Electromagnetic Imaging of Buried Objects

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joint work with

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Introduction

The inverse scattering problem we are considering is to determine the shape of an obstacle imbedded in a known inhomogeneous background from a knowledge of the scattered field due to the scattering of an incident time-harmonic electromagnetic wave at fixed frequency.

- A solution is needed in real time.
- The scattering obstacle may be either penetrable, a perfect conductor or partially coated but such information is not known a priori.
- Often only partial information on the scatterer is needed.

Piecewise Homogeneous Background

The time harmonic electric field $\mathcal{E}$ and magnetic field $\mathcal{H}$ with frequency ω satisfy

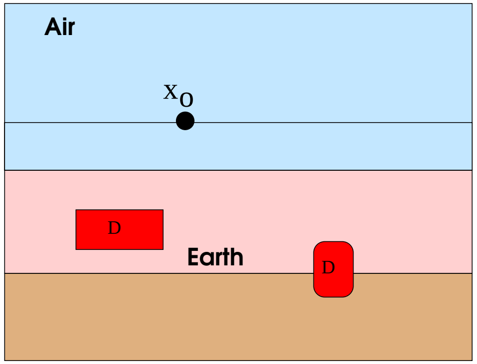
$$\begin{cases} \nabla \times \mathcal{E} - i\omega\mu_0\mathcal{H} = 0 \\ \nabla \times \mathcal{H} + (i\omega\epsilon(x) - \sigma(x))\,\mathcal{H} = 0 \end{cases}$$

- Magnetic permeability μ_0 is a positive constant.
- Electric permitivity $\epsilon(x)$ and conductivity $\sigma(x)$ are piecewise (positive) constants .
  In the air we have $\epsilon(x) = \epsilon_0$ and $\sigma(x) = 0$.

Denote by $\mathcal{E} = 1/\sqrt{\epsilon_0}E$, $\mathcal{H} = 1/\sqrt{\mu_0}H$, $k^2 = \epsilon_0\mu_0\omega^2$ and

$$n_b(x) = \frac{1}{\epsilon_0}\left(\epsilon(x) + i\frac{\sigma(x)}{\omega}\right)$$

The Scattering Problem

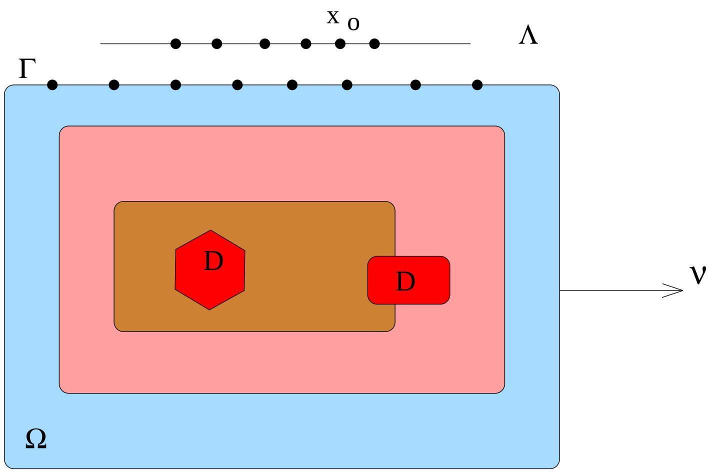


The total electric field $E = E^s + E^i$ satisfies

1. $\nabla \times \nabla \times E - k^2 n_b(x) E = 0$ in $\mathbb{R}^3 \setminus \overline{D} \cup \{x_0\}$
2. a boundary condition on ∂D
3. E decays appropriately as $|x| \to \infty$.

- Incident field $E^i = E_e(\cdot\,; x_0, p) + E_b^i(\cdot) = \mathbb{G}(\cdot, x_0)p$ where $E_e(\cdot\,; x_0, p)$ is the electric dipole located at x_0 with polarization p and E_b^i is the scattered field due to the background. $\mathbb{G}(\cdot, x_0)p$ is known as the Green's function of the background.
- $E = E^s + E^i$ where $E^s := (\cdot\, x_0, p)$ is the scattered field due to the buried objects D.

Inverse Scattering Problem



The inverse scattering problem is:

Determine D from a knowledge of $\nu \times E(x, x_0, p)$ or/and $\nu \times H(x, x_0, p)$ for $x \in \Gamma$, $x_0 \in \Lambda$, and two linearly independent polarizations p tangential to Λ at x_0.

The Linear Sampling Method

Find a solution $\varphi := \varphi^z \in L^2_t(\Lambda)$ to the near field equation

$$\mathcal{F}\varphi(x) := \int_\Lambda \nu(x) \times E^s(x, y, \varphi_z(y))ds(y) = \nu(x) \times \mathbb{G}(\cdot, z)q$$

$x \in \Gamma$, $z \in \Omega$ and $q \in \mathbb{R}^3$ is an artificial polarization.

By superposition $\mathcal{F}\varphi$ is the tangential component on $\partial\Omega$ of the scattered field corresponding to the potential

$$S\varphi(x) = \int_\Lambda \varphi(y)\mathbb{G}(x, y)ds(y)$$

as incident wave.

The Linear Sampling Method

Theorem: For every $\epsilon > 0$, there exists an approximate solution φ_ϵ^z satisfying

$$\|\mathcal{F}\varphi_\epsilon^z - \nu \times \mathbb{G}(\cdot, z)q\|_{L_t^2(\Gamma)} < \epsilon \qquad z \in \Omega$$

that behaves as follows:

- For $z \in D$, $\lim\limits_{\epsilon \to 0} \|S\varphi_\epsilon^z\|_{H(D,curl)} < \infty$.
- For each $\epsilon > 0$, $\lim\limits_{z \to \partial D} \|S\varphi_\epsilon^z\|_{H(D,curl)} = \infty$ and $\lim\limits_{z \to \partial D} \|\varphi_\epsilon^z\|_{L_t^2(\Lambda)} = \infty$.
- For $z \in \Omega \setminus \overline{D}$, $\lim\limits_{\epsilon \to 0} \|S\varphi_\epsilon^z\|_{H(D,curl)} = \infty$ and $\lim\limits_{z \to \partial D} \|\varphi_\epsilon^z\|_{L_t^2(\Lambda)} = \infty$.

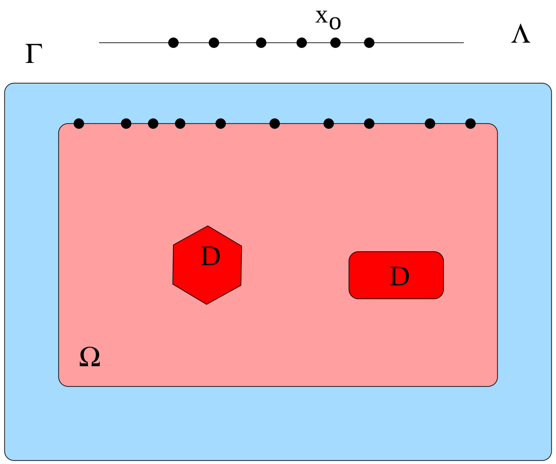
Linear Sampling Method

The linear sampling method determines φ from the near field equation $\mathcal{F}\varphi_\epsilon^z = \nu \times \mathbb{G}(\cdot, z)q$.

The support D can be determined by the behavior of φ. In particular, $\|S\varphi_\epsilon^z\|_{H(D,curl)} \to \infty$ implies $\|\varphi\|_{L^2(\Lambda)} \to \infty$.

Open Problem: In practice is obtained by using a regularization method such as Tikhonov regularization. Does this regularized solution behave in the same way as the approximate solution φ whose existence is given by the previous theorem?

Avoiding the need for Green's function



Assuming that

- the medium inside the surrounding box Ω is homogeneous.

- we can measure both $\nu \times E(x, x_0, p)$ and $\nu \times H(x, x_0, p)$ for $x \in \Gamma$, $x_0 \in \Lambda$, and two linearly independent polarizations p tangential to Λ at x_0.

We set $k_b^2 = k^2 n_b$ where n_b is the constant index of refraction of the medium inside Ω .

Reciprocity Gap Fuctional

Let $\mathbb{H} = \{W \in H(curl, \Omega) : \nabla \times \nabla \times W - k_b^2 W = 0\}$.

If $E(x, x_0, p)$ and $H(x, x_0, p) = \frac{1}{ik_b}\nabla_x \times E(x, x_0, p)$ is the total field then the reciprocity gap functional $\mathcal{R}(E, \cdot) : \mathbb{H} \to L_t^2(\Lambda)$ is defined by

$$\mathcal{R}(E, W) = \int_\Gamma \left[(\nu \times E) \cdot (\nabla \times W) - (\nu \times W) \cdot (\nabla \times E)\right] ds$$

Instead of the whole space $\mathbb{H}$, it suffices to consider a dense subset .

Reciprocity Gap Equation

We take $W = A\phi$ with $A\phi$ being the single layer potential given by

$$(A\phi)(x) = \nabla \times \nabla \times \int_\Gamma \phi(y) \frac{e^{ik_b|x-y|}}{4\pi|x-y|} ds(y), \qquad \phi \in L^2_{div}(\Gamma).$$

In this case $\mathcal{R}(E,\, A\phi) = \mathcal{R}\phi : \phi \in L^2_{div}(\Gamma) \to L^2_t(\Lambda)$.

We then look for a solution ϕ to the equation

$$\boxed{\mathcal{R}\phi = \mathcal{R}(E,\, E_e(\cdot\, ; z, q, k_b)) \qquad z \in \Omega}$$

where $E_e(x; z, q, k_b) = \frac{i}{k_s} \nabla_x \times \nabla_x \times q\, \frac{e^{ik_s|x-z|}}{4\pi|x-z|}$ is the electric dipole located at $z \in \Omega$.

Solving the Inverse Scattering Problem

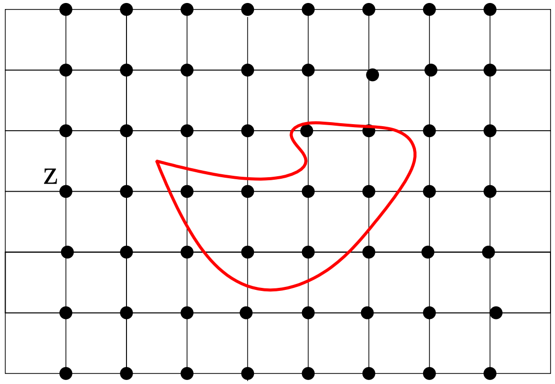
Theorem: For every $\epsilon > 0$, there exists an approximate solution ϕ_ϵ^z satisfying

$$\|\mathcal{R}\phi_\epsilon^z - \mathcal{R}(E,\ E_e(\cdot\,; z, q, k_b))\|_{L^2(\Lambda)} < \epsilon \qquad z \in \Omega$$

that behaves as follows:

- For $z \in D$, $\lim\limits_{\epsilon \to 0} \|A\phi_\epsilon^z\|_{H(D,curl)} < \infty$.
- For each $\epsilon > 0$,
  $\lim\limits_{z \to \partial D} \|A\phi_\epsilon^z\|_{H(D,curl)} = \infty$ and $\lim\limits_{z \to \partial D} \|\phi_\epsilon^z\|_{L^2_{div}(\Gamma)} = \infty$.
- For $z \in \Omega \setminus \overline{D}$
  $\lim\limits_{\epsilon \to 0} \|A\phi_\epsilon^z\|_{H(D,curl)} = \infty$ and $\lim\limits_{z \to \partial D} \|\phi_\epsilon^z\|_{L^2_{div}(\Gamma)} = \infty$.

Numerical Implementation



- Construct a grid $\mathcal{G}$.
- For $z_i \in \mathcal{G}$, solve the regularized equation

$$(\alpha I + \mathcal{R}^*\mathcal{R}\phi_q^z) = F_{z,q}$$

- Evaluate

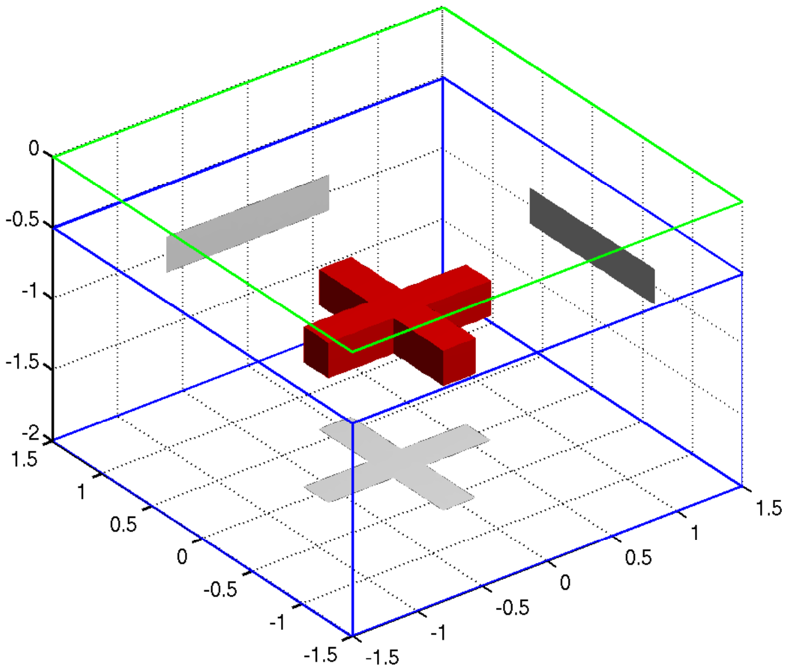
$$G(z_i) = \frac{1}{3}\left(\|\phi_{q_1}^{z_i}\|_{\ell^2}^{-1} + \|\phi_{q_2}^{z_i}\|_{\ell^2}^{-1} + \|\phi_{q_1}^{z_i}\|_{\ell^2}^{-1}\right)$$

for $z_i \in \mathcal{G}$ and three linearly independent vectors $q_1, q_2, q_3 \in \mathbb{R}^3$.

- Fix $C > 0$ and visualize the boundary by plotting

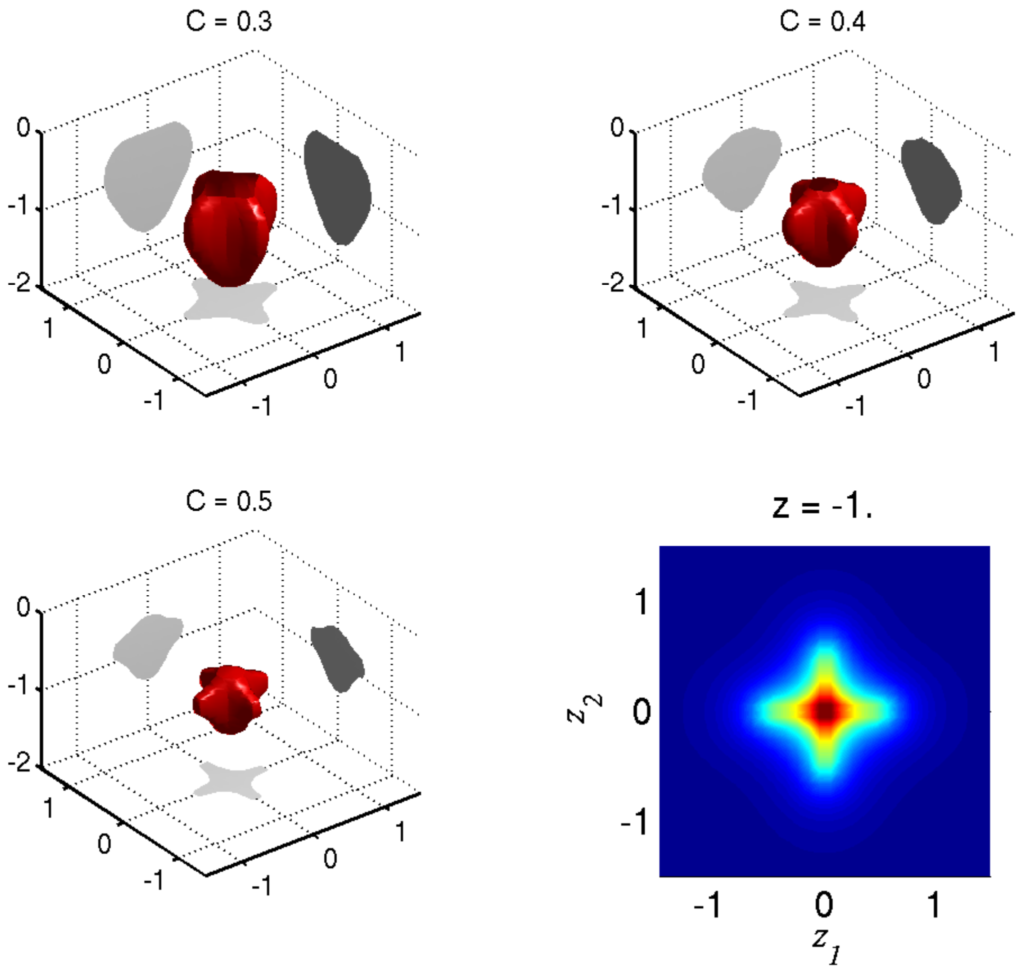
$$\mathcal{G}(z) = C \max_{z_i \in \mathcal{G}} G(z_i).$$

Numerical Examples



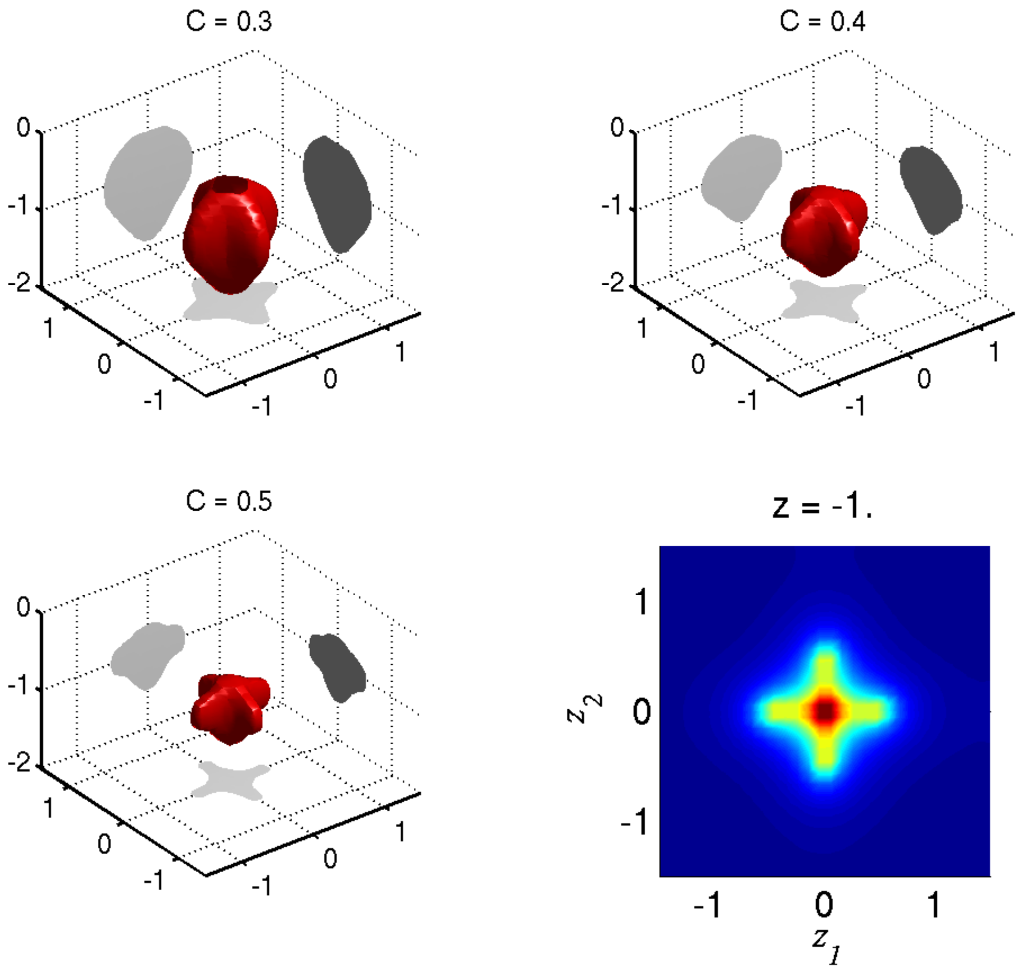
Example of a perfectly conducting cross.
The interface earth-air is at $z = 0$. The reconstructions correspond to $n = 2 + 0.5$ and 5% random noise.

Numerical Examples



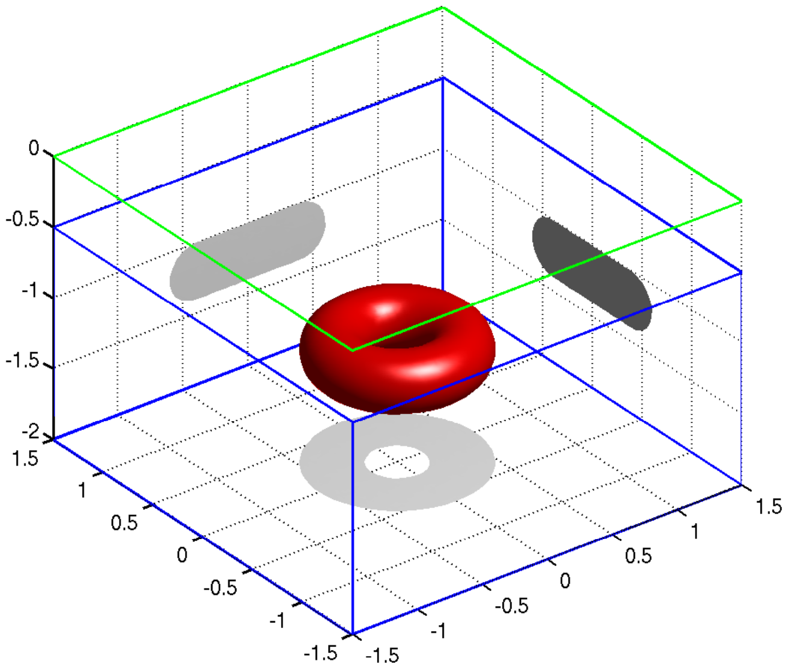
Reconstruction by using the Linear Sampling Method

Numerical Examples



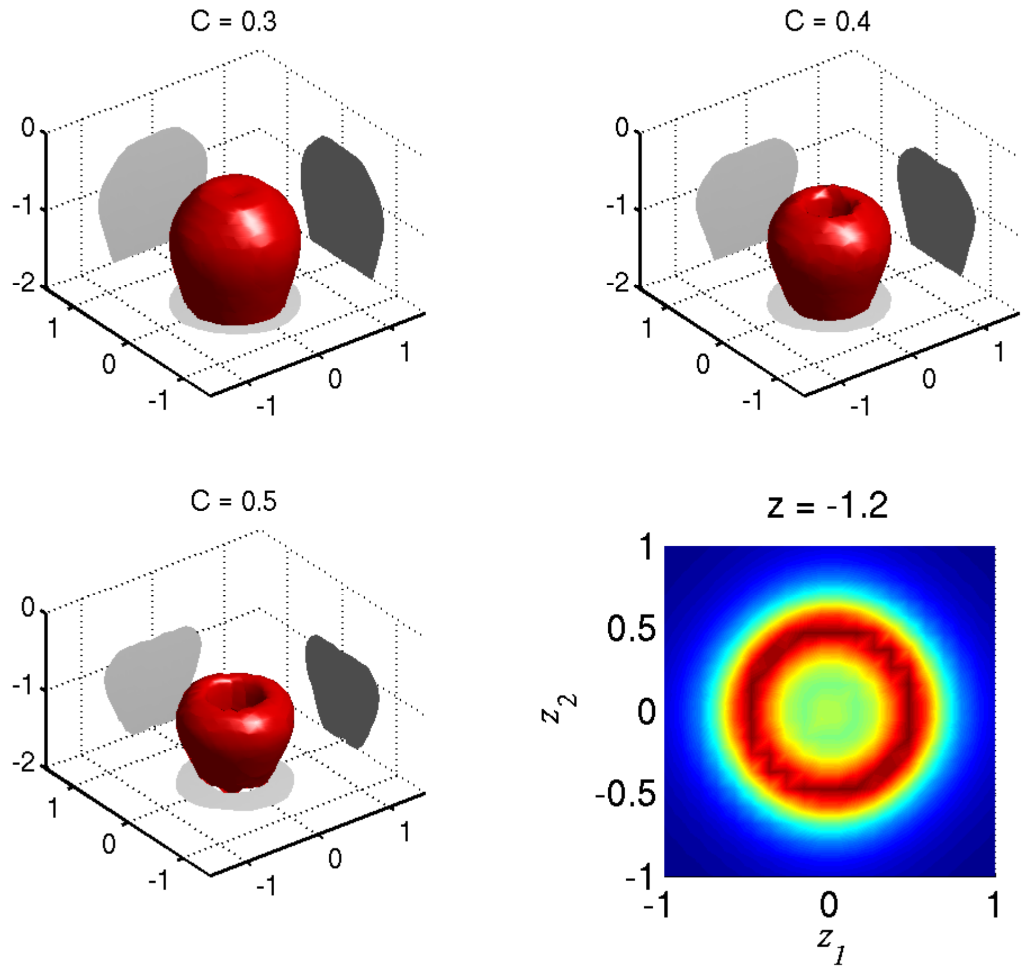
Reconstruction by using the Reciprocity Gap Functional

Numerical Examples



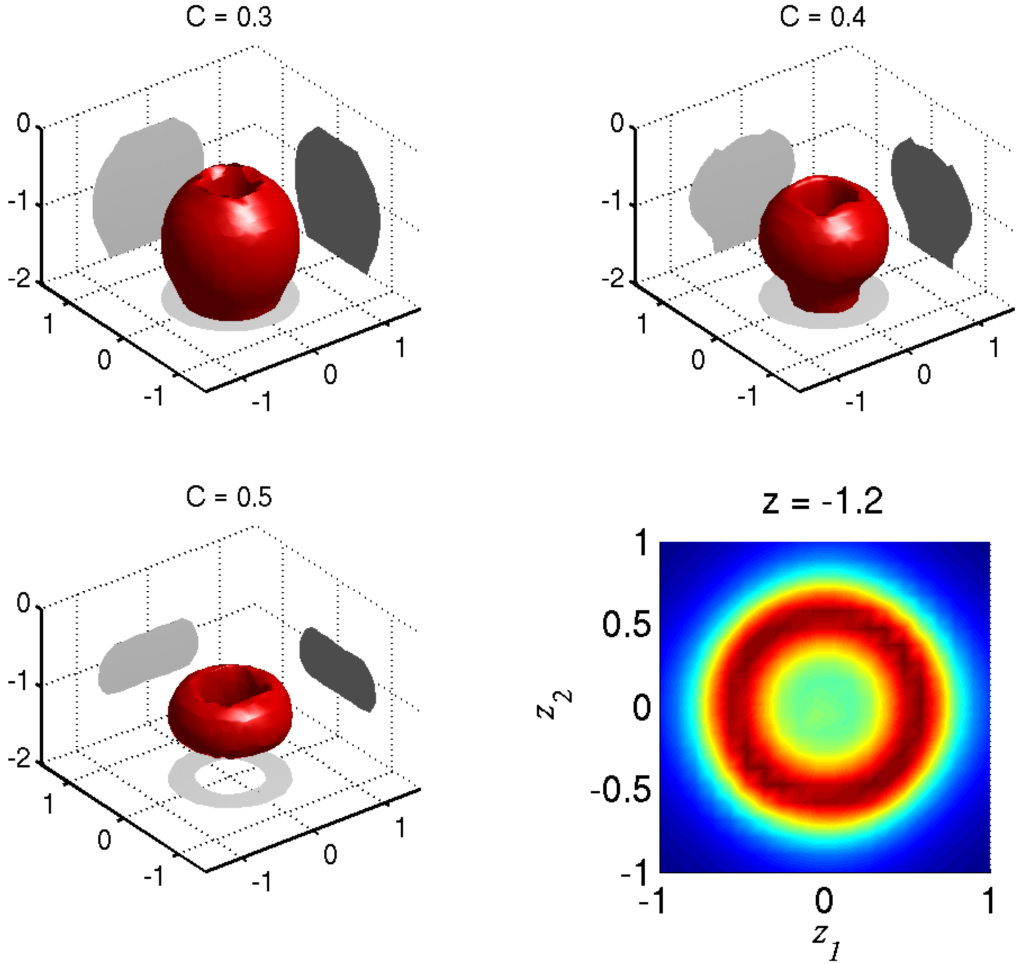
Example of a perfectly conducting torus.
The interface earth-air is at $z = 0$. The reconstructions correspond to $n = 2 + 0.5$ and 5% random noise.

Numerical Examples



Reconstruction by using the Linear Sampling Method

Numerical Examples



Reconstruction by using the Reciprocity Gap Functional