Qualitative Methods in Imaging by Electromagnetic Waves

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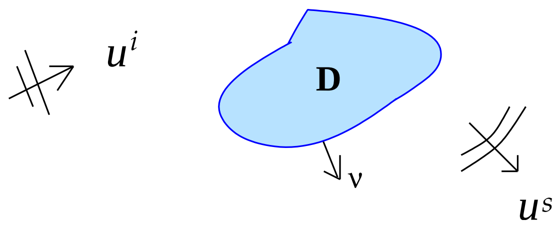
Introduction

- Recent developments in inverse electromagnetic scattering theory have led to results that are of potential interest in non-destructive testing.
- In this talk we will survey some of these results. In order to avoid technicalities we will focus our attention on the scalar case of scattering by TE or TM waves by an infinite cylinder. Furthermore for simplicity of presentation we will normally consider plane waves interrogation and far field measurements.
- All of our results remain valid for Maxwell's equations in 3D.
- All of our results remain valid for point source incident waves and near field measurements.

For details on Maxwell's equations see *Cakoni-Colton-Monk, The Linear Sampling Method in Inverse Electromagnetic Scattering, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM Publications*, (to appear).

Detection of Anomalies in an Isotropic Medium



$$\Delta u + k^2 n^2(x) u = 0 \qquad \text{in } \mathbb{R}^3$$

$$u = u^s + u^i \qquad \text{in } \mathbb{R}^3$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

where $u^i(x) := e^{ikx\cdot d}$, $|d| = 1$ and $u \in H^1_{loc}(\mathbb{R}^2)$.
We assume that $n^2 - 1$ has compact support $\overline{D}$ and n is piecewise continuous. The scattered field u^s has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O\left(r^{-3/2}\right)$$

as $r \to \infty$ where $\hat{x} = x/|x|$, $r = |x|$ and $k > 0$ is the wave number.
$u_\infty(\hat{x}, d)$ is the far field pattern of the scattered field u^s.

Detection of Anomalies in an Isotropic Medium

Let $\Omega := \{x : |x| = 1\}$.

- Assume: The far field pattern $u_\infty(\hat{x}, d)$ is known for $\hat{x} \in \Omega_0 \subset \Omega$ and $d \in \Omega_1 \subset \Omega$. Let $u_b^s(x, y)$ be the scattered field with no anomaly present and assume its far field pattern $u_\infty^b(\hat{x}, d)$ is known. The index of refraction of the background medium is denoted by $n_b(x)$
- Problem: Determine the shape and the nature of anomalies if that may be present in D.
- Measurements: The measured (multistatic data) is $u_\infty(\hat{x}, d)$ for $\hat{x} \in \Omega_0 \subset \Omega$ and $d \in \Omega_1 \subset \Omega$.

Detection of Anomalies in an Isotropic Medium

Let $\mathbb{G}(x,z)$, $z \in D$, be the Green's function for the background medium, i.e. $\mathbb{G}$ satisfies

$$\Delta_2 u + k^2 n_b^2 u = \delta(x-z) \qquad \text{in } \mathbb{R}^2$$

$$u = u^s + u^i$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

and let $\mathbb{G}_\infty(\hat{x},y)$ be its far field pattern. Consider the far field equation

$$\int_\Omega \left[u_\infty(\hat{x},d) - u_\infty^b(\hat{x},d)\right] g(d)\,ds_d = \mathbb{G}_\infty(\hat{x},z), \qquad \text{for} \quad x \in \Omega.$$

Note that this is an ill-posed integral equation of the first kind that must be solved using regularization methods.

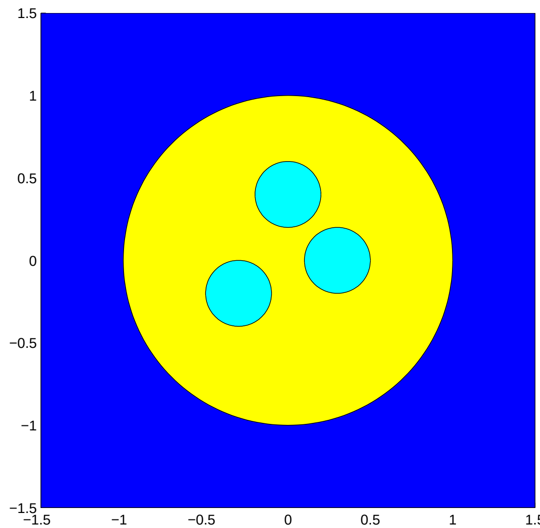
Detection of Anomalies in an Isotropic Medium

Suppose there exists an anomaly in D, i.e. there exists a region $D_0 \subset D$ where $n(x) \neq n_b$.

Then the L^2-norm of the (regularized) solution $g(d) = g(d; z)$ of the far field equation will be large for $z \in D \setminus D_0$ and small for $z \in D_0$. This method for determining D_0 is called the linear sampling method.

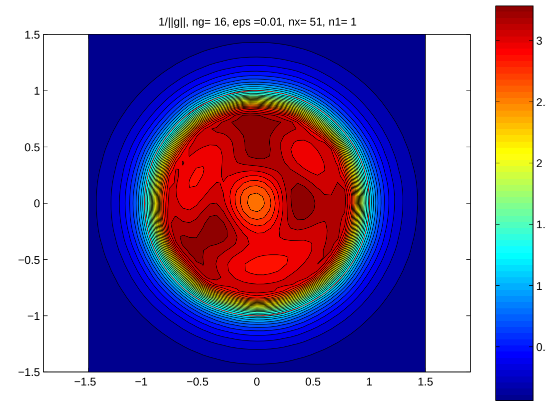
F. Cakoni and D.Colton, Qualitative Methods in Inverse Scattering Theory, Springer, Berlin, 2006.

Numerical Example: Detection of Anomalies

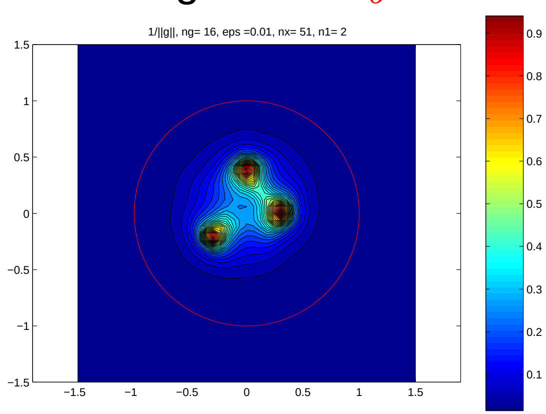


Anomalies: $n = 1.5 + 0.1i$,

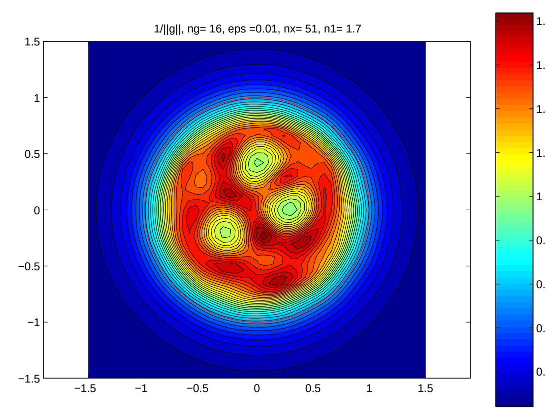
Background: $n_b = 2$



No background removal

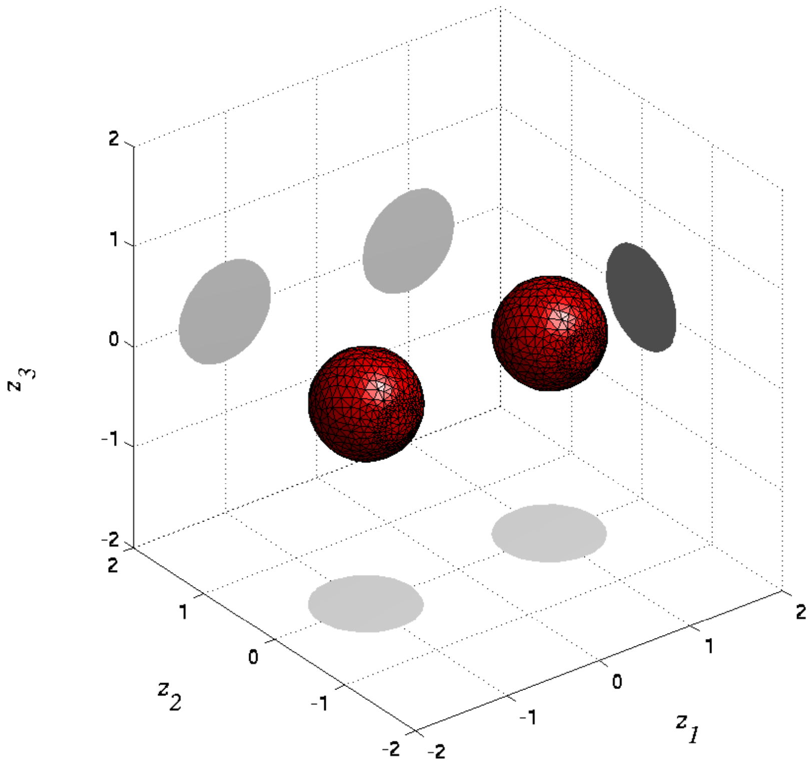


Exact background removal

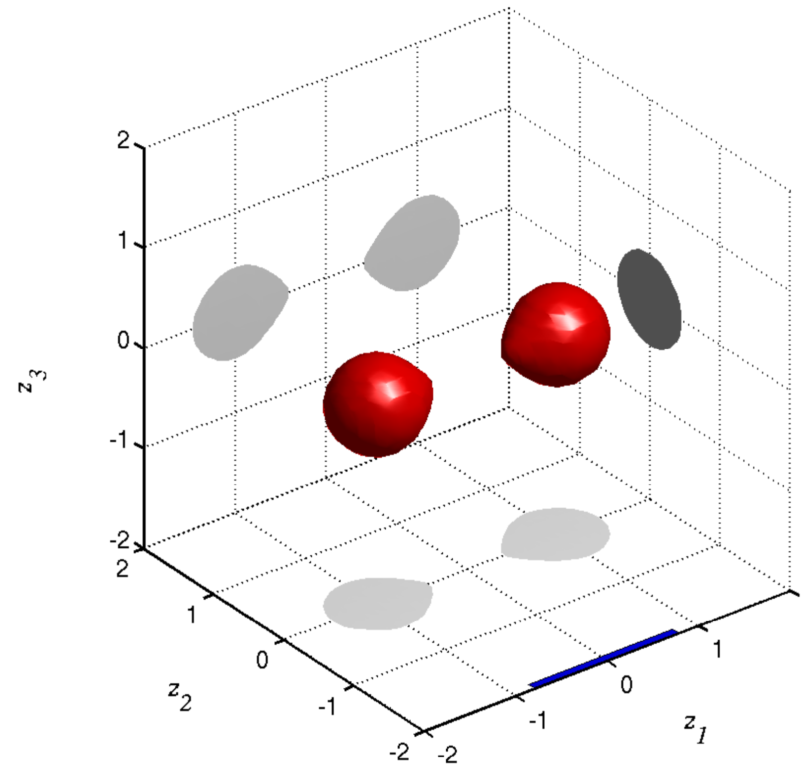


Inaccurate background removal; $n_b = 1.7$

Numerical Example: Detection of Anomalies



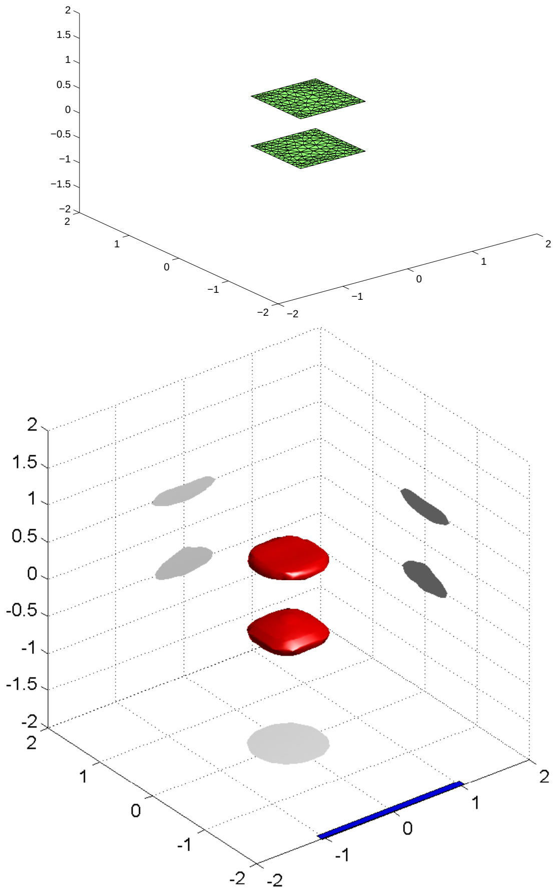
Exact Geometry

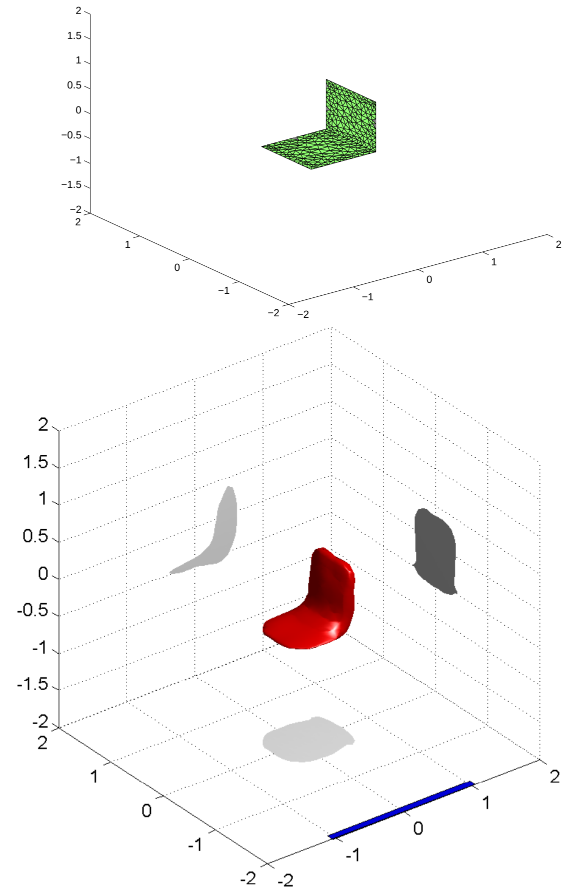


Reconstruction

$N = 3I$, $k = 4$

Numerical Example: Detection of Anomalies





Detection of Anomalies in an Isotropic Medium

Problem:

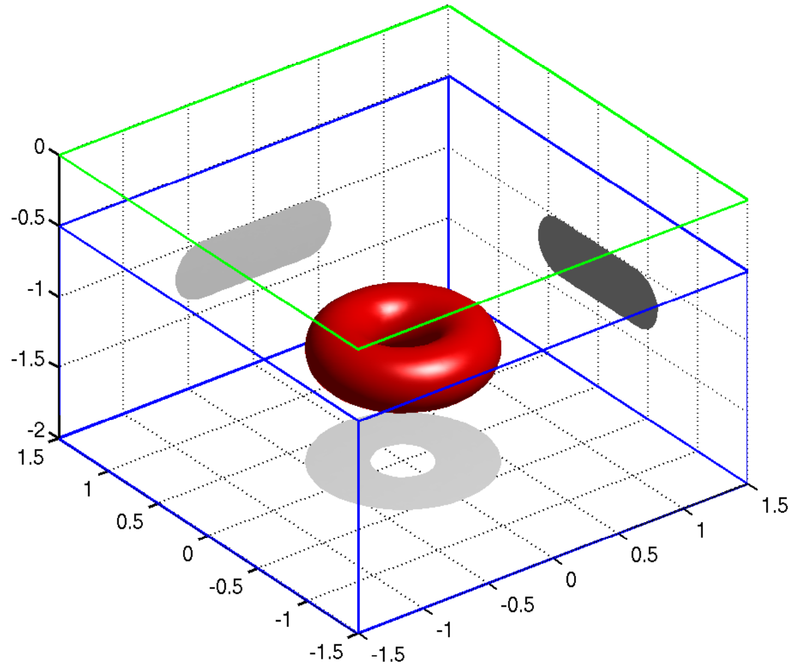
- Is there a method to avoid the construction of the Green's function but still obtain information about possible anomalies?

One Approach:

- A method related to the Linear Sampling Method is the Reciprocity Gap Functional method which determines the support of anomalies D_0 and avoids computation of the Green's function at the expanse of measuring both the electric and magnetic field on the boundary of D.

  For more details see Colton-Haddar, Inverse Problems, 21, (2005) and Cakoni-Fares-Haddar, Inverse Problems, 22, (2006).

Numerical Examples



Example of a perfectly conducting torus.

The reconstructions correspond to $n = 2 + 0.5$ and 5% random noise.

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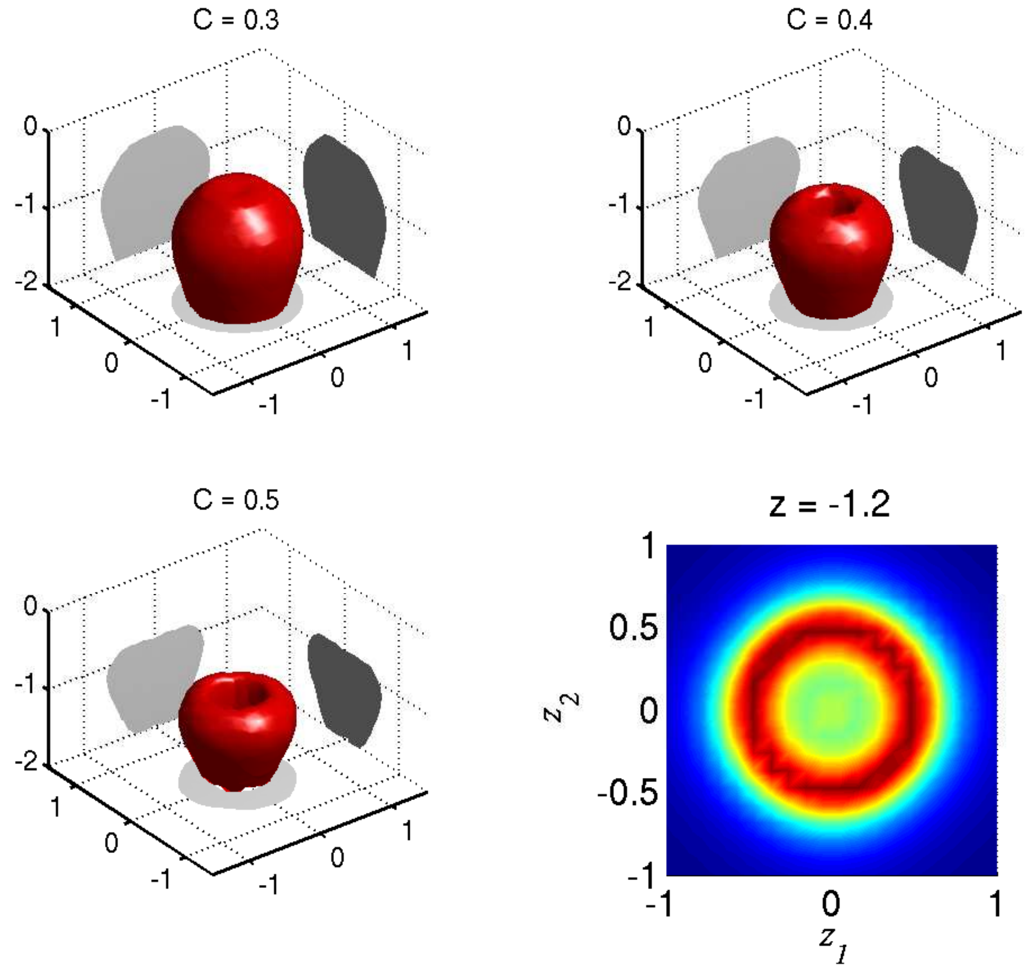
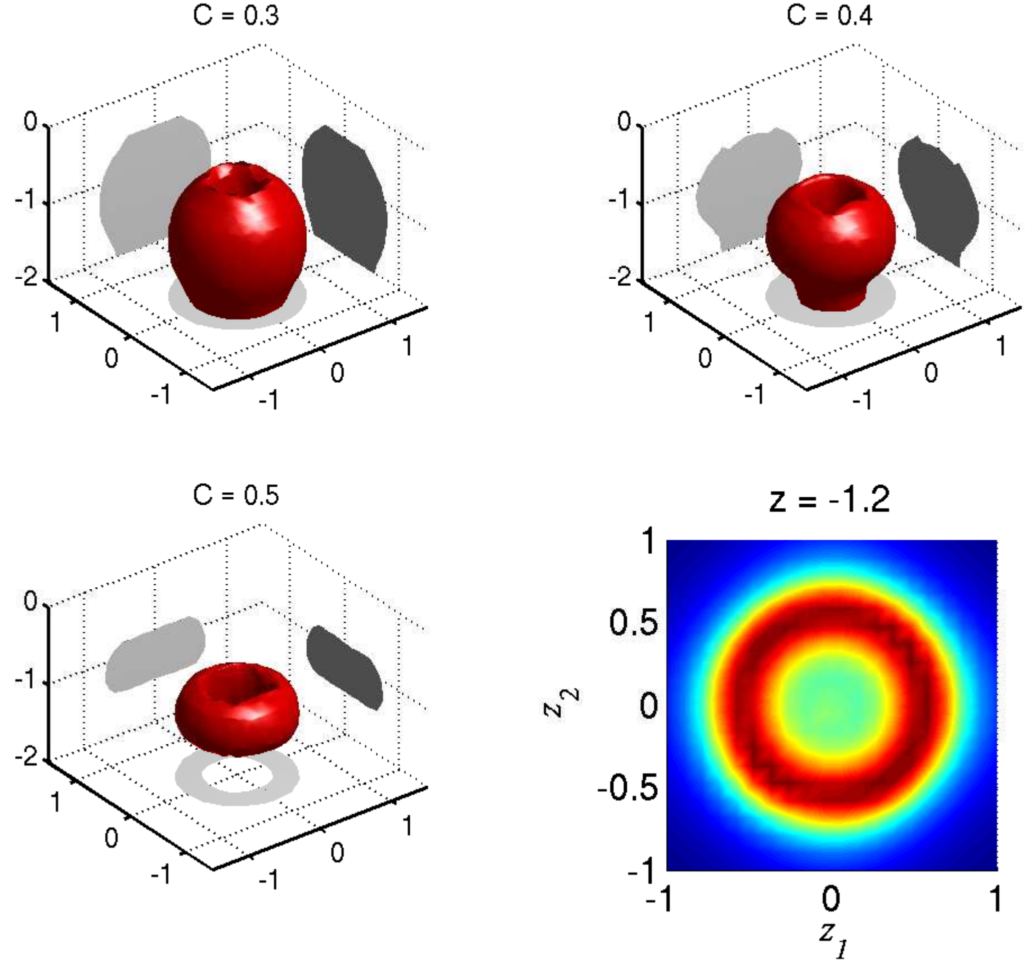
Numerical Examples



Reconstruction by using the Linear Sampling Method

Numerical Examples



Reconstruction by using the Reciprocity Gap Functional

Detection of Anomalies in an Isotropic Medium

In the following we discuss a different approach. Our goal is to obtain information about the index of refraction $n(x)$ from a knowledge of $u_\infty(\hat{x}, d)$ for $\hat{x} \in \Omega_0 \subset \Omega$ and $d \in \Omega_1 \subset \Omega$. That with be done

- without the need to compute the Green's function
- using only the electric field as measured data.

From now on we will assume that $n > 0$, i.e. the medium to be tested is a dielectric.

Detection of Anomalies in an Isotropic Medium

Problem: We want to obtain information about the index of refraction $n(x)$ from a knowledge of $u_\infty(\hat{x}, d)$ for $\hat{x} \in \Omega_0 \subset \Omega$ and $d \in \Omega_1 \subset \Omega$

Define the far field operator $F : L^2(\Omega) \to L^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d) ds(d).$$

For $z \in D$ the far field equation is

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z), \quad g \in L^2(\Omega)$$

where $\Phi_\infty(\hat{x}, z) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot z}$ is the far field pattern of the fundamental solution $\Phi(x, z) := H_0^{(1)}(k|x - y|)$.

Detection of Anomalies in an Isotropic Medium

The far field equation is again ill-posed and is typically solved using Tikhonov regularization, i.e. for small $\alpha > 0$ a function $g \in L^2(\Omega)$ is found that minimizes the Tikhonov functional

$$\|Fg - \Phi_\infty(\cdot, z)\|_{L^2}^2 + \alpha \|g\|_{L^2}^2.$$

In practice $u_\infty(\hat{x}, d)$ is noisy and hence the operator F is in fact F^δ i.e. the far field operator with kernel $u_\infty^\delta(\hat{x}, d)$ where δ is such that

$$\|u_\infty(\cdot, d) - u_\infty^\delta(\cdot, d)\|_{L^2} \leq \delta$$

and $\alpha = \alpha(\delta)$ is chosen such that $\alpha(\delta) \to 0$ as $\delta \to 0$. Assume therefore that $g_{z,\delta}$ minimizes the Tikhonov functional

$$\|F^\delta g - \Phi_\infty(\cdot, z)\|_{L^2}^2 + \alpha(\delta) \|g\|_{L^2}^2.$$

Detection of Anomalies in an Isotropic Medium

It turns out that a knowledge of $g_{z,\delta}$ gives information about the index of refraction $n(x)$ if the noise level δ is sufficiently small! To explain why this is true we need to introduce transmission eigenvalues and Herglotz wave functions.

Definition: $k > 0$ is a transmission eigenvalue if there exists a nontrivial solution $v \in L^2(D)$, $w \in L^2(D)$, $v - w \in H_0^2(D)$ of the interior transmission problem

$$\begin{aligned} \Delta w + k^2 n^2(x) w = 0 & \quad \text{in} \quad D \\ \Delta v + k^2 v = 0 & \quad \text{in} \quad D \\ w = v & \quad \text{on} \quad \partial D \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} & \quad \text{on} \quad \partial D \end{aligned}$$

ν is the unit outward normal to ∂D.

Transmission Eigenvalues

Remark: Note that if $n = 1$ the interior transmission problem is degenerate.

Question: Is there an incident wave u^i that does not scatter? (this question is also related to the injectivity of the far field operator)

Answer: If $k > 0$ is a transmission eigenvalue and $w \neq 0$, $v \neq 0$ satisfy

$$\begin{aligned} \Delta w + k^2 n^2(x) w = 0 \quad & \text{in} \quad D \\ \Delta v + k^2 v = 0 \quad & \text{in} \quad D \\ w = v \quad & \text{on} \quad \partial D \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad & \text{on} \quad \partial D \end{aligned}$$

such that v can be extended outside D as a solution to the Helmholtz equation $\tilde{v}$, then the scattered field due to $\tilde{v}$ as incident wave is identically zero.

Detection of Anomalies in an Isotropic Medium

Definition: A solution of the Helmholtz equation $\Delta u + k^2 u = 0$ of the form

$$v_g(x) := \int_\Omega e^{ikx\cdot d} g(d)\, ds_d$$

is called a Herglotz wave function with kernel $g \in L^2(\Omega)$.

Now assume that $g_{z,\delta}$ is the Tikhonov regularized solution of

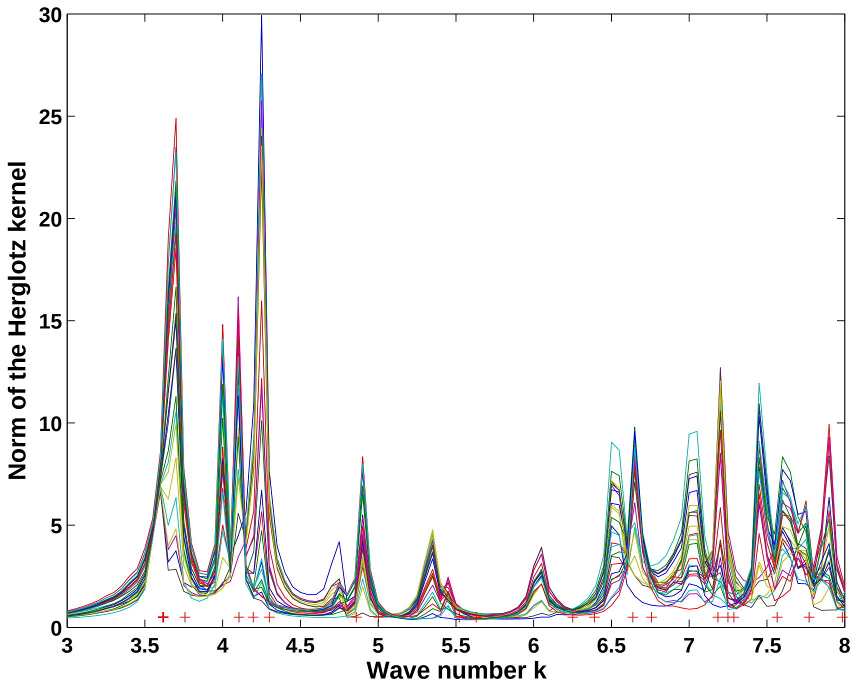
$$(F^\delta g)(\hat{x}) = \Phi(\hat{x}, z)$$

and $v_{g_{z,\delta}}$ is the Herglotz wave function with kernel $g_{z,\delta}$

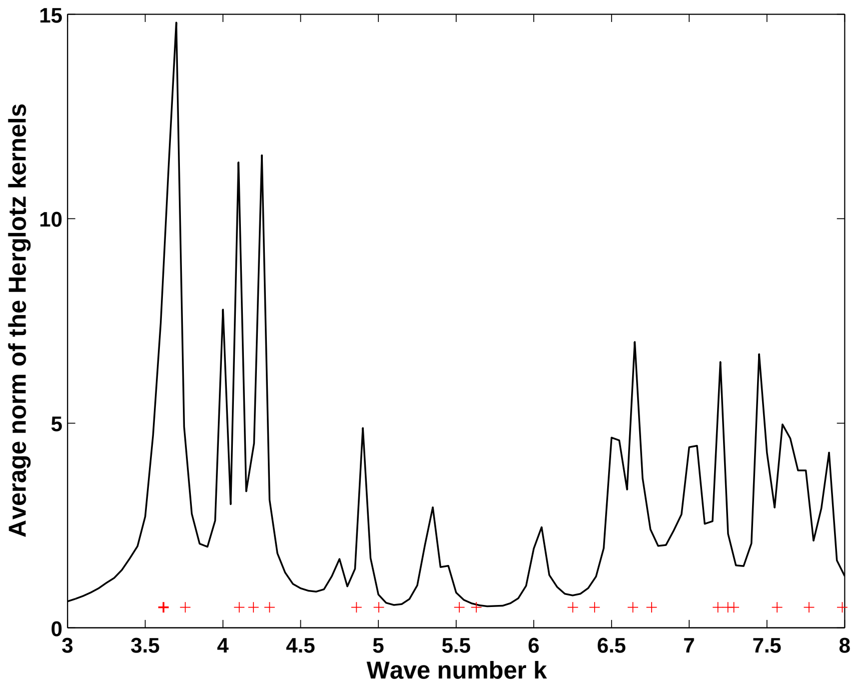
- If k is not a transmission eigenvalue then $\lim\limits_{\delta \to 0} \|v_{g_{z,\delta}}\|_{L^2(D)}$ exists.
- If k is a transmission eigenvalue then for almost every $z \in D$ $\lim\limits_{\delta \to 0} \|v_{g_{z,\delta}}\|_{L^2(D)} = \infty$.

Computation of Transmission Eigenvalues

In particular, transmission eigenvalues can be determined from the far field pattern!



A composite plot of $\|g_{z_i}\|_{L^2(\Omega)}$ against k for 25 random points $z_i \in D$



The average of $\|g_{z_i}\|_{L^2(\Omega)}$ over all choices of $z_i \in D$.

Computation of the transmission eigenvalues for a square D.

Detection of Anomalies in an Isotropic Medium

How does a knowledge of transmission eigenvalues give us information about the index of refraction $n(x)$?
Let $k_{1,n(x)}$ be the first transmission eigenvalue corresponding to $n(x)$ (which as we have just seen can be determined from the measure data).
Consider (for sake of presentation) the case of $n(x) > 1$.

It can be shown that:

- $$k_{1,n(x)} \geq \frac{\lambda_1(D)}{\sup_D n(x)}$$

  where $\lambda_1(D)$ is the first Dirichlet eigenvalue for $-\Delta$ in D.

- If $1 + \alpha \leq n_* \leq n(x) \leq n^* < \infty$ for $x \in \overline{D}$ then

$$0 < k_{1,n^*} \leq k_{1,n(x)} \leq k_{1,n_*}.$$

Detection of Anomalies in an Isotropic Medium

We find the constant n_0 such that the first transmission eigenvalue of

$$\begin{aligned} \Delta w + k^2 n_0^2 w = 0 \quad & \text{in} \quad D \\ \Delta v + k^2 v = 0 \quad & \text{in} \quad D \\ w = v \quad & \text{on} \quad \partial D \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad & \text{on} \quad \partial D \end{aligned}$$

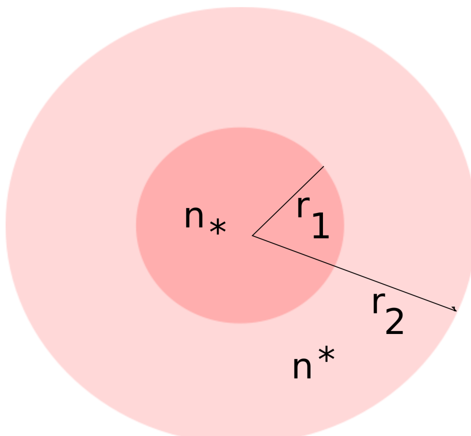
is $k_{1,n(x)}$ (which as we have seen, can be determined from the measure data). This can be done in a simple way since the problem can be reduced to a simple linear eigenvalue problem.

Then from the previous slide we have that $n_* \leq n_0 \leq n^*$ where $n_* = \inf_D(n)$ and $n^* = \sup_D(n)$.

Numerical Example: Inhomogeneous Isotropic Media

We reconstruct $n_0 > 0$ such that

$$n_* \leq n_0 \leq n^*$$



$r_2 = 2$ $n_* = 2$, $n^* = 4$

r_1	n_0
0	4
0.2	3.7
0.7	3.4
1	3.1
1.5	2.7
1.7	2.48
2	2

Detection of Anomalies in an Anisotropic Medium

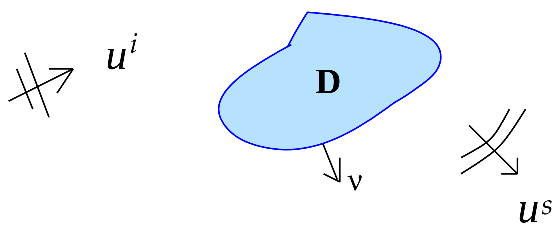
For anisotropic media the (matrix) index of refraction is not uniquely determined from either the near field or the far field data even if such data is collected over an interval of frequencies!

Furthermore, in many cases, e.g. the nondestructive testing of airplanes canopies, what is to be tested is not the presence of cavities/cracks/inhomogeneities but rather a change in the anisotropic structure.

For more details see

Cakoni-Colton-Monk-Sun, The inverse electromagnetic scattering problem for anisotropic media, Inverse Problems, 26, 074004, (2010).

Detection of Anomalies in an Anisotropic Medium



$$\nabla \cdot A \nabla u + k^2 u = 0 \qquad \text{in } \mathbb{R}^2$$

$$u = u^s + u^i \qquad \text{in } \mathbb{R}^2$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

where $u^i(x) = e^{ikx\cdot d}$, $|d| = 1$ and $u \in H^1_{loc}(\mathbb{R}^2)$.
A is a positive, real valued 2×2 matrix whose entries are piecewise continuously differentiable in $\overline{D}$ and $A - I$ has compact support $\overline{D}$.

The scattered field u^s again has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O\left(r^{-3/2}\right)$$

where $u_\infty(\hat{x}, d)$ is the far field pattern of the scattered field u^s.

Detection of Anomalies in an Anisotropic Medium

The far field operator $F : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$(Fg)(\hat{x}) := \int_{\Omega} u_\infty(\hat{x}, d) g(d) ds(d).$$

For $z \in D$ the far field equation is again

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z), \quad g \in L^2(\Omega)$$

where

$$\Phi_\infty(\hat{x}, z) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot z}$$

is the far field pattern of the fundamental solution

$$\Phi(x, z) := \tfrac{i}{4} H_0^{(1)} \left(k|x - z|\right).$$

Detection of Anomalies in an Anisotropic Medium

Transmission eigenvalues are the values of $k >$ such there exists a nontrivial solution $v, w \in H^1(D)$ of

$$
\begin{aligned}
\nabla \cdot A \nabla w + k^2 w = 0 \qquad & \text{in} \quad D \\
\Delta v + k^2 v = 0 \qquad & \text{in} \quad D \\
w = v \qquad & \text{on} \quad \partial D \\
\nu \cdot A \nabla w = \nu \cdot \nabla v \qquad & \text{on} \quad \partial D.
\end{aligned}
$$

Transmission eigenvalues can be determined as before from a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ and in the following the first transmission eigenvalue will be denoted by $k_{1,A(x)}$.

Detection of Anomalies in an Anisotropic Medium

Recall that, in the isotropic case, the first transmission eigenvalue $k_{1,n(x)}$ led to the determination of a constant n_0 such that

$$n_* \leq n_0 \leq n^*$$

$$\text{where} \quad n_* = \inf_D(n) \quad \text{and} \quad n^* = \sup_D(n).$$

In the anisotropic case from a knowledge of $k_{1,A(x)}$ we can compute a constant a_0 such that

$$a_* \leq a_0 \leq a^*$$

where $a_* = \inf_D(a_*(x))$ and $a^* = \sup_D(a^*(x))$
$a_*(x) :=$ smallest eigenvalue of $A^{-1}(x)$
$a^*(x) :=$ largest eigenvalue of $A^{-1}(x)$.

Detection of Anomalies in an Anisotropic Medium

In particular, $a_0 > 0$ is such that the first transmission eigenvalue of

$$\begin{aligned} \Delta w + k^2 a_0 w = 0 \quad & \text{in} \quad D \\ \Delta v + k^2 v = 0 \quad & \text{in} \quad D \\ w = v \quad & \text{on} \quad \partial D \\ \frac{1}{a_0}\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad & \text{on} \quad \partial D. \end{aligned}$$

is $k_{1,A(x)}$, the transmission transmission eigenvalue that is determined from the measured data.

The computation of a_0 can again be done in a simple way by means of a linear eigenvalue problem.

Numerical Examples: Homogeneous Anisotropic Media

We consider D to be the unit square $[-1/2,\ 1/2] \times [-1/2,\ 1/2]$ and

$$A_1^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \quad A_2^{-1} = \begin{pmatrix} 6 & 0 \\ 0 & 8 \end{pmatrix} \quad A_{2r}^{-1} = \begin{pmatrix} 7.4136 & -0.9069 \\ -0.9069 & 6.5834 \end{pmatrix}$$

Matrix	Eigenvalues $n_*,\ n^*$	Predicted n_0
A_1^{-1}	2, 8	5.319
A_2^{-1}	6, 8	7.407
A_{2r}^{-1}	6, 8	6.896