

Transmission Eigenvalues and Inverse Scattering Theory

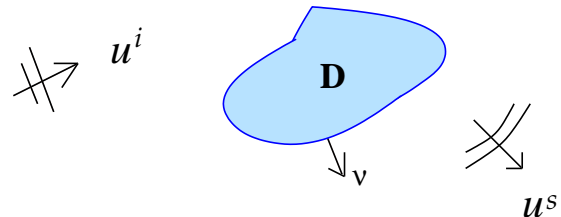
Fioralba Cakoni

colton@math.udel.edu

Department of Mathematical Sciences,
University of Delaware, USA

Research supported by AFOSR

Scattering by an Isotropic Medium



$$\Delta u + k^2 n(x) u = 0 \quad \text{in } \mathbb{R}^3$$

$$u = u^s + u^i \quad \text{in } \mathbb{R}^3$$

$$\lim_{r \rightarrow \infty} r \left(\frac{\partial u^s}{\partial r} - i k u^s \right) = 0$$

where $u^i(x) := e^{i k x \cdot d}$, $|d| = 1$ and $u \in H_{loc}^1(\mathbb{R}^3)$.

We assume that $n - 1$ has compact support \overline{D} , $n(x) > 0$ for $x \in \overline{D}$ and n is piecewise continuous. The **scattered field** u^s has the asymptotic behavior

$$u^s(x) = \frac{e^{i k r}}{r} u_\infty(\hat{x}, d) + O(r^{-2})$$

as $r \rightarrow \infty$ where $\hat{x} = x/|x|$, $r = |x|$ and $k > 0$ is the wave number.

$u_\infty(\hat{x}, d)$ is the **far field pattern** of the scattered field u^s .

The Far Field Operator

Let $\Omega := \{x : |x| = 1\}$ and define the **far field operator**
 $F : L^2(\Omega) \rightarrow L^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_{\Omega} u_{\infty}(\hat{x}, d)g(d)ds(d).$$

For $z \in D$ the **far field equation** is

$$(Fg)(\hat{x}) = \Phi_{\infty}(\hat{x}, z), \quad g \in L^2(\Omega)$$

where

$$\Phi_{\infty}(\hat{x}, z) = \frac{1}{4\pi} e^{-ik\hat{x} \cdot z}$$

is the far field pattern of the fundamental solution

$$\Phi(x, z) := \frac{e^{ik|x-z|}}{4\pi|x-z|}.$$

Transmission Eigenvalue Problem

Question: Is there an incident wave u^i that does not scatter? (this question is also related to the injectivity of the far field operator)

If there exists a nontrivial solution to the **homogeneous interior transmission problem**

$$\begin{aligned}\Delta w + k^2 n(x)w &= 0 && \text{in } D \\ \Delta v + k^2 v &= 0 && \text{in } D \\ w &= v && \text{on } \partial D \\ \frac{\partial w}{\partial \nu} &= \frac{\partial v}{\partial \nu} && \text{on } \partial D\end{aligned}$$

such that v can be extended outside D as a solution to the Helmholtz equation \tilde{v} , then the scattered field due to \tilde{v} as incident wave is identically zero.

Transmission Eigenvalues

Definition: $k > 0$ is a **transmission eigenvalue** if there exists a nontrivial solution $v \in L^2(D)$, $w \in L^2(D)$, $v - w \in H_0^2(D)$ of the **interior transmission problem**

$$\begin{aligned}\Delta w + k^2 n(x)w &= 0 && \text{in } D \\ \Delta v + k^2 v &= 0 && \text{in } D \\ w &= v && \text{on } \partial D \\ \frac{\partial w}{\partial \nu} &= \frac{\partial v}{\partial \nu} && \text{on } \partial D\end{aligned}$$

Remark: Note that if $n = 1$ the interior transmission problem is degenerate.

Transmission Eigenvalues

Theorem: Assume $|m(x)| > 0$ for $x \in \overline{D}$ where $m := 1 - n$. Then transmission eigenvalues exist and form a discrete set whose only accumulation point is infinity.

Proof: Päivärinta-Sylvester (SIMA 2008), Kirsch (IPI 2009), Cakoni-Gintides-Haddar (SIMA 2010), Cakoni-Kirsch (IJCSM to appear).

Definition: A solution of the Helmholtz equation of the form

$$v_g(x) := \int_{\Omega} e^{ikx \cdot d} g(d) ds(d), \quad g \in L^2(\Omega)$$

is called a **Herglotz wave function** with **kernel** g .

Transmission Eigenvalues

Theorem: Assume $|m| > 0$ for $x \in \overline{D}$ where $m := 1 - n$. Let $u_\infty^\delta(\hat{x}, d)$ be the measured "noisy" far field pattern and for $z \in D$ let $g_{z,\delta}$ be the **Tikhonov regularized solution** of the far field equation

$$(F^\delta g)(\hat{x}) = \Phi_\infty(\hat{x}, z).$$

- If k is not a transmission eigenvalue then

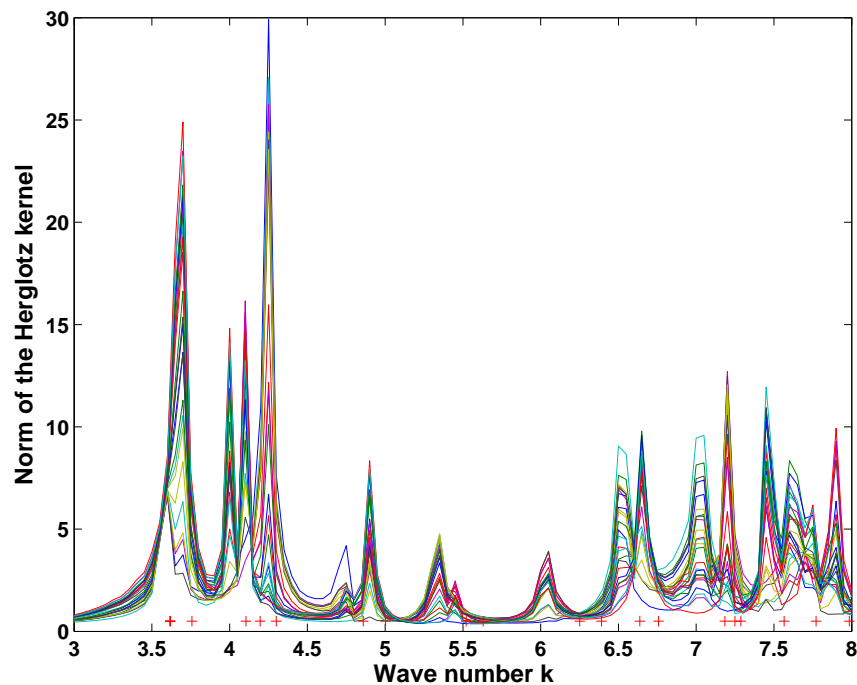
$$\lim_{\delta \rightarrow 0} \|v_{g_{z,\delta}}\|_{L^2(D)} \quad \text{exists.}$$

- If k is a transmission eigenvalue and the far field operator F has dense range then for almost every $z \in D$

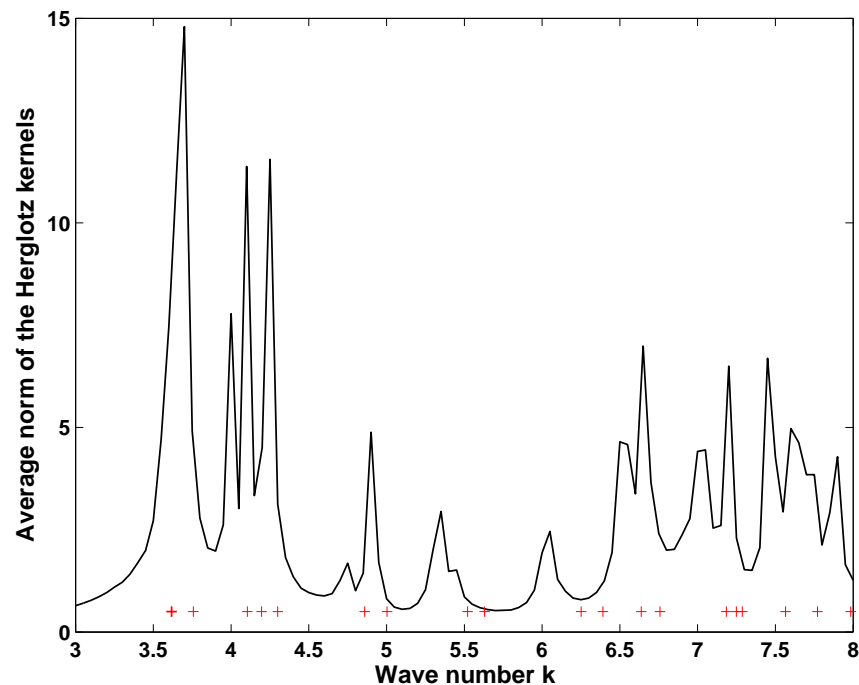
$$\lim_{\delta \rightarrow 0} \|v_{g_{z,\delta}}\|_{L^2(D)} = \infty.$$

Arens (IP 2004), Cakoni-Colton-Haddar (Comptes Rendus Math. 2010).

Computation of Transmission Eigenvalues



A composite plot of $\|g_{z_i}\|_{L^2(\Omega)}$ against k
for 25 random points $z_i \in D$



The average of $\|g_{z_i}\|_{L^2(\Omega)}$
over all choices of $z_i \in D$.

Computation of the transmission eigenvalues from the far field equation
for a square D .

Faber-Krahn Inequalities

What do transmission eigenvalues say about $n(x)$?

Let k_1 be the first transmission eigenvalue and let $\lambda_1(D)$ be the first Dirichlet eigenvalue for $-\Delta$ in D .

Theorem: If $n(x) > \alpha > 1$ for $x \in \overline{D}$. Then

$$k_1^2 \geq \frac{\lambda_1(D)}{\sup_D n}.$$

Faber-Krahn Inequalities

Proof: Consider the interior transmission problem and set $u := w - v \in H_0^2(D)$. Then

$$(\Delta + k^2)u = k^2 m w$$

where $m = 1 - n$ and hence

$$(\Delta + k^2 n) \frac{1}{m} (\Delta + k^2)u = 0 \quad \text{i.e. since } u \in H_0^2(D)$$

$$\begin{aligned} 0 &= \int_D \frac{1}{m} (\Delta u + k^2 u) (\Delta \bar{u} + k^2 n \bar{u}) dx = \\ &= \int_D \frac{1}{m} |\Delta u + k^2 n u|^2 dx + k^2 \int_D |\nabla u|^2 dx - k^4 \int_D n |u|^2 dx. \end{aligned}$$

Faber-Krahn Inequalities

But

$$\inf_{u \in H_0^2(D)} \frac{\int_D |\nabla u|^2 dx}{\int_D |u|^2 dx} \geq \inf_{u \in H_0^1(D)} \frac{\int_D |\nabla u|^2 dx}{\int_D |u|^2 dx} = \lambda_1(D)$$

and hence

$$\int_D k^2 (|\nabla u|^2 - k^2 n |u|^2) dx \geq k^2 \|u\|_{L^2(D)}^2 \left(\lambda_1(D) - k^2 \sup_D n \right).$$

Thus if $k^2 \leq \lambda_1(D) / \sup_D n$ then $u = 0$, i.e. k cannot be a transmission eigenvalue.

Faber-Krahn Inequalities

Theorem: Let $n_* = \inf_D(n)$ and $n^* = \sup_D(n)$.

● If $1 + \alpha \leq n_* \leq n(x) \leq n^* < \infty$ for $x \in \overline{D}$ then

$$0 < k_{1,n^*} \leq k_{1,n(x)} \leq k_{1,n_*}.$$

● If $0 < n_* \leq n(x) \leq n^* < 1 - \beta$ for $x \in \overline{D}$ then

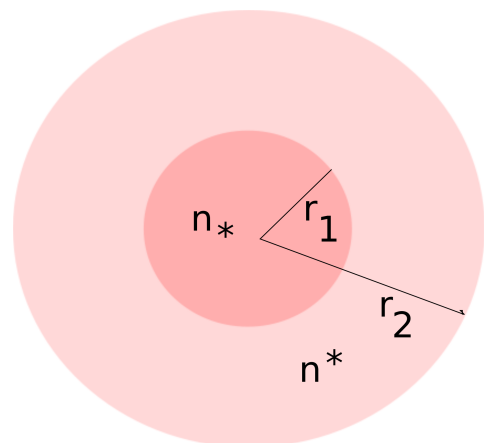
$$0 < k_{1,n_*} \leq k_{1,n(x)} \leq k_{1,n^*}.$$

Proof: Cakoni-Gintides-Haddar (SIMA 2010).

Numerical Example: Inhomogeneous Isotropic Media

We reconstruct $n_0 > 0$ such that

$$n_* \leq n_0 \leq n^*$$



$$r_2 = 2 \quad n_* = 2, \quad n^* = 4$$

r_1	n_0
0	4
0.2	3.7
0.7	3.4
1	3.1
1.5	2.7
1.7	2.48
2	2

Anisotropic Media

The corresponding interior transmission eigenvalue problem reads $v, w \in H^1(D)$ such that

$$\begin{aligned}\nabla \cdot A \nabla w + k^2 w &= 0 && \text{in } D \\ \Delta v + k^2 v &= 0 && \text{in } D \\ w &= v && \text{on } \partial D \\ \nu \cdot A \nabla w &= \nu \cdot \nabla v && \text{on } \partial D\end{aligned}$$

Cakoni-Gintides-Haddar (SIMA 2010), Cakoni-Colton-Haddar (JIEA to appear), Cakoni-Kirsch (IJCSM to appear).

Note that $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ does **not** uniquely determine non-constant $A(x)$ even if it is known for all wave numbers $k > 0$.

Faber-Krahn Inequalities

Let

$\sigma_*(x) :=$ smallest eigenvalue of $A(x)$ and

$\sigma^*(x) :=$ largest eigenvalue of $A(x)$.

Define $a_* := \inf_D(\sigma_*(x))$ and $a^* = \sup_D(\sigma^*(x))$ and denote by k_{1,n_0} be the first transmission eigenvalue of

$$a_0 \Delta w + k^2 w = 0 \quad \text{in } D$$

$$\Delta v + k^2 v = 0 \quad \text{in } D$$

$$w = v \quad \text{on } \partial D$$

$$a_0 \nu \cdot \nabla w = \nu \cdot \nabla v \quad \text{on } \partial D.$$

Faber-Krahn Inequalities

Then it can be shown

• If $\|A\|_2 > 1 + \alpha$ for $x \in \overline{D}$ then

$$0 < k_{1,a^*} \leq k_{1,A(x)} \leq k_{1,a_*}.$$

• If $\|A^{-1}\|_2 < 1 - \alpha$ for $x \in \overline{D}$ then

$$0 < k_{1,a_*} \leq k_{1,A(x)} \leq k_{1,a^*}.$$

Given the first transmission eigenvalue $k_{1,A(x)}$ and the domain D our aim is to obtain information about $A(x)$. From above we have that k_{1,a_0} is a monotonic decreasing continuous function of a_0 .

Setting k_1 equal to the measured transmission eigenvalue now gives a_0 where $a_* \leq a_0 \leq a^*$ i.e. a_0 lies between the infimum of the smallest eigenvalue and the supremum of the largest eigenvalue of $A(x)$.

Numerical Examples: Homogeneous Anisotropic Media

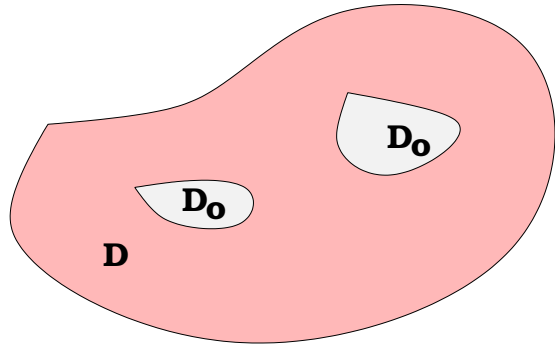
We consider D to be the unit square $[-1/2, 1/2] \times [-1/2, 1/2]$ and

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \quad A_2 = \begin{pmatrix} 6 & 0 \\ 0 & 8 \end{pmatrix} \quad A_{2r} = \begin{pmatrix} 7.4136 & -0.9069 \\ -0.9069 & 6.5834 \end{pmatrix}$$

Matrix	Eigenvalues a_* , a^*	Predicted a_0
A_{iso}	4, 4	4.032
A_1	2, 8	5.319
A_2	6, 8	7.407
A_{2r}	6, 8	6.896

Cakoni-Colton-Monk-Sun (IP to appear)

Faber-Krahn Inequalities



$$n = 1 \text{ in } D_0$$

$$n - 1 \geq \alpha > 0 \text{ in } D \setminus D_0$$

The case when there are regions in D where $n = 1$ (i.e. cavities) is more delicate.

Let $\lambda_1(D)$ be again the first eigenvalue of $-\Delta$ in D , $n_* = \inf_{D \setminus D_0}(n)$ and $n^* = \sup_{D \setminus D_0}(n)$.

Theorem: If $1 + \alpha \leq n_* \leq n(x) \leq n^* < \infty$ for $x \in \overline{D} \setminus D_0$ and $n(x) = 1$ in D_0 then

$$0 < \sqrt{\frac{\lambda_1(D)}{n^*}} \leq k_{1,n(x),D} \leq k_{1,n_*,\overline{D} \setminus D_0}.$$

Proof: Cakoni-Colton-Haddar (SIMA 2010), Cakoni-Gintides-Haddar (SIMA 2010).

Open Questions

- Do there exist complex transmission eigenvalues? Do there exist real eigenvalues if both the background medium and the scatterer are absorbing?
- Does the above theory apply if $m := n - 1$ changes sign in D ?
- Do transmission eigenvalues/ (eigenfunctions ?) uniquely determine the index of refraction? Inverse Eigenvalue Problem.

Spherically Stratified Media

Now consider the case when $n(x) = n(|x|) = n(r)$. The solutions of the interior transmission problem

$$\begin{aligned} \Delta w + k^2 n(r) w &= 0 & \Delta v + k^2 v &= 0 & \text{in } B \\ w &= v & \frac{\partial w}{\partial r} &= \frac{\partial v}{\partial r} & \text{on } \partial B \end{aligned}$$

where $B := \{x : |x| < a\}$ can be found in the form

$$v(r, \theta) = a_\ell j_\ell(kr) P_\ell(\cos \theta) \quad w(r, \theta) = b_\ell y_\ell(kr) P_\ell(\cos \theta) \quad \text{where}$$

$$y_\ell'' + \frac{2}{r} y_\ell' + \left(k^2 n(r) - \frac{\ell(\ell+1)}{r^2} \right) y_\ell = 0$$

is such that

$$\lim_{r \rightarrow 0} r^{-\ell} y_\ell(r) = \frac{\sqrt{\pi} k^\ell}{2^{\ell+1} \Gamma(\ell + 3/2)}.$$

The Inverse Eigenvalue Problem

k is a **transmission eigenvalue** if and only if $\operatorname{Re} k > 0$ and

$$d_\ell(k) = \det \begin{vmatrix} y_\ell(a) & -j_\ell(ka) \\ y'_\ell(a) & -kj'_\ell(ka) \end{vmatrix} = 0.$$

- Noting that $d_\ell(k)$ is an even entire function of k of exponential type.
- Using **Hadamard's factorization theorem**.
- Asymptotic behavior of $d_\ell(k)$ for k large.
- **Müntz's theorem**.

Theorem (Cakoni-Colton-Gintides): If $n(0)$ is given then $n(r)$ is uniquely determined from a knowledge of the transmission eigenvalues.

*Partial results are given by McLaughlin-Polyakov (JDE1994),
McLaughlin-Polyakov-Sacks (SIAP 1994)*

Complex Eigenvalues

Do complex eigenvalues exist? The following example suggests that the answer is yes:

Consider the **interior transmission problem** in a disk B of radius one in \mathbb{R}^2 with constant index of refraction $n^2 > 1$:

$$\begin{aligned} \Delta_2 w + k^2 n^2 w &= 0 & \Delta_2 v + k^2 v &= 0 & \text{in } & B \\ w &= v & \frac{\partial w}{\partial r} &= \frac{\partial v}{\partial r} & \text{on } & \partial B \end{aligned}$$

Then $k \neq 0$ is a transmission eigenvalue if and only if

$$d(k, n) = k [J_1(k) J_0(kn) - n J_0(k) J_1(kn)] = 0.$$

We will show that there exist complex zeros of $d(k, n)$ provided n is sufficiently close to one.

Complex Eigenvalues

Using $J_0'(t) = -J_1(t)$ and $(tJ_1(t))' = tJ_0(t)$ we have that

$$\left. \frac{\partial}{\partial n} d(k, n) \right|_{n=1} = -k^2 (J_1^2(k) + J_0^2(k))$$

$$\text{i.e. } f(k) = \lim_{n \rightarrow 1^+} \frac{d(k, n)}{n-1} = -k^2 (J_1^2(k) + J_0^2(k)).$$

Note that

- By **Hadamard's factorization theorem** $f(k)$ has an infinite number of complex zeros.
- By **Montel's theorem** the convergence as $n \rightarrow 1^+$ is uniform.
- By **Hurwitz's theorem** if $f(k_0) = 0$ and $\epsilon > 0$ then for n sufficiently close to one there exists a zero of $d(k, n)$ in $|k - k_0| < \epsilon$.