Transmission Eigenvalues in Inverse Scattering Theory

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Research supported in part by the US Air Force Office of Scientific Research

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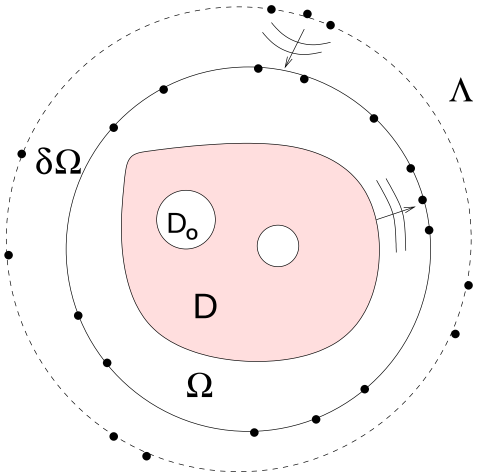
Motivation

Nondestructive testing of materials using electromagnetic interrogation.

- Detect the presence of cavities, cracks, and inclusions in both dielectric and absorbing (possibly anisotropic) materials.
- Detect changes in the anisotropic structure of non-homogeneous materials.



The imaging tool is a set of eigenvalues associated with inhomogeneous media known as transmission eigenvalues

Motivation

Two important issues:

- Transmission eigenvalues can be determined from the scattered data.
- Transmission eigenvalues carry information about material properties.

The goal is

- to quantify how the presence of abnormalities affects transmission eigenvalues and use this information to test the integrity of materials.

Outline of the Talk

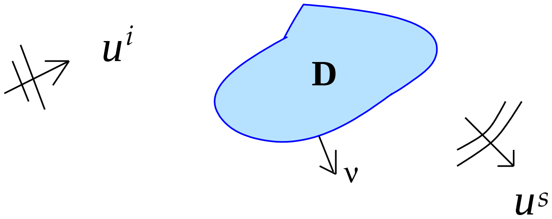
- How the transmission eigenvalue problem arises in scattering theory.
- How transmission eigenvalues can be computed from scattering data.
- An exposé of the most recent results on transmission eigenvalues.
- What can be done in terms of applications.
- Open questions.

This presentation is done for the scalar case and far field measurements.

Similar developments for Maxwell's equations in $\mathbb{R}^3$ can be found in **Cakoni- Colton-Monk**, *The Linear Sampling Method in Inverse Electromagnetic Scattering*, CBMS-NSF, **80**, SIAM Publication (2010).

Bellis-Cakoni-Guzina (to appear) has extended this discussion to full anisotropic elasticity.

Scattering by an Inhomogeneous Media



$$\Delta u + k^2 n(x) u = 0 \qquad \text{in } \mathbb{R}^2$$

$$u = u^s + u^i \qquad \text{in } \mathbb{R}^2$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

We assume that $n - 1$ has compact support $\overline{D}$ and $n \in L^\infty(D)$ is such that $\Re(n) \geq \gamma > 0$ and $\Im(n) \geq 0$ in $\overline{D}$.

Question: Is there an incident wave u^i that does not scatter?

The answer to this question leads to the transmission eigenvalue problem.

Transmission Eigenvalue Problem

If there exists a nontrivial solution to the homogeneous interior transmission problem

$$
\begin{aligned}
\Delta w + k^2 n(x) w = 0 &\qquad \text{in} \qquad D \\
\Delta v + k^2 v = 0 &\qquad \text{in} \qquad D \\
w = v &\qquad \text{on} \qquad \partial D \\
\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} &\qquad \text{on} \qquad \partial D
\end{aligned}
$$

such that v can be extended outside D as a solution to the Helmholtz equation $\tilde{v}$, then the scattered field due to $\tilde{v}$ as incident wave is identically zero.

Remark: Note that if $n = 1$ the interior transmission problem is degenerate.

Transmission Eigenvalue Problem

Definition: $k^2 \in \mathbb{C}$ is a transmission eigenvalue if there exists a nontrivial solution $v \in L^2(D)$, $w \in L^2(D)$, $w - v \in H_0^2(D)$ of the homogeneous interior transmission problem

$$\begin{array}{rll} \Delta w + k^2 n(x) w = 0 & \text{in} & D \\ \Delta v + k^2 v = 0 & \text{in} & D \\ w = v & \text{on} & \partial D \\ \dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} & \text{on} & \partial D \end{array}$$

Corresponding nontrivial solutions (v, w) are called eigenpairs

Note: If $\Im(n) > 0$ in $\overline{D}$, there are no real transmission eigenvalues.

Measurements

Let $u^i(x) = e^{ikx\cdot d}$. The corresponding scattered field assumes the asymptotic behavior:

$$u^s(x,d,k) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x},d,k) + O\left(\frac{1}{r^{3/2}}\right)$$

as $r \to \infty$ where $\hat{x} = x/|x|$, $r = |x|$ and $k > 0$ is the wave number.

$u_\infty(\hat{x},d,k)$ is called the far field pattern of the scattered field u^s.

Our data is $u_\infty(\hat{x},d,k)$ for $\hat{x} \in \Omega_0 \subset \Omega$, $d \in \Omega_1 \subset \Omega$, $k \in [k_0,\ k_1]$ where $\Omega := \{x \in \mathbb{R}^2 :\ |x| = 1\}$.

For the sake of simplicity we assume that the data is available on all of Ω.

The Far Field Operator

Define the far field operator $F : L^2(\Omega) \to L^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d, k) g(d) ds(d), \qquad \left(S = I + \frac{ik}{\sqrt{2\pi k}} e^{-i\pi/4} F \right)$$

F is injective and has dense range if and only if k^2 is not a transmission eigenvalue such that for a corresponding eigenpair (v, w), v takes the form a Herglotz wave function

$$v_g(x) := \int_\Omega e^{ikx\cdot d} g(d) ds(d), \qquad g \in L^2(\Omega).$$

Note that Herglotz wave functions with $g \in L^2(\Omega)$ are dense in the space of the solutions to Helmholtz equation with respect to the $L^2(D)$-norm.

The Far Field Equation

From now on we assume that D (or a reconstruction of D) is known.

For $z \in D$ the far field equation is

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z, k), \quad g \in L^2(\Omega)$$

where $\Phi_\infty(\hat{x}, z) = \frac{e^{i\pi/4}}{\sqrt{4\pi k}} e^{-ik\hat{x}\cdot z}$ is the far field pattern of the fundamental solution $\Phi(x, z, k) := 1/4 H_0^{(1)}(k|x-z|)$.

In fact only the measured "noisy" far field pattern $u_\infty^\delta(\hat{x}, d, k)$ is available, where $\delta > 0$ is the noise level, which leads to the noisy far field equation

$$(F^\delta g)(\hat{x}) := \int_\Omega u_\infty^\delta(\hat{x}, d, k) g(d) ds(d) = \Phi_\infty(\hat{x}, z, k).$$

Computation of Real TE

Theorem: Assume that either $n > 1$ or $n < 1$ for $x \in \overline{D}$. For $z \in D$, let $g_{z,\delta}$ be the Tikhonov regularized solution of the far field equation

$$(F^\delta g)(\hat{x}) = \Phi_\infty(\hat{x}, z, k).$$

- If k^2 is not a transmission eigenvalue then

$$\lim_{\delta \to 0} \|v_{g_{z,\delta,k}}\|_{L^2(D)} \qquad \text{exists.}$$

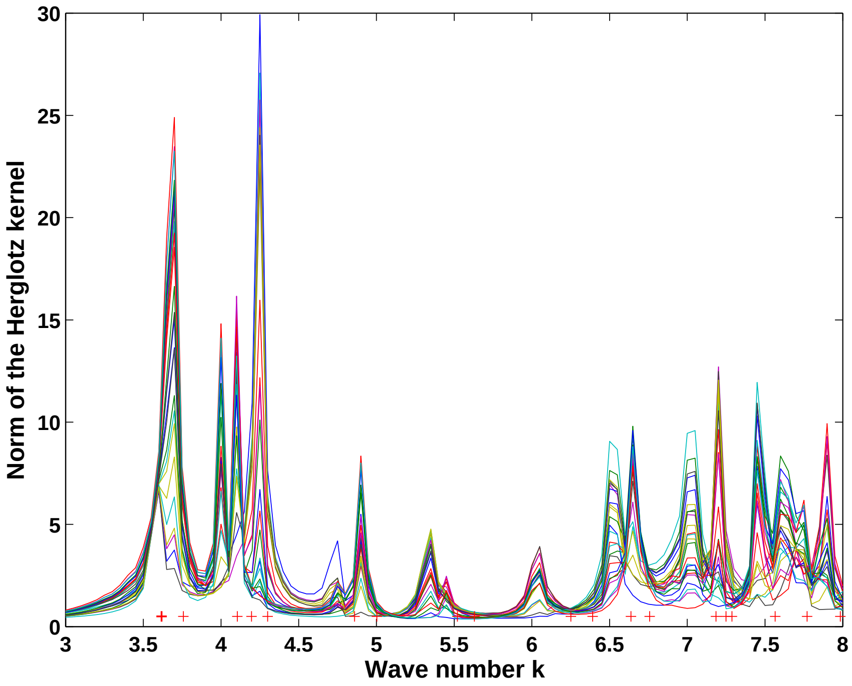
Arens (Inverse Problems 2004),

- If k^2 is a transmission eigenvalue then for almost every $z \in D$

$$\lim_{\delta \to 0} \|v_{g_{z,\delta,k}}\|_{L^2(D)} = \infty.$$

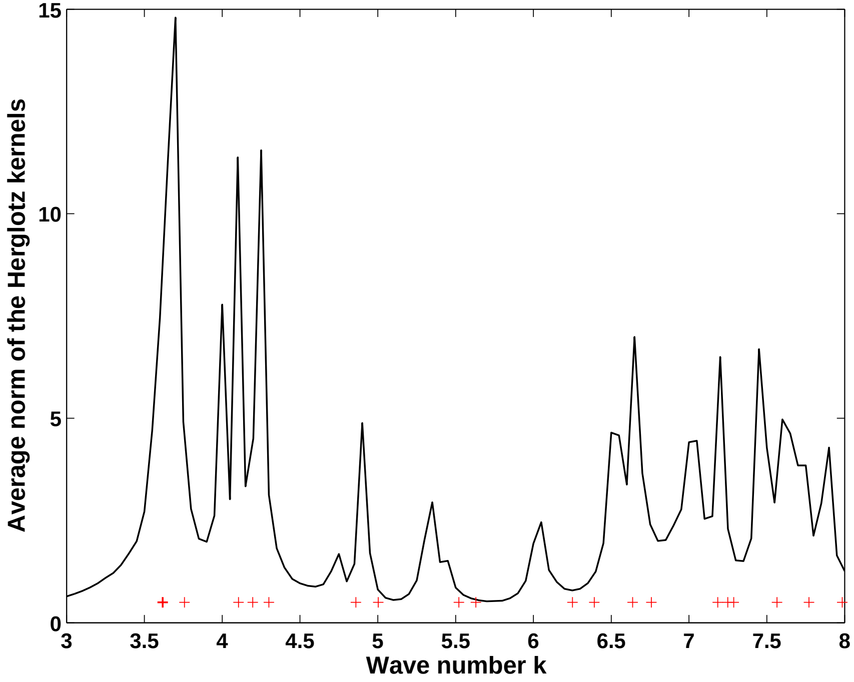
Cakoni-Colton-Haddar (Comptes Rendus Math. 2010).

Computation of Real TE



A composite plot of $\|g_{z_i}\|_{L^2(\Omega)}$ against k

for 25 random points $z_i \in D$



The average of $\|g_{z_i}\|_{L^2(\Omega)}$

over all choices of $z_i \in D$.

Computation of the transmission eigenvalues from the far field equation for the unit square D.

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Historical Overview

- The transmission eigenvalue problem in scattering theory was introduced by **Kirsch (1986)** and **Colton-Monk (1988)**
- Research was focused on the discreteness of transmission eigenvalues for variety of scattering problems: **Colton-Kirch-Päivärinta (1989), Rynne-Sleeman (1991), Cakoni-Haddar (2007), Colton-Päivärinta-Sylvester (2007), Kirsch (2009)** all this assuming $n-1>\alpha>0$ (or $1-n>\alpha>0$), **Hickmann (to appear)** $n-1>0, n-1\in C^2(\overline{D})$; **Cakoni-Colton-Haddar (2010)** the case when $n=1$ in $D_0\subset D$ and $n-1>\alpha>0$ in $D\setminus\overline{D}_0$.
- The first proof of existence of transmission eigenvalues for large enough contrast is due to **Päivärinta-Sylvester (2009)**. The existence of an infinite set of transmission eigenvalues is proven by **Cakoni-Gintides-Haddar (2010)** again uder the above assumption on n.
- ... In the past year the area has been growing rapidly ....
- **Hitrik-Krupchyk-Ola-Päivärinta** in a series of papers have extended the transmission eigenvalue problem to more general class of differential operators with constant coefficients.
- **Finch** has connected the discreteness of the transmission spectrum to a uniqueness question in thermo-acoustic imaging: interesting case is $n-1\in C^\alpha(\overline{D})$, $|n-1|\geq 0$. **Nguyen talk yesterday**

Radially Symmetric Case

We consider the interior eigenvalue problem for a ball of radius a with index of refraction $n(r)$ being a function of $r := |x|$

$$\begin{aligned} \Delta w + k^2 n(r) w = 0 &\qquad \text{for} \qquad |x| < a \\ \Delta v + k^2 v = 0 &\qquad \text{for} \qquad |x| < a \\ w = v &\qquad \text{for} \qquad |r| = a \\ \frac{\partial w}{\partial r} = \frac{\partial v}{\partial r} &\qquad \text{for} \qquad |r| = a \end{aligned}$$

We assume that $n \in C^2(\overline{D})$, $n(r) > 0$ and $\int_0^a [n(r)]^{1/2}\, dr \neq a$.

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Radially Symmetric Case

The solutions of the transmission eigenvalue problem in general can be found in the form

$v(r,\theta) = a_\ell j_\ell(kr) P_\ell(\cos\theta) \qquad w(r,\theta) = b_\ell y_\ell(kr) P_\ell(\cos\theta)$ where

$$y_\ell'' + \frac{2}{r} y' + \left(k^2 n(r) - \frac{\ell(\ell+1)}{r^2} \right) y_\ell = 0$$

is such that

$$\lim_{r\to 0} r^{-\ell} y_\ell(r) = \frac{\sqrt{\pi}\, k^\ell}{2^{\ell+1}\Gamma(\ell+3/2)}.$$

Then k^2 is a transmission eigenvalue if and only if $\mathcal{R}e\, k > 0$ and

$$d_\ell(k) = \text{det} \begin{vmatrix} y_\ell(a) & -j_\ell(ka) \\ y_\ell'(a) & -kj_\ell'(ka) \end{vmatrix} = 0.$$

Radially Symmetric Case

$$d_\ell(k) = \frac{1}{a^2 k [n(0)]^{\ell/2+1/4}} \sin k \left(a - \int_0^a [n(r)]^{1/2}\, dr \right) + O\left(\frac{\ln k}{k^2} \right)$$

We first observe that there exists infinitely many positive real zeros of $d_\ell(k)$ for k large and these zeros are the transmission eigenvalues.

Question: Do transmission eigenvalues uniquely determine $n(r)$?

Theorem: If $n(0)$ is given then $n(r)$ is uniquely determined from a knowledge of the transmission eigenvalues.

Cakoni-Colton-Gintides, SIAM J. Math. Anal. to appear.

Previous results in this direction were given by *McLaughlin-Polyakov, Jour. Diff. Eqns, 1994, McLaughlin-Polyakov-Sacks, SIAP, 1994.*

Complex Eigenvalues

Do complex eigenvalues exist? The following example suggests that the answer is yes:

Consider the interior transmission problem in a disk B of radius one in $\mathbb{R}^2$ with constant index of refraction $n^2 > 1$:

$$\begin{array}{cccc} \Delta_2 w + k^2 n^2 w = 0 & \Delta_2 v + k^2 v = 0 & \text{in} & B \\ w = v & \dfrac{\partial w}{\partial r} = \dfrac{\partial v}{\partial r} & \text{on} & \partial B \end{array}$$

Then $k^2 \neq 0$ is a transmission eigenvalue if and only if

$$d(k, n) = k\left[J_1(k)J_0(kn) - nJ_0(k)J_1(kn)\right] = 0.$$

We will show that there exist complex zeros of $d(k, n)$ provided n is sufficiently close to one.

Complex Eigenvalues

Using $J_0'(t) = -J_1(t)$ and $(tJ_1(t))' = tJ_0(t)$ we have that

$$\frac{\partial}{\partial n} d(k,n)\Big|_{n=1} = -k^2\left(J_1^2(k) + J_0^2(k)\right)$$

i.e $$f(k) = \lim_{n \to 1^+} \frac{d(k,n)}{n-1} = -k^2\left(J_1^2(k) + J_0^2(k)\right).$$

Note that

- By Hadamard's factorization theorem $f(k)$ has an infinite number of complex zeros.
- By Montel's theorem the convergence as $n \to 1^+$ is uniform.
- By Hurwitz's theorem if $f(k_0) = 0$ and $\epsilon > 0$ then for n sufficiently close to one there exists a zero of $d(k,n)$ in $|k - k_0| < \epsilon$.

Cakoni-Colton-Gintides, SIAM J. Math. Anal., to appear.

Transmission Eigenvalues

We assume that $n \in L^\infty(D)$ is such that $n - 1 \geq \alpha > 0$.

The transmission eigenvalue problem can be written for the difference $u := w - v \in H_0^2(D)$ as an eigenvalue problem for the fourth order equation:

$$(\Delta + k^2)\frac{1}{n-1}(\Delta + k^2 n)u = 0$$

i.e. in the variational form

$$\int_D \frac{1}{n-1}(\Delta u + k^2 n u)(\Delta\overline{\varphi} + k^2\overline{\varphi})\,dx = 0 \qquad \text{for all } \varphi \in H_0^2(D)$$

Transmission Eigenvalues

Let $1 < n_* = \inf_D (n)$ and $n^* = \sup_D (n)$. Then we have

$$0 = \int_D \frac{1}{n-1} \left|(\Delta u + k^2 n u)\right|^2 \, dx + k^2 \int_D \left(|\nabla u|^2 - k^2 n |u|^2\right) dx.$$

Poincare inequality yields the Faber-Krahn type inequality for the first transmission eigenvalue (not isoperimetric)

$$k^2_{1,D,n} > \frac{\lambda_1(D)}{n^*}.$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D.

In particular there are no real transmission eigenvalues in the interval $(0, \lambda_1(D)/n^*)$.

Transmission Eigenvalues

Letting $k^2 := \tau$, the transmission eigenvalue problem can be written as a quadratic pencil operator

$$u - \tau K_1 u + \tau^2 K_2 u = 0, \qquad u \in H_0^2(D)$$

with selfadjoint compact operators $K_1 = T^{-1/2} T_1 T^{-1/2}$ and $K_2 = T^{-1/2} T_2 T^{-1/2}$ where

$$(Tu,\varphi)_{H^2(D)} = \int_D \frac{1}{n-1} \Delta u \, \Delta\overline{\varphi} \, \mathrm{d}x \qquad \text{coercive}$$

$$(T_1 u,\varphi)_{H^2(D)} = -\int_D \frac{1}{n-1} \left(\Delta u \, \overline{\phi} + n u \, \Delta\overline{\varphi}\right) \, \mathrm{d}x$$

$$(T_2 u,\varphi)_{H^2(D)} = \int_D \frac{n}{n-1} u \, \overline{\varphi} \, \mathrm{d}x \qquad \text{non-negative.}$$

Transmission Eigenvalues

The transmission eigenvalue problem can be transformed to the eigenvalue problem

$$(\mathbb{K} - \xi\mathbb{I})U = 0, \qquad U = \begin{pmatrix} u \\ \tau K_2^{1/2} u \end{pmatrix}, \qquad \xi := \frac{1}{\tau}$$

for the non-self-adjoint compact operator
$\mathbb{K}\colon H_0^2(D) \times H_0^2(D) \to H_0^2(D) \times H_0^2(D)$ given by

$$\mathbb{K} := \begin{pmatrix} K_1 & -K_2^{1/2} \\ K_2^{1/2} & 0 \end{pmatrix}.$$

However from here one can see that the transmission eigenvalues form a discrete set with $+\infty$ as the only possible accumulation point.

Transmission Eigenvalues

To obtain existence of transmission eigenvalues and isoperimetric Faber-Krahn type inequalities we rewrite the transmission eigenvalue problem in the form

$$(\mathbf{A}_\tau - \tau\mathbf{B})u = 0 \quad \text{in} \quad H_0^2(D)$$

$$(\mathbf{A}_\tau u, \varphi)_{H^2(D)} = \int_D \frac{1}{n-1}(\Delta u + \tau u)(\Delta\overline{\varphi} + \tau\overline{\varphi})\,dx + \tau^2 \int_D u \cdot \overline{\varphi}\,dx$$

$$(\mathbf{B}u, \varphi)_{H^2(D)} = \int_D \nabla u \cdot \nabla\overline{\varphi}\,dx$$

Observe that

- The mapping $\tau \longrightarrow \mathbf{A}_\tau$ is continuous from $(0, +\infty)$ to the set of self-adjoint coercive operators from $H_0^2(D) \rightarrow H_0^2(D)$.
- $\mathbf{B} : H_0^2(D) \rightarrow H_0^2(D)$ is self-adjoint, compact and non-negative.

Transmission Eigenvalues

Now we consider the generalized eigenvalue problem

$$(\mathbf{A}_\tau - \lambda(\tau)\mathbf{B})u = 0 \quad \text{in} \quad H_0^2(D)$$

Note that $k^2 = \tau$ is a transmission eigenvalue if and only if satisfies $\lambda(\tau) = \tau$

For a fixed $\tau > 0$ there exists an increasing sequence of eigenvalues $\lambda_j(\tau)_{j \geq 1}$ such that $\lambda_j(\tau) \to +\infty$ as $j \to \infty$.

These eigenvalues satisfy

$$\lambda_j(\tau) = \min_{W \subset \mathcal{U}_j} \left(\max_{u \in W \setminus \{0\}} \frac{(\mathbf{A}_\tau u, u)}{(\mathbf{B}u, u)} \right).$$

Transmission Eigenvalues

Hence, if there exists two positive constants $\tau_0 > 0$ and $\tau_1 > 0$ such that

- $\mathbf{A}_{\tau_0} - \tau_0 \mathbf{B}$ is positive on $H_0^2(D)$,
- $\mathbf{A}_{\tau_1} - \tau_1 \mathbf{B}$ is non positive on a m dimensional subspace of $H_0^2(D)$

then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1, \ldots, m$, has at least one solution in $[\tau_0, \tau_1]$ meaning that there exists m transmission eigenvalues (counting multiplicity) within the interval $[\tau_0,\, \tau_1]$.

It is now obvious that determining such constants τ_0 and τ_1 provides the existence of transmission eigenvalues as well as the desired isoperimetric inequalities.

Faber-Krahn Inequalities

Theorem Assume that $n \in L^\infty(D)$, and either $1 < n_* \le n(x) \le n^*$ almost everywhere in $\overline{D}$. Then, there exists an infinite discrete set of real transmission eigenvalues accumulating at infinity $+\infty$. Furthermore

- $k^2_{1,n^*} \le k^2_{1,n(x)} \le k^2_{1,n_*}$.

One can prove that, for n constant, the first transmission eigenvalue $k^2_{1,n}$ depends strictly monotonically increasing on n and is continuous on n. In particular, this shows that the first transmission eigenvalue determine uniquely the constant index of refraction, provided that it is known a priori that either $n > 1$.

Similar results can be obtained for the case when $0 < n_* \le n(x) \le n^* < 1$.

Cakoni-Gintides-Haddar, SIAM J. Math. Anal. (2010)

Detection of Anomalies in an Isotropic Medium

We find the constant n_0 such that the first transmission eigenvalue of

$$\begin{aligned} \Delta w + k^2 n_0 w &= 0 && \text{in} \quad D \\ \Delta v + k^2 v &= 0 && \text{in} \quad D \\ w &= v && \text{on} \quad \partial D \\ \frac{\partial w}{\partial \nu} &= \frac{\partial v}{\partial \nu} && \text{on} \quad \partial D \end{aligned}$$

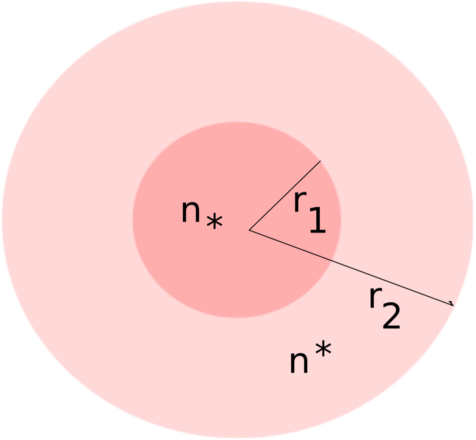
is $k^2_{1,n(x)}$ (which can be determined from the measure data).

Then from the previous discussion we have that $n_* \le n_0 \le n^*$.

Numerical examples show in fact that $n_0 \approx \frac{1}{|D|} \int_D n(x)\, dx$

Numerical Example: Inhomogeneous Isotropic Media

We reconstruct $n_0 > 0$ such that
$n_* \leq n_0 \leq n^*$



$r_2 = 2$ $n_* = 2$, $n^* = 4$

r_1	n_0
0	4.2
0.2	3.7
0.7	3.4
1	3.1
1.5	2.7
1.7	2.48
2	2.05

Anisotropic Media

The corresponding transmission eigenvalue problem is to find $v, w \in H^1(D)$ such that

$$\begin{aligned} \nabla \cdot A^{-1} \nabla w + k^2 n w = 0 &\qquad \text{in} \qquad D \\ \Delta v + k^2 v = 0 &\qquad \text{in} \qquad D \\ w = v &\qquad \text{on} \qquad \partial D \\ \nu \cdot A^{-1} \nabla w = \nu \cdot \nabla v &\qquad \text{on} \qquad \partial D \end{aligned}$$

Cakoni-Gintides-Haddar, SIMA, (2010), Cakoni-Colton-Haddar, JIEA (2010) both for $n = 1$, *Cakoni-Kirsch, IJCSM (2010)*.

Note that $u_\infty(\hat{x}, d, k)$ for $\hat{x}, d \in \Omega$ does not uniquely determine $A(x)$ even if it is known for all wave numbers $k > 0$.

Faber-Krahn Inequalities

Let

$$\sigma_*(x) := \text{smallest eigenvalue of } A(x) \qquad \text{and}$$

$$\sigma^*(x) := \text{largest eigenvalue of } A(x).$$

Define $a_* := \inf_D(\sigma_*(x))$ and $a^* = \sup_D(\sigma^*(x))$ and denote by k_{1,a_0} be the first transmission eigenvalue of

$$\begin{aligned} \Delta w + k^2 a_0 w = 0 \qquad & \text{in} \quad D \\ \Delta v + k^2 v = 0 \qquad & \text{in} \quad D \\ w = v \qquad & \text{on} \quad \partial D \\ \frac{1}{a_0}\nu \cdot \nabla w = \nu \cdot \nabla v \qquad & \text{on} \quad \partial D. \end{aligned}$$

Faber-Krahn Inequalities

Then it can be shown

- If $\overline{\xi} \cdot A(x)\, \xi \geq \alpha|\xi|^2$ for $x \in \overline{D}$, and $\alpha > 1$ then

$$k_{1,a^*}^2 \leq k_{1,A(x)}^2 \leq k_{1,a_*}^2.$$

Given the first transmission eigenvalue $k_{1,A(x)}^2$ and the domain D our aim is to obtain information about $A(x)$. It can be shown that k_{1,a_0}^2 is strictly monotonic increasing and continuous with respect to a_0.

Setting k_1^2 equal to the measured transmission eigenvalue now gives a_0 where $a_* \leq a_0 \leq a^*$ i.e. a_0 lies between the infimum of the smallest eigenvalue and the supremum of the largest eigenvalue of $A(x)$.

Numerical Examples: Homogeneous Anisotropic Media

We consider D to be the unit square $[-1/2,\, 1/2] \times [-1/2,\, 1/2]$ and

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \quad A_2 = \begin{pmatrix} 6 & 0 \\ 0 & 8 \end{pmatrix} \quad A_{2r} = \begin{pmatrix} 7.4136 & -0.9069 \\ -0.9069 & 6.5834 \end{pmatrix}$$

Matrix	Eigenvalues $a_*,\ a^*$	Predicted a_0
A_{iso}	4, 4	4.032
A_1	2, 8	5.319
A_2	6, 8	7.407
A_{2r}	6, 8	6.896

Cakoni-Colton-Monk-Sun, Inverse Problems, (2010)

Complex Eigenvalues

We are interested in understanding complex eigenvalues in order to deal with absorbing/dispersive media (in particular if we want to apply these ideas in the imaging of human tissues).

In this case

$$\begin{aligned} \Delta w + k^2 n_1(x) w = 0 \qquad & \text{in} \qquad D \\ \Delta v + k^2 n_0(x) v = 0 \qquad & \text{in} \qquad D \end{aligned}$$

$w - v \in H_0^2(D)$.

- If $\Im(n_1) > 0$, $\Im(n_0) > 0$ one may have real transmission eigenvalues (evidence of such for a combination of material properties have been given by *Guzina-Bellis, J. Elast. (2009)* for the elastic case.

Complex Eigenvalues

Current results on complex transmission eigenvalues for media of general shape are limited to identifying eigenvalue free zones in the complex plane.

- The first result for homogeneous media is given in *Cakoni-Colton-Gintides SIMA (to appear)*.
- The best result to date is due *Hitrik-Krupchyk-Ola-Päivärinta, arcXiv*. In particular they have shown that k^2 are confined to a parabolic neighborhood of the positive real axis. More specifically:

  Theorem: For $n \in C^\infty(\overline{D}, \mathbb{R})$ and $1 < \alpha \leq n \leq \beta$, there exists a $0 < \delta < 1$ and $C > 1$ both independent of n such that all transmission eigenvalues $\tau := k^2 \in \mathbb{C}$ with $|\tau| > C$ satisfies $\Re(\tau) > 0$ and $\Im(\tau) \leq C|\tau|^{1-\delta}$.

Open Questions

- What information about the index of refraction do the higher real transmission eigenvalues carry?
- Do complex transmission eigenvalues exist in general?
- Any progress toward the solution of the inverse spectral problem for general media is desirable. A step in this direction has been made by *Hitrik-Krupchyk-Ola-Päivärinta* who proved the completeness of $u = v - w$ in an appropriate space where (v, w) are the eigenpairs for differential operators of even order greater than 3.
- The analysis of the transmission eigenvalue problem for media with contrast $n - 1$ that changes sign is an open problem. In the case when $|n - 1| \geq \alpha > 0$ progress can be made using $\top$-coercivity. Interesting is the case when $n - 1$ smoothly changes sign, since it is related to a uniqueness question in thermo-acoustic imaging and is more challenging problem.