Identification of buried objects using electromagnetic Cauchy data

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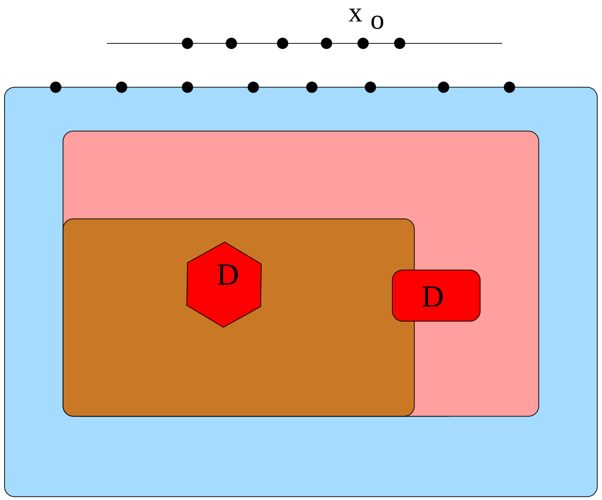
Introduction

The inverse scattering problem we are considering is to determine the shape and some of physical properties of an obstacle buried in a known inhomogeneous background from a knowledge of the tangential component of electric and magnetic total field on a given surface due to time harmonic dipoles as incident fields.



The Scattering Problem

- In the background medium the electric field satisfies

$$\nabla \times \nabla \times \mathcal{E} - k^2 n(x)\mathcal{E} = 0$$

where $k^2 = \epsilon_0 \mu_0 \omega^2$ and $n(x) = \frac{\epsilon(x)}{\epsilon_0} + i\frac{\sigma(x)}{\omega\epsilon_0}$ is a piecewise constant function s.th. $\mathcal{R}e(n) > 0$ and $\mathcal{I}m(n) \geq 0$.

- The scatterer is the support D of an anisotropic object with index of refraction given by a symmetric matrix-valued function $N(x)$, $x \in \overline{D}$ with bounded entries s.th.

$$\bar{\xi} \cdot \mathcal{I}m(N)\,\xi \geq 0 \ \text{ and } \ \bar{\xi} \cdot \mathcal{R}e(N)\,\xi \geq \gamma|\xi|^2 \text{ for all } \xi \in \mathbb{C}^3.$$

On the part Γ_2 of the boundary $\partial D = \overline{\Gamma_1} \cup \overline{\Gamma_2}$, the scatterer may be coated by a thin layer of highly conductive material with surface conductivity given by the bounded function $\eta > 0$.

The Scattering Problem

We define $E^i(x) = \mathbb{G}(x, x_0)p$ with $\mathbb{G}(x, x_0)$ the Green's tensor of the background. Then the exterior total electric field $E = E^s + E^i$ and interior electric field E^{int} satisfy

$$\begin{aligned}
\nabla \times \nabla \times E^{int} - k^2 N(x) E^{int} &= 0 && \text{in } D, \\
\nabla \times \nabla \times E - k^2 n(x) E &= p\,\delta(x - x_0) && \text{in } \mathbb{R}^3 \setminus \overline{D} \\
\nu \times E - \nu \times E^{int} &= 0 && \text{on } \partial D \\
\nu \times (\nabla \times E) - \nu \times (\nabla \times E^{int}) &= 0 && \text{on } \Gamma_1 \\
\nu \times (\nabla \times E) - \nu \times (\nabla \times E^{int}) &= ik\eta(\nu \times E) \times \nu && \text{on } \Gamma_2.
\end{aligned}$$

E^s is an outgoing radiating field.

The Scattering Problem

- If $\eta = 0$ we have the classical transmission problem.
- If $\eta = \infty$ and $\Gamma_1 = \emptyset$ we obtain the boundary value problem for a perfect conductor.

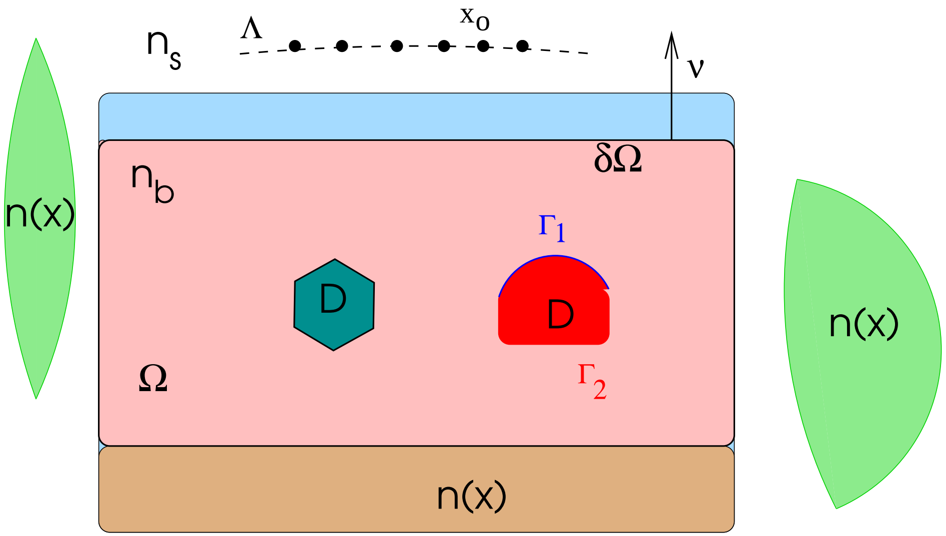
Note that $E^i(x) = E_e(x, x_0, p, k_s) + E_b^s(x)$ where

$$E_e(x, x_0, p, k_s) := \frac{i}{k_s} \nabla_x \times \nabla_x \times p \, \frac{e^{ik_s|x-x_0|}}{4\pi|x-x_0|}$$

is the electric field of the dipole located at x_0 with polarization p and $k_s^2 = k^2 n_s$, $n_s = n(x_0)$,

and E_b^s is the scattered field due to the background medium.

Inverse Scattering Problem



The inverse scattering problem is:

Determine D and η from a knowledge of $\nu \times E(x, x_0, p)$ and $ik\nu \times H(x, x_0, p) = \nu \times (\nabla \times E(x, x_0, p))$ for $x \in \partial\Omega$, $x_0 \in \Lambda$, and two linearly independent polarizations p tangential to Λ at x_0.

The Reciprocity Gap Operator

The measured data defines the reciprocity gap operator $R : \mathbb{H}(\Omega) \to L^2_t(\Lambda)$ by

$$R(W)(x_0) = \mathcal{R}(E(\cdot, x_0, p(x_0)), W)p(x_0)$$

where

$$\mathcal{R}(E, W) := \int_{\partial\Omega} (\nu \times E) \cdot (\nabla \times W) - (\nu \times W) \cdot (\nabla \times E)\, ds$$

and

$$W \in \mathbb{H}(\Omega) := \left\{W \in H(curl, \Omega) : \; \nabla \times \nabla \times W - k_b^2 W = 0 \,\text{in}\, \Omega\right\}$$

with $k_b^2 = k^2 n_b$.

A Simple Idea

For $z \in D$ we observe that W satisfies

$$R(W)(x_0) = \mathcal{R}(E(\cdot, x_0, p(x_0)), E_e(\cdot, z, q, k_b))p(x_0), \quad \text{for all } x_0 \in \Lambda$$

iff $W = E^0$ where (E^0, E^{int}) solves the interior transmission problem

$$\begin{aligned}
\nabla \times \nabla \times E^{int} - k^2 N(x) E^{int} &= 0 && \text{in } D \\
\nabla \times \nabla \times E^0 - k^2 n_b E^0 &= 0 && \text{in } D \\
\nu \times E^0 - \nu \times E^{int} &= E_e(\cdot, z, q, k_b) && \text{on } \partial D \\
\nu \times (\nabla \times E^0) - \nu \times (\nabla \times E^{int}) &= \nu \times (\nabla \times E_e(\cdot, z, q, k_b)) && \text{on } \Gamma_1 \\
\nu \times (\nabla \times E^0) - \nu \times (\nabla \times E^{int}) &= \nu \times (\nabla \times E_e(\cdot, z, q, k_b)) && \\
&\quad + ik\eta(\nu \times (E^0 - E_e(\cdot, z, q, k_b)) \times \nu && \text{on } \Gamma_2
\end{aligned}$$

If E^0 depends continuously on $E_e(\cdot, z, q, k_b)$ then $\lim\limits_{z \to \partial D} \|W\|_D \to \infty$

Interior Transmission Problem

Theorem (Cakoni-Haddar) *Provided that the uniqueness holds, the interior transmission problem has a solution* $E^{int} \in L^2(D)$*,* $E^0 \in L^2(D)$ *such that* $\nu \times E^0|_{\Gamma_2} \in L^2_t(\Gamma_2)$*, and this solution depends continuously on the data* $E_e(\cdot, z, q, k_b)$ *in the respective norms.*

- If $\eta = 0$ see H. HADDAR, *Math Meth. Appl. Sci* 27, 2004, 2111-2129.
- if $\eta = \infty$ and $\Gamma_1 = \emptyset$ the interior transmission problem becomes the interior Dirichlet problem for Maxwell's equation.
- For a survey on the interior transmission problem see D.COLTON, L. PÄIVÄRINTA AND J. SYLVESTER (to appear).

The values of k for which the uniqueness of the interior transmission problem does not hold are called interior transmission eigenvalues.

The Reciprocity Gap Operator

Note that E^0 is not in $\mathbb{H}(\Omega)$.

To overcome this issue we consider a parametric family of functions in $\mathbb{H}(\Omega)$ which form a dense subset of

$$H_{inc}(D, \Gamma_2) = \{u \in L^2(D),\ \nu \times u \in L^2_t(\Gamma_2),\ \nabla \times \nabla \times u - k_b^2 u = 0\}$$

A example of such parametric family is

$$(A\varphi)(x) := \nabla_x \times \nabla_x \times \int_{\mathcal{M}} \varphi(y)\Phi(x, y, k_b)\, ds$$

where $\mathcal{M}$ is a part of the analytic boundary of some domain containing Ω and $\Phi(x, y, k_b) = \frac{e^{ik_s|x-y|}}{4\pi|x-y|}$.

Sampling Integral Equation

Our sampling algorithm now consists in seeking for each sampling point z an approximate solution $\varphi_z \in L^2(\mathcal{M})$ of the ill-posed first kind integral equation

$$\mathcal{S}\varphi_z = \ell_z \quad \text{in} \quad L^2_t(\Lambda)$$

where

$$\mathcal{S}\varphi = RA\varphi \quad \text{and} \quad \ell_z := \mathcal{R}(E(\cdot, x_0, p(x_0)), E_e(\cdot, z, q, k_b))p(x_0)$$

where q is an artificial polarization.

Lemma: *The operator* $\mathcal{S} : L^2(\mathcal{M}) \to L^2_t(\Lambda)$ *is compact, injective and has dense range provided that* k *is not a transmission eigenvalue*

Solving the Inverse Problem

Theorem: Assume that k is not a transmission eigenvalue. Then

- For $z \in D$ and every $\epsilon > 0$, there exists a $\varphi_z^\epsilon \in L^2(\mathcal{M})$ satisfying

$$\|\mathcal{S}\varphi_z^\epsilon - \ell_z\|_{L_t^2(\Lambda)} < \epsilon$$

  such that $\lim\limits_{\epsilon \to 0} \|A\varphi_z^\epsilon - E^0\|_{H_{inc}(D,\Gamma_2)} = 0$. For a fixed $\epsilon > 0$

$$\lim_{z \to \partial D} \|A\varphi_z^\epsilon\|_{H_{inc}(D,\Gamma_2)} = \infty \text{ and } \lim_{z \to \partial D} \|\varphi_z^\epsilon\|_{L^2(\mathcal{M})} = \infty.$$

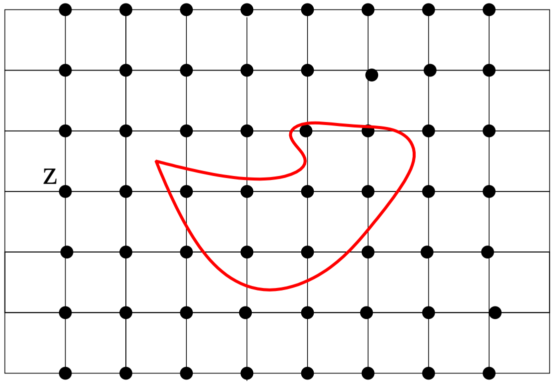
- For $z \in \Omega \setminus \overline{D}$, every $\varphi_z^\epsilon \in L^2(\mathcal{M})$ satisfying

$$\|\mathcal{S}\varphi_z^\epsilon - \ell_z\|_{L_t^2(\Lambda)} < \epsilon \quad \text{is such that}$$

$$\lim_{\epsilon \to 0} \|A\varphi_z^\epsilon\|_{H_{inc}(D,\Gamma_2)} = \infty \text{ and } \lim_{\epsilon \to 0} \|\varphi_z^\epsilon\|_{L^2(\mathcal{M})} = \infty.$$

Determination of D

D can be determined from the above behaviour of φ_z.



- Construct a grid $\mathcal{G}$.
- For $z_i \in \mathcal{G}$, solve the regularized equation

$$(\alpha I + \mathcal{S}^*\mathcal{S})\varphi_{z_i,q} = \ell_{z_i,q}$$

- Evaluate

$$\Phi(z_i) = \frac{1}{3}\left(\|\varphi_{z_i,q_1}\|_{\ell^2}^{-1} + \|\varphi_{z_i,q_2}\|_{\ell^2}^{-1} + \|\varphi_{z_i,q_3}\|_{\ell^2}^{-1}\right), z_i \in \mathcal{G}, q_i \in \mathbb{R}$$

- Visualize the boundary by plotting the isoclines of

$$\Phi(z) = C \max_{z_i \in \mathcal{G}} \Phi(z_i) \quad \text{for a fixed } C > 0.$$

Determination of η

If $\mathcal{I}m(N) = 0$ by using the Green's formulas we obtain that

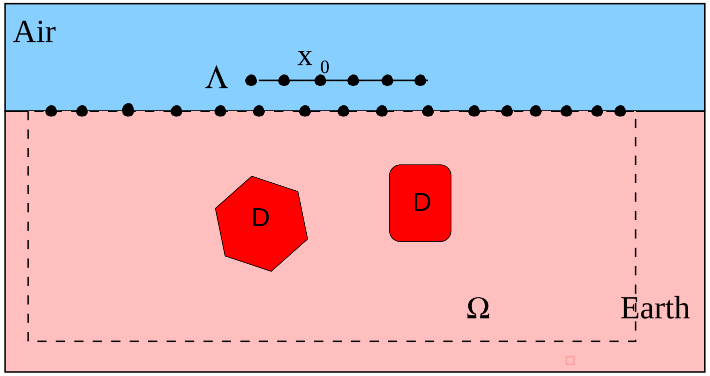
$$\int_{\Gamma_2} \eta \left|\nu \times \left(E^0 - E_e(\cdot, z, q, k_b)\right)\right|^2 ds = A(z, \Omega, k_b, q) - \mathcal{R}e(\sqrt{n_b} q \cdot E^0$$

$$+k\mathcal{I}m\,(n_b) \left(\int_{\Omega\setminus\overline{D}} |E_e(\cdot, z, q, k_b)|^2 \; dy + \int_D \left|E_z^0\right|^2 dy\right)$$

where $z \in D$ and (E^0, E^{int}) is the solution of the interior transmission problem with boundary data coming from $E_e(\cdot, z, q, k_b)$.

Recalling that E^0 can be approximated in the $L^2(D) \times L_t^2(\Gamma_2)$-norm by $A\varphi_z$ where φ_z is the regularized solution of $\mathcal{S}\varphi_z = \ell_z$ used to determine D, this equation gives information about the surface conductivity η.

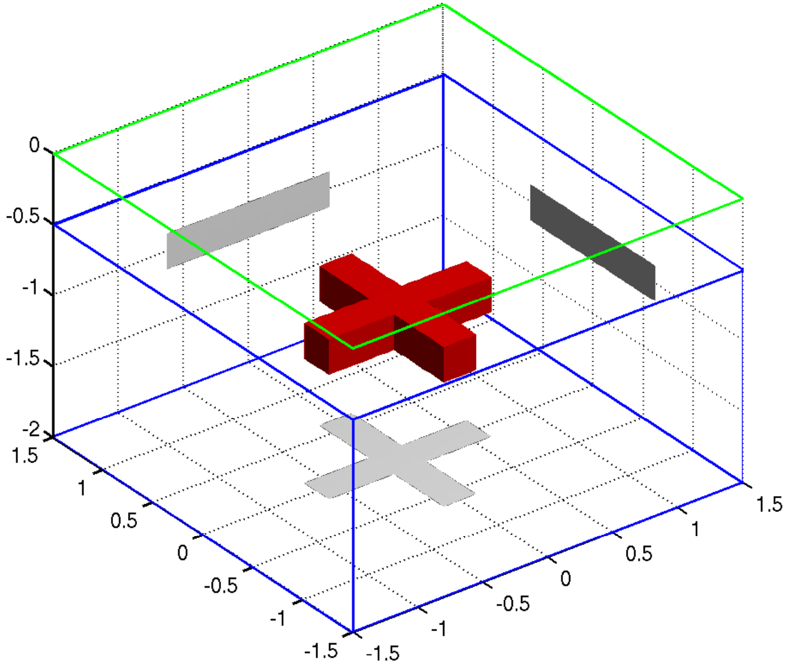
Application to Buried Objects



We assume

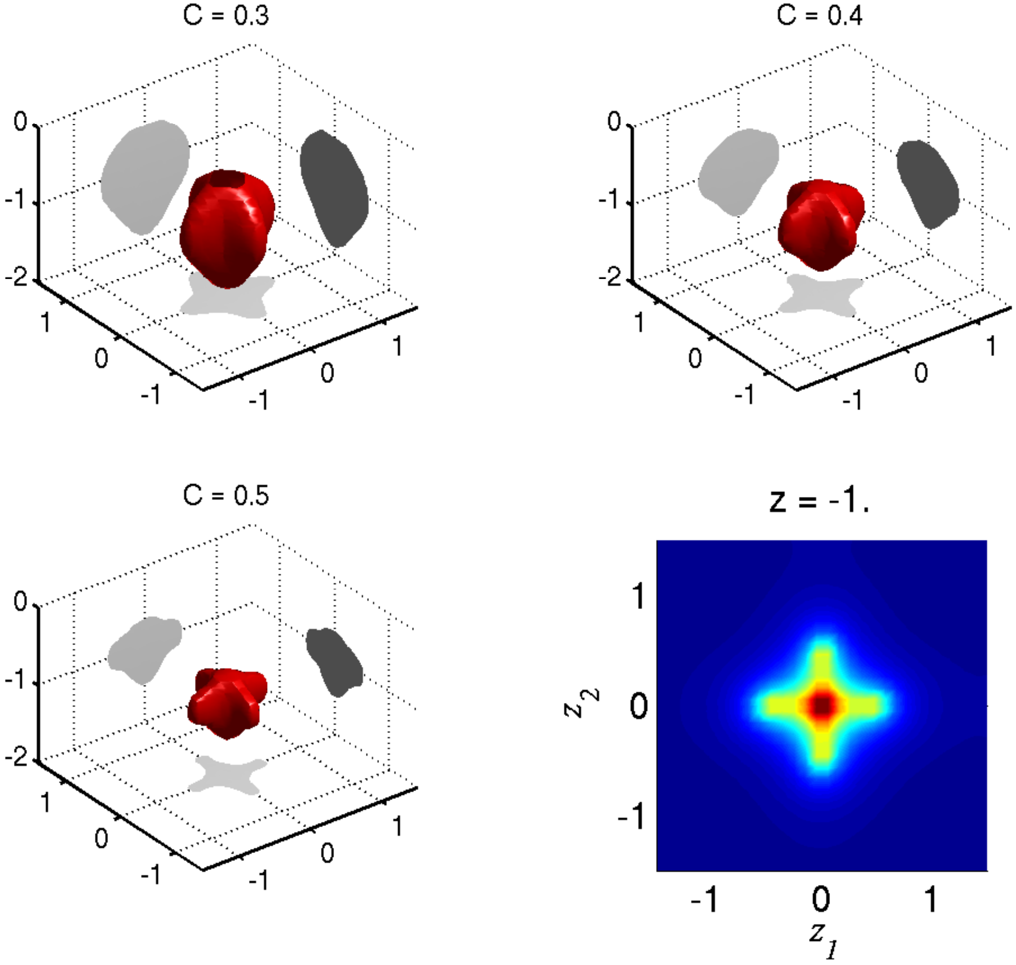
- The background medium inside Ω is absorbing, i.e. $\mathcal{I}m(n_b) > 0$ considerably.
- The box Ω containing the scatterer is big enough so that the respond of the dipoles located on Λ from the sides and the bottom is negligible.

Numerical Examples



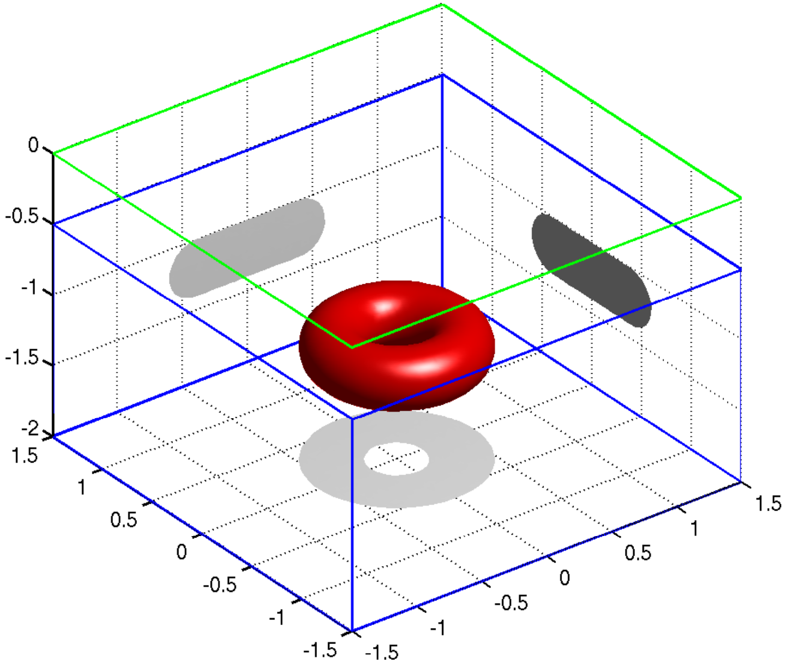
Example of a perfectly conducting cross.
The interface earth-air is at $z = 0$. The reconstructions correspond to $n = 2 + 0.5i$ and 5% random noise.

Numerical Examples



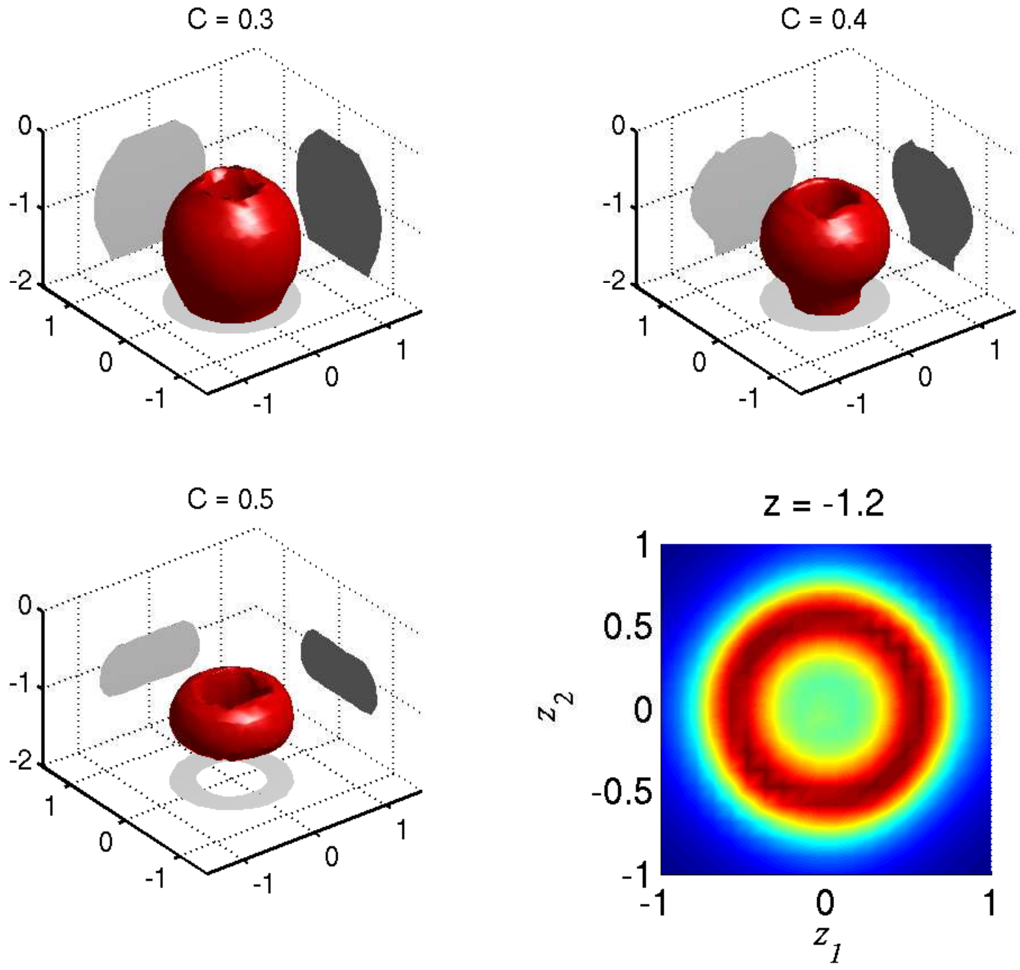
Reconstruction by using the Reciprocity Gap Functional

Numerical Examples



Example of a perfectly conducting torus.
The interface earth-air is at $z = 0$. The reconstructions correspond to $n = 2 + 0.5i$ and 5% random noise.

Numerical Examples



Reconstruction by using the Reciprocity Gap Functional

References

The method is introduced in

- D. COLTON AND H. HADDAR, *Inverse Problems*, 21 (2005), 383-398.
- F. CAKONI, M.B. FARES AND H. HADDAR, *Inverse Problems*, 22 (2006), 845-867.

The case of partially coated obstacles with determination of surface parameters is developed in

- F. CAKONI AND D. COLTON, *J. Applied and Computational Math.* (to appear).
- F. CAKONI AND H. HADDAR, *J. Integral Equations and Applications* (to appear).
- D.COLTON, T. HUTTUNEN AND P. MONK, (in preparation).