotz wave function $E_{g_\epsilon^z}$ in the proof of Lemma 1 is the ϵ-approximate solution to the far field equation (39). Hence the result of the theorem follows from Lemma 1. □

A draw back of (52) is that the extent of the coating Γ_2 is not known. So, in practice this expression only provides a lower bound for η. In addition, due to the accuracies in the determination of Γ by the linear sampling method, the computation of the outward normal ν can be problematic hence the most reliable lower bound for η is the following estimate

$$\eta \geq \frac{-\frac{k^2}{6\pi}\|q\|^2 + \Re\left(E_{g^z}(z)\right)}{\|(E_{g^z} + E_e(\cdot,\, z, q))\|^2_{L^2_t(\Gamma)}} \tag{53}$$

where g_z is the regularized solution of the far field equation (39) which is previously computed to determine D.

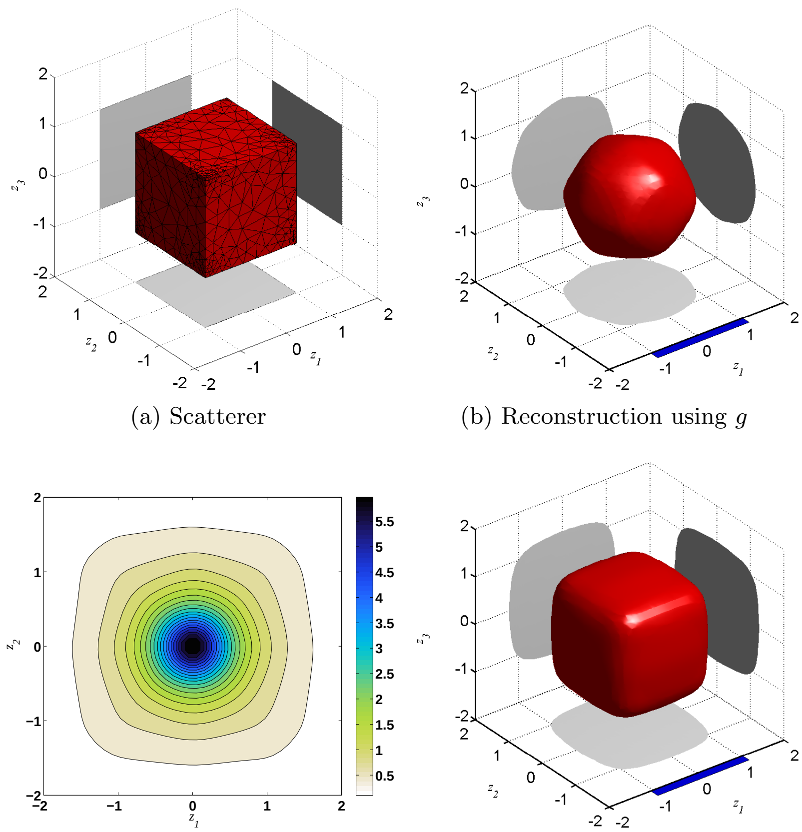
4 Numerical examples

For detailed numerical examples of shape reconstruction for coated objects, and also of estimating the surface conductivity the reader can consult [12]. Here we will give a single numerical example that illustrates our more general experience with the method. The numerical experiment is performed on synthetic far field data computed using the Ultra Weak Variational Formulation of Maxwell's equations as described in [16]. This (already approximate) far field data is further corrupted by noise as described in [12]. The far field data is then used first to reconstruct the shape of the scatterer using the standard Linear Sampling Method. This involves computing an approximate solution to the far field equation (39) for many sampling points z by discretizing g on the unit sphere and applying Tikhonov regularization and Morozov's principle to this ill-posed problem.

Once an approximation to the boundary of the scatterer is determined, the conductivity η can be approximated using (52) or a lower bound estimated using (53).

For this example we choose as test object the cube $[-1,1]^3$. Outside this cube $N = 1$ and within the cube $N = 2$. The entire cube is coated with $\eta = .1$ and $k = 3$ so the wavelength of the radiation is $\lambda = 2.09$. Figure 1 shows the result of reconstructing the cube using the Linear Sampling Method with 96 incoming waves (and 96 measurements) for each of two linearly independent polarizations (the other parameters in the method including the surface chosen for display are as in [12]). It is interesting to see that the Herglotz wave function gives a much better reconstruction of the scatterer than the Herglotz kernel.

Using the reconstructed surface in panel (c) of Fig. 1 we can estimate η. Alternatively we can test the formula (52) using the exact boundary in (a). The exact value is $\eta = 0.1$ using (52) gives $\eta \approx 0.14$ and using (53) gives the same approximation. Of course the reconstructed scatterer is not very accurate and this accounts for the rather poor approximation to η (the lower bound is an overestimate for this reason also).



(a) Scatterer

(b) Reconstruction using g

(c) Reconstruction using g

(d) Reconstruction using E_g

Fig. 1. Reconstructing the cube: (a) the original scatterer showing the surface mesh, (b) The reconstructed surface using the Linear Sampling Method and g, (c) A contour map of $1/\|g_z\|$ in the plane $z_3 = 0$ showing how the surface in (b) is obtained, (d) A reconstruction of the scatterer using $|E_{g_z}(z)|$. Surprisingly, use of the Herglotz wave function E_{g_z} gives a much better reconstruction of the scatterer than use of the kernel.

5 Conclusion

We have given some mathematical theory to substantiate the use of the Linear Sampling Method for reconstructing the shape of coated dielectrics. Assuming a conjecture on the existence of solutions of an interior transmission problem we have also derived a formula for the surface conductivity. Numerical results here and elsewhere show that the method can be applied in practice. We hope that the conjectured existence theory will be proved shortly.

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References

1. Angell T S and Kirsch A, 1992 The conductive boundary condition for Maxwell's equations *SIAM J. Appl. Math.* **52** pp. 1597-1610.
2. Buffa A and Ciarlet P Jr 2001 On traces for functional spaces related to Maxwell's equations. Part I: An integration by parts formula in Lipschitz polyhedra *Math. Meth. Appl. Sci.* **24** pp. 9-30.
3. Buffa A and Ciarlet P Jr 2001 On traces for functional spaces related to Maxwell's equations. Part II: Hodge decompositions on the boundary of Lipschitz polyhedra and applications *Math. Meth. Appl. Sci.* 24 pp. 31-48.
4. Cessenat M 1996 *Mathematical Methods in Electromagnetism* World Sciences, Singapore.
5. Cakoni F, Colton D and Monk P (2004) The determination of the surface conductivity of a partially coated dielectric, SIAM J. Appl. Math. **64**, 709-723.
6. Cakoni F, Colton D and Monk P The Inverse Electromagnetic Scattering Problem for a Partially Coated Dielectric , submitted for publication.
7. Cakoni F, Colton D and Monk P 2004 The electromagnetic inverse scattering problem for partially coated Lipschitz domains *Proc. Royal Soc. Edinburgh*, **134 A** 661-682.
8. Cakoni F and Colton D 2004 The determination of the surface impedance of a partially coated obstacle from far field data *SIAM J. Appl. Math.* **64** 709-723.
9. Cakoni F and Colton D 2003 A uniqueness theorem for an inverse electomagnetic scattering problem in inhomogeneous anisotropic media *Proc. Edinburgh Math. Soc.* **46** 293-314.
10. Colton D, Haddar H and Piana M 2003 The linear sampling method in inverse electromagnetic scattering theory *Inverse Problems* **19** S105-S137.
11. Colton D and Kress R 1998 *Inverse Acoustic and Electromagnetic Scattering Theory* 2nd ed. Springer Verlag.
12. Colton D and Monk P 3006, Target Identification of Coated Objects, to appear in *IEEE Trans. Antennas and Propagation*
13. Haddar H 2004 The interior transmission problem for anisotropic Maxwell's equations and its applications to the inverse problem *Math. Methods Appl. Sci.* **27** 2111-2129.
14. Hähner P 1994 An approximation theorem in inverse electromagnetic scattering *Math. Methods Appl. Sci.* **17** 293-303.
15. Hoppe D J and Rahmat-Samii Y 1995 *Impedance Boundary Conditions in Electromagnetics* Taylor&Francis Publishers.
16. Huttunen T, Malinen M, and Monk P, Solving Maxwell's Equations Using the Ultra Weak Variational Formulation, submitted for publication.
17. McLean W 2000 *Strongly Elliptic Systems and Boundary Integral Equations* Cambridge University Press.

18. Monk P 2003 *Finite Element Methods for Maxwell's Equations* Oxford University Press, Oxford.