# Inverse Scattering

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**Abstract**

We give a survey of the mathematical basis of inverse scattering theory, concentrating on the case of time harmonic acoustic waves. After an introduction and historical remarks we give an outline of the direct scattering problem. This is then followed by sections on uniqueness results in inverse scattering theory and iterative and decomposition methods to reconstruct the shape and material properties of the scattering object. We conclude by discussing qualitative methods in inverse scattering theory, in particular the linear sampling method and its use in obtaining lower bounds on the constitutive parameters of the scattering object.

## 1 Introduction and historical remarks

Scattering theory is concerned with the effects that obstacles and inhomogeneities have on the propagation of waves and in particular time-harmonic waves. In the context of this book, scattering theory provides the mathematical tools for imaging via acoustic and electromagnetic waves with applications to such fields as radar, sonar, geophysics, medical imaging and nondestructive testing.

For reasons of brevity, in this survey we focus our attention on the case of acoustic waves and only give passing references to the case of electromagnetic waves. We will furthermore give few proofs, referring the reader interested in further details to [23]. Since the literature in the area is enormous, we have only referenced a limited number of papers and hope that the reader can use these as starting point for further investigations.

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Mathematical acoustics begins with the modelling of acoustic waves, i.e., sound waves. The two main media for the propagation and scattering of sound waves are air and water (underwater acoustics). A third important medium with properties close to those of water is the human body, i.e., biological tissue (ultrasound). Since sound waves are considered as small perturbations in a gas or a fluid, the equation of acoustics, i.e., the wave equation

$$\frac{1}{c^2}\,\frac{\partial^2 p}{\partial t^2} = \Delta p \tag{1.1}$$

for the pressure $p = p(x,t)$ is obtained by linearization of the equations for the motion of fluids. Here, $c = c(x)$ denotes the local speed of sound and the fluid velocity is proportional to $\operatorname{grad} p$. For time-harmonic acoustic waves of the form

$$p(x,t) = \operatorname{Re}\left\{u(x)\,e^{-i\omega t}\right\} \tag{1.2}$$

with frequency $\omega > 0$, it follows that the complex-valued space dependent part $u$ satisfies the reduced wave equation

$$\Delta u + \frac{\omega^2}{c^2}\,u = 0. \tag{1.3}$$

Here we emphasize that the physical quantity describing the sound wave is the real-valued sound pressure $p(x,t)$ and not the complex-valued amplitude $u(x)$ in the representation $u(x)\,e^{-i\omega t}$. For a homogeneous medium the speed of sound $c$ is constant and (1.3) becomes the *Helmholtz equation*

$$\triangle u + k^2 u = 0, \tag{1.4}$$

where the wave number $k$ is given by the positive constant $k = \omega/c$.

A solution to the Helmholtz equation whose domain of definition contains the exterior of some sphere is called radiating if it satisfies the *Sommerfeld radiation condition*

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{1.5}$$

where $r = |x|$ and the limit holds uniformly in all directions $x/|x|$. Here, and in the sequel, $|x| := \sqrt{x_1^2 + x_2^2 + x_3^2}$ denotes the Euclidean norm of $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. For more details on the physical background of linear acoustic waves the reader is referred to [67].

We will confine our presentation of scattering theory for time-harmonic acoustic waves to two basic problems, namely scattering by a bounded impenetrable obstacle and scattering by a penetrable inhomogeneous medium of compact support. For a vector $d \in \mathbb{R}^3$ with $|d| = 1$, the function $e^{ik\,x\cdot d}$ satisfies the Helmholtz equation for all $x \in \mathbb{R}^3$. It is called a *plane wave*, since $e^{i(k\,x\cdot d - \omega t)}$ is constant on the planes $k\,x\cdot d - \omega t = \text{const}$. Note that these wave fronts travel with velocity $c$ in the direction $d$. Assume that an incident field is given by the plane wave $u^i(x) = e^{ik\,x\cdot d}$. Then the simplest obstacle scattering problem is to find the scattered field $u^s$ as a radiating solution to the Helmholtz equation in the exterior of a bounded scatterer $D$ such that the total field $u = u^i + u^s$ satisfies the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial D \tag{1.6}$$

corresponding to a sound-soft obstacle with the total pressure, i.e., the excess pressure over the static pressure $p_0$, vanishing on the boundary. Concerning the geometry of scattering obstacles, for simplicity, we always will assume that $D$ is a bounded domain with a connected boundary $\partial D$ of class $C^2$. In particular, this implies that the complement $\mathbb{R}^3 \setminus \bar{D}$ is connected. However, our results remain valid for a finite number of scattering obstacles.

Boundary conditions other than the Dirichlet condition also need to be considered such as the Neumann or sound-hard boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D \tag{1.7}$$

or, more generally, the impedance boundary condition

$$\frac{\partial u}{\partial \nu} + i\lambda u = 0 \quad \text{on } \partial D, \tag{1.8}$$

where $\nu$ is the outward unit normal to $\partial D$ and $\lambda$ is a positive constant called the surface impedance. More generally the impedance $\lambda$ can also vary on $\partial D$. Since $\operatorname{grad} u$ is proportional to the fluid velocity, the impedance boundary condition describes obstacles for which the normal velocity of the fluid on the boundary is proportional to the excess pressure on the boundary. The Neumann condition corresponds to a vanishing normal velocity on the boundary. In order to avoid repetitions by considering all possible types of boundary conditions, we will in general confine ourselves to presenting the basic ideas in acoustic obstacle scattering for the case of a sound-soft obstacle.

The simplest scattering problem for an inhomogeneous medium assumes that the speed of sound is constant outside a bounded domain $D$. Then the total field $u = u^i + u^s$ satisfies

$$\triangle u + k^2 n u = 0 \quad \text{in } \mathbb{R}^3 \tag{1.9}$$

and the scattered wave $u^s$ fulfills the Sommerfeld radiation condition (1.5), where the wave number is given by $k = \omega / c_0$ and $n = c_0^2 / c^2$ is the refractive index $n$ given by the ratio of the square of the sound speeds $c = c_0$ in the homogeneous host medium and $c = c(x)$ in the inhomogeneous medium. The *refractive index* is positive, satisfies $n(x) = 1$ for $x \notin D$ and we assume $n$ to be continuously differentiable in $\mathbb{R}^3$ (our results are also in general valid for $n$ being merely piecewise continuous in $\mathbb{R}^3$). An absorbing medium is modeled by adding an absorption term which leads to a refractive index with a positive imaginary part of the form

$$n = \frac{c_0^2}{c^2} + i\,\frac{\gamma}{k}$$

in terms of a possibly space dependent absorption coefficient $\gamma$.

Summarizing, given the incident wave and the physical properties of the scatterer, the *direct scattering problem* is to find the scattered wave and in particular its behavior at large distances from the scattering object, i.e., its far field behavior. The *inverse scattering problem* takes this answer to the direct scattering problem as its starting point and asks what is the nature of the scatterer that gave rise to such far field behavior?

To be more specific, it can be shown that radiating solutions $u^s$ to the Helmholtz equation have the asymptotic behavior

$$u^s(x) = \frac{e^{ik|x|}}{|x|}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty, \tag{1.10}$$

uniformly for all directions $\hat{x} = x/|x|$, where the function $u_\infty$ defined on the unit sphere $S^2$ is known as the *far field pattern* of the scattered wave. For plane wave incidence we indicate the dependence of the far field pattern on the incident direction $d$ and the observation direction $\hat{x}$ by writing $u_\infty = u_\infty(\hat{x}, d)$. The inverse scattering problem can now be formulated as the problem of determining either the sound-soft obstacle $D$ or the index of refraction $n$ (and hence also $D$) from a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for $\hat{x}$ and $d$ on the unit sphere $S^2$ (or a subset of $S^2$).

One of the earliest mathematical results in inverse scattering theory was Schiffer's proof in 1967 that the far field pattern $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in S^2$ uniquely determines the shape of a sound–soft obstacle $D$. Unfortunately, Schiffer's proof does not immediately generalize to other boundary conditions. This problem was remedied by Kirsch and Kress in 1993 who, using an idea originally proposed by Isakov, showed that $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in S^2$ uniquely determines the shape of $D$ as long as the solution of the direct scattering problem depends continuously on the boundary data [55]. In particular, it is not necessary to know the boundary condition a priori in order to guarantee uniqueness! The uniqueness problem for inverse scattering by an inhomogeneous medium was solved by Nachman [69], Novikov [71] and Ramm [80] in 1988 who based their analysis on the fundamental work of Sylvester and

Uhlmann [89]. Their uniqueness proof was subsequently considerably simplified by Hähner [36].

The first attempt to reconstruct the shape of a sound-soft scattering obstacle from a knowledge of the far field pattern in a manner acknowledging the nonlinear and ill-posed nature of the problem was made by Roger in 1981 using Newton's iteration method [82]. A characterization and rigorous proof of the existence of the Fréchet derivative of the solution operator in Newton's method was then established by Kirsch [48] and Potthast [76] in 1993 and 1994, respectively. An alternative approach to solving the inverse scattering problem was proposed by Colton and Monk in 1986 and by Kirsch and Kress in 1987 who broke up the inverse scattering problem into a linear, ill-posed problem and a nonlinear, well posed problem [25, 54]. The optimization method of Kirsch and Kress has the attractive feature of needing only a single incident field for its implementation. On the other hand, to use such methods it is necessary to know the number of components of the scatterer as well as the boundary condition satisfied by the field on the surface of the scatterer. These problems were overcome by Colton and Kirsch in 1996 through the derivation of a *linear* integral equation with the far field data as its kernel (i.e. multi-static data is needed for its implementation) [20]. This method, subsequently called the *linear sampling method*, was further developed by Colton, Piana and Potthast [30] and numerous other researchers. A significant development in this approach to the inverse scattering problem was the introduction of the *factorization method* by Kirsch in 1998 [49]. For further historical information on these "sampling" methods in inverse scattering theory, we refer the reader to another chapter of this book as well as the monographs [7] and [53].

Optimization methods and sampling methods for the inverse scattering problem for inhomogeneous media have been extensively investigated by numerous authors. In general, the optimization methods are based on rewriting the scattering problem corresponding to (1.9) as the *Lippmann–Schwinger integral equation*

$$u(x) = e^{ikx\cdot d} - \frac{k^2}{4\pi}\int_{\mathbb{R}^3} \frac{e^{ik|x-y|}}{|x-y|}\, m(y)u(y)\,dy, \quad x \in \mathbb{R}^3, \tag{1.11}$$

where $m := 1 - n$ and the object is to determine $m$ from a knowledge of

$$u_\infty(\hat{x}, d) = -\frac{k^2}{4\pi}\int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} m(y)u(y)\,dy, \quad \hat{x}, d \in S^2. \tag{1.12}$$

On the other hand, sampling methods have also been used to study the inverse scattering problem associated with (1.9) where now the object is to only determine the support of $m$. For details and further references see [7, 23, 53].

Finally, as pointed out in [22], an alternative direction in inverse scattering theory than that discussed above is to only try to obtain lower and upper bounds on a few relevant features of the scattering object rather than attempting a complete reconstruction. This relatively new direction in inverse scattering theory will be discussed in Section 5.

# 2 Direct scattering problems

## 2.1 The Helmholtz equation

Most of the basic properties of solutions to the Helmholtz equation (1.4) can be deduced from the *fundamental solution*

$$\Phi(x,y) := \frac{1}{4\pi}\,\frac{e^{ik|x-y|}}{|x-y|}\,, \quad x \neq y. \tag{2.1}$$

For fixed $y \in \mathbb{R}^3$ it satisfies the Helmholtz equation in $\mathbb{R}^3 \setminus \{y\}$. In addition, it satisfies the radiation condition (1.5) uniformly with respect to $y$ on compact subsets of $\mathbb{R}^3$. Physically speaking, the fundamental solution represents an acoustic point source located at the point $y$. In addition to plane waves, point sources will also occur as incident fields in scattering problems.

Green's integral theorems provide basic tools for investigating the Helmholtz equation. As an immediate consequence they imply the *Helmholtz representation*

$$u(x) = \int_{\partial D} \left\{ \frac{\partial u}{\partial \nu}\,(y)\,\Phi(x,y) - u(y)\,\frac{\partial \Phi(x,y)}{\partial \nu(y)} \right\} ds(y), \quad x \in D, \tag{2.2}$$

for solutions $u \in C^2(D) \cap C^1(\bar{D})$ to the Helmholtz equation. The representation (2.2) implies that solutions to the Helmholtz equation inherit analyticity from the fundamental solution. Any solution $u$ to the Helmholtz equation in $D$ satisfying

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma \tag{2.3}$$

for some open subset $\Gamma \subset \partial D$ must vanish identically in $D$. This can be seen via extending the definition of $u$ by (2.2) for $x \in (\mathbb{R}^3 \setminus \bar{D}) \cup \Gamma$. Then, by Green's integral theorem, applied to $u$ and $\Phi(x,\cdot)$, we have $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. Clearly $u$ solves the Helmholtz equation in $(\mathbb{R}^3 \setminus \partial D) \cup \Gamma$ and therefore by analyticity $u = 0$ in $D$ since $D$ and $\mathbb{R}^3 \setminus \bar{D}$ are connected through the gap $\Gamma$ in $\partial D$.

As a consequence of the radiation condition (1.5) the Helmholtz representation is also valid in the exterior domain $\mathbb{R}^3 \setminus \bar{D}$, i.e., we have

$$u(x) = \int_{\partial D} \left\{ u(y)\,\frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}\,(y)\,\Phi(x,y) \right\} ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{2.4}$$

for radiating solutions $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C^1(\mathbb{R}^3 \setminus D)$ to the Helmholtz equation. From (2.4) we observe that radiating solutions $u$ to the Helmholtz equation satisfy *Sommerfeld's finiteness condition*

$$u(x) = O\left(\frac{1}{|x|}\right), \quad |x| \to \infty, \tag{2.5}$$

uniformly for all directions and that the validity of the Sommerfeld radiation condition (1.5) is invariant under translations of the origin.

We are now in a position to introduce the fundamental notion of the *far field pattern* of radiating solutions to the Helmholtz equation.

**Theorem 2.1** *Every radiating solution $u$ to the Helmholtz equation has an asymptotic behavior of the form*

$$u(x) = \frac{e^{ik|x|}}{|x|}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty, \tag{2.6}$$

*uniformly in all directions $\hat{x} = x/|x|$, where the function $u_\infty$ defined on the unit sphere $S^2$ is called the far field pattern of $u$. Under the assumptions of* (2.4) *we have*

$$u_\infty(\hat{x}) = \frac{1}{4\pi}\int_{\partial D}\left\{u(y)\,\frac{\partial e^{-ik\,\hat{x}\cdot y}}{\partial\nu(y)} - \frac{\partial u}{\partial\nu}(y)\,e^{-ik\,\hat{x}\cdot y}\right\}ds(y), \quad \hat{x} \in S^2. \tag{2.7}$$

*Proof.* This follows from (2.4) by using the estimates

$$\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|x|}}{|x|}\left\{e^{-ik\,\hat{x}\cdot y} + O\left(\frac{1}{|x|}\right)\right\}, \quad \frac{\partial}{\partial\nu(y)}\,\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|x|}}{|x|}\left\{\frac{\partial e^{-ik\,\hat{x}\cdot y}}{\partial\nu(y)} + O\left(\frac{1}{|x|}\right)\right\}$$

which hold uniformly for all $y \in \partial D$ and all directions $x/|x|$ as $|x| \to \infty$. □

From the representation (2.7) it follows that the far field pattern is an analytic function on $S^2$. As extension of (2.6) each radiating solution $u$ to the Helmholtz equation has an *Atkinson–Wilcox* expansion of the form

$$u(x) = \frac{e^{ik|x|}}{|x|}\sum_{\ell=0}^{\infty}\frac{1}{|x|^\ell}\,F_\ell\left(\frac{x}{|x|}\right) \tag{2.8}$$

that converges absolutely and uniformly on compact subsets of $\mathbb{R}^3 \setminus B$ where $B \supset \bar{D}$ is a ball centered at the origin. The coefficients in the expansion (2.8) are determined in terms of the far field pattern $F_0 = u_\infty$ by the recursion

$$2ik\ell F_\ell = \ell(\ell-1)F_{\ell-1} + BF_{\ell-1}, \quad \ell = 1, 2, \ldots, \tag{2.9}$$

where $B$ denotes the Laplace–Beltrami operator for the unit sphere. The following consequence of the expansion (2.8) is known as *Rellich's lemma*.

**Lemma 2.2** *Let $u$ be a radiating solution to the Helmholtz equation for which the far field pattern $u_\infty$ vanishes identically. Then $u$ vanishes identically.*

*Proof.* This follows from (2.8) and (2.9) together with the analyticity of solutions to the Helmholtz equation. □

**Corollary 2.3** *Let $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C^1(\mathbb{R}^3 \setminus D)$ be a radiating solution to the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}$ for which*

$$\operatorname{Im} \int_{\partial D} u \, \frac{\partial \bar{u}}{\partial \nu} \, ds \geq 0. \tag{2.10}$$

*Then $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$.*

*Proof.* Using Green's integral theorem, the radiation condition can be utilized to establish that

$$\lim_{r \to \infty} \int_{|x|=r} \left\{ \left| \frac{\partial u}{\partial \nu} \right|^2 + k^2 |u|^2 \right\} ds = -2k \operatorname{Im} \int_{\partial D} u \, \frac{\partial \bar{u}}{\partial \nu} \, ds.$$

Now the assumption (2.10) implies $\lim_{r\to\infty} \int_{|x|=r} |u|^2 \, ds = 0$ and the statement follows from Lemma 2.2. □

Scattering from infinitely long cylindrical obstacles or inhomogeneities leads to the Helmholtz equation in $\mathbb{R}^2$. The two-dimensional case can be used as an approximation for scattering from finitely long cylinders, and more importantly, it can serve as a model case for testing numerical approximation schemes in direct and inverse scattering. Without giving details, we can summarize that our analysis remains valid in two dimensions after appropriate modifications of the fundamental solution and the radiation condition. The fundamental solution to the Helmholtz equation in two dimensions is given by

$$\Phi(x, y) := \frac{i}{4} \, H_0^{(1)}(k|x - y|), \quad x \neq y, \tag{2.11}$$

in terms of the Hankel function $H_0^{(1)}$ of the first kind of order zero. In $\mathbb{R}^2$ the Sommerfeld radiation condition has to be replaced by

$$\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u}{\partial r} - iku \right) = 0, \quad r = |x|, \tag{2.12}$$

uniformly for all directions $x/|x|$. According to the form (2.12) of the radiation condition, the definition of the far field pattern (2.6) has to be replaced by

$$u(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} \left\{ u_\infty(\hat{x}) + O\left( \frac{1}{|x|} \right) \right\}, \quad |x| \to \infty, \tag{2.13}$$

and the representation (2.7) assumes the form

$$u_\infty(\hat{x}) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{8\pi k}} \int_{\partial D} \left\{ u(y)\, \frac{\partial e^{-ik\,\hat{x}\cdot y}}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}\,(y)\, e^{-ik\,\hat{x}\cdot y} \right\} ds(y) \tag{2.14}$$

for $\hat{x} = x/|x|$.

## 2.2 Obstacle scattering

After renaming the unknown functions, the direct scattering problem for sound-soft obstacles is a special case of the following exterior Dirichlet problem: Given a function $f \in C(\partial D)$, find a radiating solution $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ to the Helmholtz equation that satisfies the boundary condition

$$u = f \quad \text{on } \partial D. \tag{2.15}$$

**Theorem 2.4** *The exterior Dirichlet problem for the Helmholtz equation has at most one solution.*

*Proof.* Let $u$ satisfy the homogeneous boundary condition $u = 0$ on $\partial D$. If $u$ were continuously differentiable up to the boundary, we could immediately apply Corollary 2.3 to obtain $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. However, in the formulation of the exterior Dirichlet problem $u$ is only assumed to be in $C(\mathbb{R}^3 \setminus D)$. We refrain from discussing possibilities to overcome this regularity gap and refer to the literature [23]. □

**Theorem 2.5** *The exterior Dirichlet problem has a unique solution.*

*Proof.* The existence of a solution can be elegantly based on boundary integral equations. In the layer approach, one tries to find the solution in the form of a combined acoustic double and single-layer potential

$$u(x) = \int_{\partial D} \left\{ \frac{\partial \Phi(x,y)}{\partial \nu(y)} - i\Phi(x,y) \right\} \varphi(y)\, ds(y) \tag{2.16}$$

for $x \in \mathbb{R}^3 \setminus \bar{D}$ with a density $\varphi \in C(\partial D)$. Then, after introducing the single and double-layer integral operators $S, K : C(\partial D) \to C(\partial D)$ by

$$(S\varphi)(x) := 2 \int_{\partial D} \Phi(x,y)\varphi(y)\, ds(y), \quad x \in \partial D, \tag{2.17}$$

$$(K\varphi)(x) := 2 \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, \varphi(y)\, ds(y), \quad x \in \partial D, \tag{2.18}$$

and using their regularity and jump relations it can be seen that (2.16) solves the exterior Dirichlet problem provided the density $\varphi$ is a solution of the integral equation

$$\varphi + K\varphi - iS\varphi = 2f. \tag{2.19}$$

Due to their weakly singular kernels the operators $S$ and $K$ turn out to be compact. Hence, the existence of a solution to (2.19) can be established with the aid of the Riesz–Fredholm theory for compact operators by showing that the homogeneous form of (2.19) only allows the trivial solution $\varphi = 0$.

Let $\varphi$ be a solution of the homogeneous equation and let the subscripts $\pm$ denote the limits obtained by approaching $\partial D$ from $\mathbb{R}^3 \setminus \bar{D}$ and $D$, respectively. Then the potential $u$ defined by (2.16) in all of $\mathbb{R}^3 \setminus \partial D$ satisfies the homogeneous boundary condition $u_+ = 0$ on $\partial D$ whence $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ follows by Theorem 2.4. The jump relations for single- and double-layer potentials now yield

$$-u_- = \varphi, \quad -\frac{\partial u_-}{\partial \nu} = i\varphi \quad \text{on } \partial D.$$

Hence, using Green's first integral theorem, we obtain

$$i \int_{\partial D} |\varphi|^2 ds = \int_{\partial D} \bar{u}_- \frac{\partial u_-}{\partial \nu}\, ds = \int_D \left\{ |\operatorname{grad} u|^2 - k^2 |u|^2 \right\} dx,$$

and taking the imaginary part yields $\varphi = 0$. □

We note that, in addition to existence of a solution, the Riesz–Fredholm theory also establishes well-posedness, i.e., the continuous dependence of the solution on the data. Instead of the classical setting of continuous functions spaces, the existence analysis can also be considered in the Sobolev space $H^{1/2}(\partial D)$ for the boundary integral operators leading to solutions in the energy space $H^1_{\text{loc}}(\mathbb{R}^3 \setminus \bar{D})$ (see [65, 70]).

We further note that without the single-layer potential included in (2.16) the corresponding double-layer integral equation suffers from non-uniqueness if $k$ is a so-called irregular wave number or internal resonance, i.e., if there exist nontrivial solutions $u$ to the Helmholtz equation in the interior domain $D$ satisfying homogeneous Neumann boundary conditions $\partial u / \partial \nu = 0$ on $\partial D$.

For the numerical solution of the boundary integral equations in scattering theory via spectral methods in two and three dimensions we refer to [23]. For boundary element methods we refer to [81].

In general, for the scattering problem the boundary values are as smooth as the boundary, since they are given by the restriction of the analytic function $u^i$ to $\partial D$. Therefore, we may use the Helmholtz representation (2.4) and Green's second integral theorem applied to $u^i$ and $\Phi(x, \cdot)$ to obtain the following theorem:

**Theorem 2.6** *For scattering from a sound-soft obstacle $D$ we have*

$$u(x) = u^i(x) - \int_{\partial D} \frac{\partial u}{\partial \nu}(y)\, \Phi(x,y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{2.20}$$

*and the far field pattern of the scattered field $u^s$ is given by*

$$u_\infty(\hat{x}) = -\frac{1}{4\pi} \int_{\partial D} \frac{\partial u}{\partial \nu}(y)\, e^{-ik\,\hat{x}\cdot y}\, ds(y), \quad \hat{x} \in S^2. \tag{2.21}$$

The representation (2.20) for the scattered field through the so-called secondary sources on the boundary is known as *Huygens' principle*. Here we will use it for the motivation of the *Kirchhoff* or *physical optics approximation* as an intuitive procedure to simplify the direct scattering problem. For large wave numbers $k$, i.e., for small wave lengths, in a first approximation a convex object $D$ locally may be considered at each point $x$ of $\partial D$ as a plane with normal $\nu(x)$. This suggests

$$\frac{\partial u}{\partial \nu} = 2\, \frac{\partial u^i}{\partial \nu}$$

on the part $\partial D_- := \{x \in \partial D : \nu(x) \cdot d < 0\}$ that is illuminated and

$$\frac{\partial u}{\partial \nu} = 0$$

in the shadow region $\partial D_+ := \{x \in \partial D : \nu(x) \cdot d \geq 0\}$. Thus the Kirchhoff approximation for the scattering of a plane wave with incident direction $d$ at a convex sound-soft obstacle is given by

$$u(x) \approx e^{ik\,x\cdot d} - 2 \int_{\partial D_-} \frac{\partial e^{ik\,y\cdot d}}{\partial \nu(y)}\, \Phi(x,y)\, ds(y) \tag{2.22}$$

for $x \in \mathbb{R}^3 \setminus \bar{D}$.

## 2.3 Scattering by an inhomogeneous medium

Recall the scattering problem for an inhomogeneous medium with refractive index $n$ as described by (1.9) for the total wave $u = u^i + u^s$ with incident field $u^i$ and the scattered wave $u^s$ satisfying the Sommerfeld radiation condition. The function $m := 1 - n$ has support $\bar{D}$.

The counterpart of the Helmholtz representation is given by the *Lippmann-Schwinger equation*

$$u(x) = u^i - k^2 \int_D \Phi(x,y) m(y) u(y)\, dy, \quad x \in \mathbb{R}^3, \tag{2.23}$$

that can be shown to be equivalent to the scattering problem.

In order to establish existence of a solution to (2.23) via the Riesz–Fredholm theory it must be shown that the homogeneous equation has only the trivial solution, or equivalently, that the only solution to (1.9) satisfying the radiation condition is identically zero. For this, in addition to Rellich's lemma, the following *unique continuation principle* is required: Any solution $u \in C^2(G)$ of the equation (1.9) in a domain $G \subset \mathbb{R}^3$ such that $n \in C(G)$ and $u$ vanishes in an open subset of $G$ vanishes identically. Hence we have the following result on existence and uniqueness for the inhomogeneous medium scattering problem.

**Theorem 2.7** *For a refractive index $n \in C^1(\mathbb{R}^3)$ with $\operatorname{Re} n \geq 0$ and $\operatorname{Im} n \geq 0$ the Lippmann–Schwinger equation or, equivalently, the inhomogeneous medium scattering problem has a unique solution.*

*Proof.* From Green's first integral theorem it follows that

$$\int_{\partial D} u \frac{\partial \bar{u}}{\partial \nu}\, ds = \int_D \left\{ |\operatorname{grad} u|^2 - k^2 \overline{n} |u|^2 \right\} dx.$$

Taking the imaginary part and applying Corollary 2.3 yields $u = 0$ in $\mathbb{R}^3 \setminus D$ in view of the assumptions on $n$ and the proof is finished by the unique continuation principle. □

From (2.23) we see that

$$u^s(x) = -k^2 \int_{\mathbb{R}^3} \Phi(x,y) m(y) u(y)\, dy, \quad x \in \mathbb{R}^3.$$

Hence, the far field pattern $u_\infty$ is given by

$$u_\infty(\hat{x}) = -\frac{k^2}{4\pi} \int_{\mathbb{R}^3} e^{-ik\,\hat{x}\cdot y} m(y) u(y)\, dy, \quad \hat{x} \in S^2. \tag{2.24}$$

We note that for $k^2 \|m\|_\infty$ sufficiently small, $u$ can be obtained by the method of successive approximations. If in (2.24) we replace $u$ by the first term in this iterative process, we obtain the *Born approximation*

$$u_\infty(\hat{x}) \approx -\frac{k^2}{4\pi} \int_{\mathbb{R}^3} e^{-ik\,\hat{x}\cdot y} m(y) u^i(y)\, dy, \quad \hat{x} \in S^2. \tag{2.25}$$

For numerical solutions of the inverse medium scattering problem by finite element methods coupled with boundary element methods via nonlocal boundary conditions we refer to [66].

## 2.4 The Maxwell equations

We now consider the *Maxwell equations* as the foundation of electromagnetic scattering theory. Our presentation is organized parallel to that on the Helmholtz equation, i.e., on acoustic scattering, and will be confined to homogeneous isotropic media. Consider electromagnetic wave propagation in an isotropic dielectric medium in $\mathbb{R}^3$ with constant electric permittivity $\varepsilon$ and magnetic permeability $\mu$. The electromagnetic wave is described by the electric field $\mathcal{E}$ and the magnetic field $\mathcal{H}$ satisfying the time dependent Maxwell equations

$$\operatorname{curl}\mathcal{E} + \mu\,\frac{\partial\mathcal{H}}{\partial t} = 0, \quad \operatorname{curl}\mathcal{H} - \varepsilon\,\frac{\partial\mathcal{E}}{\partial t} = 0. \tag{2.26}$$

For time-harmonic electromagnetic waves of the form

$$\mathcal{E}(x,t) = \operatorname{Re}\left\{\varepsilon^{-1/2}E(x)\,e^{-i\omega t}\right\}, \quad \mathcal{H}(x,t) = \operatorname{Re}\left\{\mu^{-1/2}H(x)\,e^{-i\omega t}\right\} \tag{2.27}$$

with frequency $\omega > 0$ the complex-valued space dependent parts $E$ and $H$ satisfy the reduced Maxwell equations

$$\operatorname{curl}E - ikH = 0, \quad \operatorname{curl}H + ikE = 0, \tag{2.28}$$

where the wave number $k$ is given by the positive constant $k = \sqrt{\varepsilon\mu}\,\omega$. We will only be concerned with the reduced Maxwell equations and will henceforth refer to them as the Maxwell equations.

A solution $E, H$ to the Maxwell equations whose domain of definition contains the exterior of some sphere is called radiating if it satisfies one of the *Silver–Müller radiation conditions*

$$\lim_{r\to\infty}(H \times x - rE) = 0 \tag{2.29}$$

or

$$\lim_{r\to\infty}(E \times x + rH) = 0, \tag{2.30}$$

where $r = |x|$ and the limits hold uniformly in all directions $x/|x|$. For more details on the physical background of electromagnetic waves, we refer to [47, 68].

For the Maxwell equations the counterpart of the Helmholtz representation (2.2) is given by the *Stratton–Chu formula*

$$\begin{aligned} E(x) = {} & -\operatorname{curl}\int_{\partial D}\nu(y)\times E(y)\,\Phi(x,y)\,ds(y) \\ & +\frac{1}{ik}\operatorname{curl}\operatorname{curl}\int_{\partial D}\nu(y)\times H(y)\,\Phi(x,y)\,ds(y), \quad x \in D, \end{aligned} \tag{2.31}$$

for solutions $E, H \in C^1(D) \cap C(\bar{D})$ to the Maxwell equations. A corresponding representation for $H$ can be obtained from (2.31) with the aid of $H = \operatorname{curl} E/ik$.

The representation (2.31) implies that each continuously differentiable solution to the Maxwell equations automatically has analytic Cartesian components. Therefore one can employ the vector identity $\operatorname{curl}\operatorname{curl} E = -\Delta E + \operatorname{grad}\operatorname{div} E$ to prove that for a solution $E, H$ to the Maxwell equations both $E$ and $H$ are divergence free and satisfy the vector Helmholtz equation. Conversely, if $E$ is a solution to the vector Helmholtz equation $\triangle E + k^2 E = 0$ satisfying $\operatorname{div} E = 0$, then $E$ and $H := \operatorname{curl} E/ik$ satisfy the Maxwell equations.

It can be shown that solutions $E, H$ to the Maxwell equations in $D$ satisfying

$$\nu \times E = \nu \times H = 0 \quad \text{on } \Gamma \tag{2.32}$$

for some open subset $\Gamma \subset \partial D$ must vanish identically in $D$.

As a consequence of the Silver–Müller radiation condition the Stratton–Chu formula is also valid in the exterior domain $\mathbb{R}^3 \setminus \bar{D}$, i.e., we have

$$\begin{aligned} E(x) = \operatorname{curl} \int_{\partial D} \nu(y) \times E(y)\, \Phi(x,y)\, ds(y) \\ -\frac{1}{ik} \operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times H(y)\, \Phi(x,y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \end{aligned} \tag{2.33}$$

for radiating solutions $E, H \in C^1(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ to the Maxwell equations. Again, a corresponding representation for $H$ can be obtained from (2.33) with the aid of $H = \operatorname{curl} E/ik$.

From (2.33) it can be seen that the radiation condition (2.29) implies (2.30) and vice versa. Furthermore, one can deduce that radiating solutions $E, H$ to the Maxwell equations automatically satisfy the *Silver–Müller finiteness conditions*

$$E(x) = O\left(\frac{1}{|x|}\right), \quad H(x) = O\left(\frac{1}{|x|}\right), \quad |x| \to \infty, \tag{2.34}$$

uniformly for all directions and that the validity of the Silver–Müller radiation conditions (2.29) and (2.30) is invariant under translations of the origin. From the Helmholtz representation (2.4) for radiating solutions to the Helmholtz equation and the Stratton–Chu formulas for radiating solutions to the Maxwell equations, it can be deduced that for solutions to the Maxwell equations the Silver–Müller radiation condition is equivalent to the Sommerfeld radiation condition for the Cartesian components of $E$ and $H$.

The Stratton–Chu formula (2.33) can be used to introduce the notion of the *electric* and *magnetic far field patterns*.

**Theorem 2.8** *Every radiating solution $E, H$ to the Maxwell equations has the asymptotic form*

$$E(x) = \frac{e^{ik|x|}}{|x|}\left\{E_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad H(x) = \frac{e^{ik|x|}}{|x|}\left\{H_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\} \tag{2.35}$$

*for $|x| \to \infty$ uniformly in all directions $\hat{x} = x/|x|$, where the vector fields $E_\infty$ and $H_\infty$ defined on the unit sphere $S^2$ are called the electric far field pattern and magnetic far field pattern, respectively. They satisfy*

$$H_\infty = \nu \times E_\infty \quad \textit{and} \quad \nu \cdot E_\infty = \nu \cdot H_\infty = 0 \tag{2.36}$$

*with the unit outward normal $\nu$ on $S^2$. Under the assumptions of* (2.33) *we have*

$$E_\infty(\hat{x}) = \frac{ik}{4\pi}\,\hat{x} \times \int_{\partial D} \{\nu(y) \times E(y) + [\nu(y) \times H(y)] \times \hat{x}\}\, e^{-ik\,\hat{x}\cdot y}\, ds(y) \tag{2.37}$$

*for $\hat{x} \in S^2$ and a corresponding expression for $H_\infty$.*

*Rellich's lemma* carries over immediately from the Helmholtz to the Maxwell equations:

**Lemma 2.9** *Let $E, H$ be a radiating solution to the Maxwell equations for which the electric far field pattern $E_\infty$ vanishes identically. Then $E$ and $H$ vanish identically.*

The electromagnetic counterpart of Corollary 2.3 is given by the following result:

**Corollary 2.10** *Let $E, H \in C^1(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ be a radiating solution to the Maxwell equations in $\mathbb{R}^3 \setminus \bar{D}$ for which*

$$\mathrm{Re} \int_{\partial D} \nu \cdot E \times \bar{H}\, ds \le 0.$$

*Then $E = H = 0$ in $\mathbb{R}^3 \setminus \bar{D}$.*

For two vectors $d, p \in \mathbb{R}^3$ with $|d| = 1$ and $p \cdot d = 0$ the *plane waves*

$$E^i(x) = p\, e^{ik\,x\cdot d}, \quad H^i(x) = d \times p\, e^{ik\,x\cdot d} \tag{2.38}$$

satisfy the Maxwell equations for all $x \in \mathbb{R}^3$. The orthogonal vectors $p$ and $d \times p$ describe the polarization direction of the electric and the magnetic field, respectively. Given the incident field $E^i, H^i$ and a bounded domain $D \subset \mathbb{R}^3$, the simplest obstacle scattering problem is to find the scattered field $E^s, H^s$ as a radiating solution to the Maxwell equations in the exterior of the scatterer $D$ such that the total field $E = E^i + E^s,\ H = H^i + H^s$ satisfies the perfect conductor boundary condition

$$\nu \times E = 0 \quad \text{on } \partial D, \tag{2.39}$$

where $\nu$ is the outward unit normal to $\partial D$. A more general boundary condition is the impedance or Leontovich boundary condition

$$\nu \times H - \lambda\,(\nu \times E) \times \nu = 0 \quad \text{on } \partial D, \tag{2.40}$$

where $\lambda$ is a positive constant or function called the surface impedance.

**Theorem 2.11** *The scattering problem for a perfect conductor has a unique solution.*

*Proof.* Uniqueness follows from Corollary 2.10. The existence of a solution again can be based on boundary integral equations. In the layer approach, one tries to find the solution in the form of a combined magnetic and electric dipole distribution

$$\begin{aligned} E^s(x) = \operatorname{curl} \int_{\partial D} a(y)\Phi(x,y)\,ds(y) \\ + i \operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times (S_0^2 a)(y)\Phi(x,y)\,ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D. \end{aligned} \tag{2.41}$$

Here $S_0$ denotes the single-layer operator (2.17) in the potential theoretic limit case $k = 0$ and the density $a$ is assumed to belong to the space $C^{0,\alpha}_{\mathrm{div}}(\partial D)$ of Hölder continuous tangential fields with Hölder continuous surface divergence. After defining the electromagnetic boundary integral operators $M$ and $N$ by

$$(Ma)(x) := 2 \int_{\partial D} \nu(x) \times \operatorname{curl}_x \{a(y)\Phi(x,y)\}\,ds(y), \quad x \in \partial D, \tag{2.42}$$

and

$$(Na)(x) := 2\,\nu(x) \times \operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times a(y)\,\Phi(x,y)\,ds(y), \quad x \in \partial D, \tag{2.43}$$

it can be shown, that $E^s$ given by (2.41) together with $H^s = \operatorname{curl} E^s/ik$ solves the perfect conductor scattering problem provided the density $a$ satisfies the integral equation

$$a + Ma + iNPS_0^2 a = -2\,\nu \times E^i. \tag{2.44}$$

Here the operator $P$ is defined by $Pb := (\nu \times b) \times \nu$ for (not necessarily tangential) vector fields $b$. Exploiting the smoothing properties of the operator $S_0$ it can be shown that $M + iNPS_0^2$ is a compact operator from $C^{0,\alpha}_{\rm div}(\partial D)$ into itself. The existence of a solution to (2.44) can now be based on the Riesz–Fredholm theory by establishing that the homogeneous form of (2.44) only has the trivial solution [23]. □

Note that, analogous to the acoustic case, without the electric dipole distribution included in (2.41) the corresponding magnetic dipole integral equation is not uniquely solvable if $k$ is a irregular wave number, i.e., if there exists a nontrivial solution $E, H$ to the Maxwell equations in $D$ satisfying the homogeneous boundary condition $\nu \times E = 0$ on $\partial D$.

Instead of the classical setting of Hölder continuous functions the integral equation can also be considered in the Sobolev space $H^{1/2}_{\rm div}(\partial D)$ of tangential fields in $H^{1/2}(\partial D)$ that have a weak surface divergence in $H^{1/2}(\partial D)$ (see [70]).

In addition to electromagnetic obstacle scattering, one can also consider scattering from an inhomogeneous medium where outside a bounded domain $D$ the electric permittivity $\varepsilon$ and magnetic permeability $\mu$ are constant and the conductivity $\sigma$ vanishes, i.e., $\varepsilon = \varepsilon_0$, $\mu = \mu_0$ and $\sigma = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. For simplicity we will assume that the magnetic permeability is constant throughout $\mathbb{R}^3$. Then, again assuming the time harmonic form (2.27) with $\varepsilon$ and $\mu$ replaced by $\varepsilon_0$ and $\mu_0$, respectively, the total fields $E = E^i + E^s$, $H = H^i + H^s$ satisfy

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + iknE = 0 \quad \text{in } \mathbb{R}^3 \tag{2.45}$$

and the scattered wave $E^s, H^s$ satisfies the Silver–Müller radiation condition, where the wave number is given by $k = \sqrt{\varepsilon_0 \mu_0}\,\omega$ and $n = (\varepsilon + i\,\sigma/\omega)/\varepsilon_0$ is the refractive index. Establishing uniqueness requires an electromagnetic analogue of the unique continuation principle and existence can be based on an electromagnetic variant of the Lippman–Schwinger equation [23].

The scattering of time-harmonic electromagnetic waves by an infinitely long cylinder with a simply connected cross section $D$ leads to boundary value problems for the two-dimensional Helmholtz equation in the exterior $\mathbb{R}^2 \setminus \bar{D}$ of $D$. If the electric field is polarized parallel to the axis of the cylinder and if the axis of the cylinder is parallel to the $x_3$ axis, then

$$E = (0, 0, u), \quad H = \frac{1}{ik}\left(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1}, 0\right)$$

satisfies the Maxwell equations if and only if $u = u(x_1, x_2)$ satisfies the Helmholtz equation. The homogeneous perfect conductor boundary condition is satisfied on the boundary of the cylinder if the homogeneous Dirichlet boundary condition $u = 0$ on $\partial D$ is satisfied. If the magnetic field is polarized parallel to the axis of the cylinder then the roles of $E$ and $H$ have to be reversed, i.e.,

$$H = (0, 0, u), \quad E = \frac{i}{k}\left(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1}, 0\right),$$

and the perfect conductor boundary condition corresponds to the Neumann boundary condition $\partial u/\partial\nu = 0$ on $\partial D$ with the unit normal $\nu$ to the boundary $\partial D$ of the cross section $D$. Hence the analysis of two-dimensional electromagnetic scattering problems coincides with that of two-dimensional acoustic scattering problems.

## 2.5 Historical remarks

Equation (1.4) carries the name of Helmholtz (1821–1894) for his contributions to mathematical acoustics. The radiation condition (1.5) was introduced by Sommerfeld in 1912 to characterize an outward energy flux. The expansion (2.8) was first established by Atkinson in 1949 and generalized by Wilcox in 1956. The fundamental Lemma 2.2 is due to Rellich (1943) and Vekua (1943). The combined single and double-layer approach (2.16) for the existence analysis was introduced independently by Leis, Brakhage and Werner, and Panich in the 1960s in order to remedy the non-uniqueness deficiency of the classical double-layer approach due to Vekua, Weyl and Müller from the 1950s. Huygens' principle is also referred to as the Huygens–Fresnel principle and named for Huygens (1629–1695) and Fresnel (1788–1827) in recognition of their contributions to wave optics. The physical optics approximation (2.22) is named for Kirchhoff (1824–1887) for his contributions to optics. The terms Lippmann–Schwinger equation and Born approximation are adopted from quantum physics. The equations (2.26) are named for Maxwell (1831–1879) for his fundamental contributions to electromagnetic theory. The radiation conditions (2.29) and (2.30) were independently introduced in the 1940s by Silver and Müller. The integral representations (2.31) and (2.33) were first presented by Stratton and Chu in 1939. Extending the ideas of Leis, Brakhage and Werner, and Panich from acoustics to electromagnetics, the approach (2.41) was introduced independently by Knauff and Kress, Jones, and Mautz and Harrington in the 1970s in order to remedy the non-uniqueness deficiency of the classical approach due to Weyl and Müller from the 1950s.

# 3 Uniqueness in inverse scattering

## 3.1 Scattering by an obstacle

The first step in studying any inverse scattering problem is to establish a uniqueness result, i.e., if a given set of data is known exactly does this data uniquely determine the support and/or the material properties of the scatterer? We will begin with the case of scattering by an impenetrable obstacle and then proceed to the case of a penetrable obstacle.

From Section 2.2 we recall that the direct obstacle scattering problem is to find $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ such that the total field $u = u^i + u^s$ satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D} \tag{3.1}$$

and the sound-soft boundary condition

$$u = 0 \quad \text{on } \partial D \tag{3.2}$$

where $u^i(x) = e^{ik\,x\cdot d}$, $|d| = 1$, and $u^s$ is a radiating solution. We also recall from Theorem 2.1 that $u^s$ has the asymptotic behavior

$$u^s(x,d) = \frac{e^{ik|x|}}{|x|}\left\{u_\infty(\hat{x},d) + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty, \tag{3.3}$$

uniformly for all directions $\hat{x} = x/|x|$, where $u_\infty$ is the far field pattern of the scattered field $u^s$. By Green's integral theorem and the far field representation (2.21) it can be shown that the far field pattern satisfies the *reciprocity relation* [23]

$$u_\infty(\hat{x},d) = u_\infty(-d,-\hat{x}), \quad \hat{x}, d \in S^2. \tag{3.4}$$

The *inverse scattering problem* we are concerned with is to determine $D$ from a knowledge of $u_\infty(\hat{x},d)$ for $\hat{x}$ and $d$ on the unit sphere $S^2$ and fixed wave number $k$. In particular, for the acoustic scattering problem (3.1)–(3.2) we want to show that $D$ is uniquely determined by $u_\infty(\hat{x},d)$ for $\hat{x}, d \in S^2$. We note that by the reciprocity relation (3.4) the far field pattern $u_\infty$ is an analytic function of both $\hat{x}$ and $d$ and hence it would suffice to consider $u_\infty$ for $\hat{x}$ and $d$ restricted to a surface patch of the unit sphere $S^2$.

**Theorem 3.1** *Assume that $D_1$ and $D_2$ are two obstacles such that the far field patterns corresponding to the exterior Dirichlet problem* (3.1)–(3.2) *for $D_1$ and $D_2$ coincide for all incident directions $d$. Then $D_1 = D_2$.*

*Proof.* Let $u_1^s$ and $u_2^s$ be the scattered fields corresponding to $D_1$ and $D_2$ respectively. By the analyticity of the scattered field as a function of $x$ and Rellich's Lemma 2.2, the scattered fields satisfy $u_1^s(\cdot,d) = u_2^s(\cdot,d)$ in the unbounded component $G$ of the complement of $\overline{D_1} \cup \overline{D_2}$ for all $d \in S^2$. This in turn implies that the scattered fields corresponding to $\Phi(\cdot,z)$ as incident field and $D_1$ or $D_2$ as the scattering obstacle satisfy $u_1^s(x,z) = u_2^s(x,z)$ for all $x, z \in G$. Now assume that

$D_1 \neq D_2$. Then, without loss of generality, there exists $x^* \in \partial G$ such that $x^* \in \partial D_1$ and $x^* \notin \overline{D_2}$. Then setting $z_n := x^* + \frac{1}{n}\nu(x^*)$ we have that $\lim_{n\to\infty} u_2^s(x^*, z_n)$ exists but $\lim_{n\to\infty} u_1^s(x^*, z_n) = \infty$ which is a contradiction and hence $D_1 = D_2$. □

An open problem is to determine if one incident plane wave at a fixed wave number $k$ is sufficient to uniquely determine the scatterer $D$. If it is known a priori that in addition to the sound–soft boundary condition (3.2) that $D$ is contained in a ball of radius $R$ such that $kR < 4.49$ then $D$ is uniquely determined by its far field pattern for a single incident direction $d$ and fixed wave number $k$ [33] (see also [23]). $D$ is also uniquely determined if instead of assuming that $D$ is contained in a ball of sufficiently small radius it is assumed that $D$ is close to a given obstacle [88]. It is also known that for a wide class of sound–soft scatterers a finite number of incident fields is sufficient to uniquely determine $D$ [83]. Finally, if it is assumed that $D$ is polyhedral, then a single incident plane wave is sufficient to uniquely determine $D$ [1], [64].

We conclude this section on uniqueness results for the inverse scattering problem for an obstacle by considering the scattering of electromagnetic waves by a perfectly conducting obstacle $D$. From Section 2.4 we recall that the direct obstacle scattering problem is to find $E, H \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ such that the total field $E = E^i + E^s$, $H = H^i + H^s$ satisfies the Maxwell equations

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikE = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D} \tag{3.5}$$

and the perfect conductor boundary condition

$$\nu \times E = 0 \quad \text{on } \partial D \tag{3.6}$$

where $E^i, H^i$ is the plane wave given by (2.38) and $E^s, H^s$ is a radiating solution. We also recall from Theorem 2.1 that $u^s$ has the asymptotic behavior

$$E^s(x, d, p) = \frac{e^{ik|x|}}{|x|}\left\{E_\infty(\hat{x}, d, p) + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty, \tag{3.7}$$

where $E_\infty$ is the electric far field pattern of the scattered field $u^s$.

The inverse scattering problem is to determine $D$ from a knowledge of $E_\infty(\hat{x}, d, p)$ for $\hat{x}$ and $d$ on the unit sphere $S^2$, three linearly independent polarizations $p$ and fixed wave number $k$. We note that $E_\infty$ is an analytic function of $\hat{x}$ and $d$ and is linear with respect to $p$. The following theorem can be proved using the same ideas as in the proof of Theorem 3.1.

**Theorem 3.2** *Assume that $D_1$ and $D_2$ are two perfect conductors such that for a fixed wave number $k$ the electric far field patterns for both scatterers coincide for all incident directions $d$ and three linearly independent polarizations $p$. Then $D_1 = D_2$.*

In the case when $D$ consists of finitely many polyhedra, a single incident wave is sufficient to uniquely determine $D$ [63].

## 3.2 Scattering by an inhomogeneous medium

We now return to scattering of acoustic waves but instead of scattering by a sound–soft obstacle we consider scattering by an inhomogeneous medium where the governing equation (see Section 2.3) is

$$\Delta u + k^2 n u = 0 \quad \text{in } \mathbb{R}^3 \tag{3.8}$$

for $u = u^i + u^s \in C^2(\mathbb{R}^2)$ where $n \in C^1(\mathbb{R}^3)$ is the refractive index satisfying $\operatorname{Re} n > 0$ and $\operatorname{Im} n \geq 0$, $u^i(x) = e^{ik\,x\cdot d}$ and $u^s$ is radiating. We let $\bar{D}$ denote the support of $m := 1 - n$. By Theorem 2.1 the scattered wave $u^s$ again has the asymptotic behavior (3.3). The inverse scattering problem we are now concerned with is to determine the index of refraction $n$ (and hence $D$) from a knowledge of $u_\infty(\hat{x}, d)$ for $\hat{x}$ and $d$ on the unit sphere $S^2$ and fixed wave number $k$. In particular, we want to show that $n$ is uniquely determined from $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in S^2$ and fixed wave number $k$.

**Theorem 3.3** *The refractive index $n$ in* (3.8) *is uniquely determined by $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in S^2$ and a fixed value of the wave number $k$.*

*Proof.* Let $B$ be an open ball centered at the origin and containing the support of $m = 1 - n$. The first step in the proof is to construct a solution of (3.8) in $B$ of the form

$$w(x) = e^{i\,z\cdot x}\,(1 + r(x)) \tag{3.9}$$

where $z \cdot z = 0,\ z \in \mathbb{C}^3$ and

$$\|r\|_{L^2(B)} \leq \frac{C}{|\operatorname{Re} z|}$$

for some positive constant $C$ and $|\operatorname{Re} z|$ sufficiently large. This is done in [36] by using Fourier series. The second step is to show that, given two open balls $B_1$ and $B_2$ centered at the origin and containing the support of $m$ such that $\overline{B_1} \subset B_2$, then the set of solutions $\{u(\cdot, d) : d \in S^2\}$ satisfying (3.8) is complete in

$$H := \left\{ w \in C^2(B_2) : \Delta w + k^2 n w = 0 \quad \text{in } B_2 \right\}$$

with respect to the norm in $L^2(B_1)$ [36]. Now assume that $n_1$ and $n_2$ are refractive indices such that the corresponding far field patterns satisfy $u_{1,\infty}(\cdot, d) = u_{2,\infty}(\cdot, d), d \in$

$S^2$, and assume that the supports of $1 - n_1$ and $1 - n_2$ are contained in $\overline{B_1}$. Then using Rellich's Lemma 2.2 and Green's integral theorem it can be shown that

$$\int_{B_1} u_1(\cdot, \tilde{d}) u_2(\cdot, d)(n_1 - n_2)\, dx = 0$$

for all $d, \tilde{d} \in S^2$ and hence

$$\int_{B_1} w_1 w_2 (n_1 - n_2)\, dx = 0 \tag{3.10}$$

for all solutions $w_1, w_2 \in C^2(B_2)$ of $\Delta w_1 + k^2 n_1 w = 0$ and $\Delta w_2 + k^2 n_2 w_2 = 0$ in $B_2$. Now choose $z_1 := y + \rho a + ib$ and $z_2 := y - \rho a - ib$ such that $\{y, a, b\}$ is an orthogonal basis in $\mathbb{R}^3$ with the properties that $|a| = 1$ and $|b|^2 = |y|^2 + \rho^2$ and substitute those values of $z$ into (3.9) arriving at functions $w_1$ and $w_2$. Substitute these functions into (3.10) and let $\rho \to \infty$ to arrive at

$$\int_{B_1} e^{2i\, y \cdot x} \left( n_1(x) - n_2(x) \right)\, dx = 0$$

for arbitrary $y \in \mathbb{R}^3$, i.e., $n_1(x) = n_2(x)$ for $x \in B_1$ by the Fourier integral theorem. □

In the case of scattering by a sound-soft obstacle, the proof of uniqueness given in Theorem 3.1 remains valid in $\mathbb{R}^2$. However this is not the case for scattering by an inhomogeneous medium. Indeed until recently the question of whether or not Theorem 3.3 remains valid in $\mathbb{R}^2$ was one of the outstanding open problems in inverse scattering theory. The problem was finally resolved in 2008 by Bukhgeim [5] who showed that in $\mathbb{R}^2$ the index of refraction $n$ is uniquely determined by $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in S^1$ and a fixed value of the wave number $k$.

We conclude this section with a few remarks on scattering by an anisotropic medium. Let $n$ be as above and recall that $\bar{D}$ is the support of $m := 1 - n$. Let $A$ be a $3 \times 3$ matrix-valued function whose entries $a_{jk}$ are continuously differentiable functions in $\bar{D}$ such that $A$ is symmetric and satisfies

$$\overline{\xi} \cdot (\operatorname{Im} A)\xi \le 0, \quad \overline{\xi} \cdot (\operatorname{Re} A)\xi > \gamma |\xi|^2$$

for all $\xi \in \mathbb{C}^3$ and $x \in D$ where $\gamma$ is a positive constant. We assume that $A(x) = I$ for $x \in \mathbb{R}^3 \setminus \bar{D}$. The anisotropic scattering problem is to find $u = u^i + u^s \in H^1_{\text{loc}}(\mathbb{R}^3)$ such that

$$\nabla \cdot A \nabla u + k^2 n u = 0 \quad \text{in } \mathbb{R}^3 \tag{3.11}$$

in the weak sense where again $u^i(x) = e^{ik\, x \cdot d}$ and $u^s$ is radiating. The existence of a unique solution to this scattering problem has been established by Hähner [37].

The scattered field again has the asymptotics (3.3). The inverse scattering problem is now to determine $D$ from a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in S^2$. We note that the matrix $A$ is not uniquely determined by $u_\infty$ and hence without further a priori assumptions the determination of $D$ is the most that can be hoped for [34, 75]. To this end we have the following theorem due to Hähner [37].

**Theorem 3.4** *Assume $\gamma > 1$. Then $D$ is uniquely determined by $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in S^2$.*

We note that Theorem 3.4 remains valid if the condition on $\operatorname{Re} A$ is replaced by the condition

$$\overline{\xi} \cdot (\operatorname{Re} A^{-1})\xi \geq \mu |\xi|^2$$

for all $\xi \in \mathbb{C}^3$ and $x \in \bar{D}$ where $\mu$ is a positive constant such that $\mu > 1$ [7]. Note that the isotropic case when $A = I$ is handled by Theorem 3.3.

Uniqueness theorems for the Maxwell equations in an isotropic inhomogeneous medium have been established by Colton and Päivärinta [28] and Hähner [38]. The proof is similar to that of Theorem 3.3 for the scalar problem except that technical problems arise due to the fact that we must now construct a solution $E, H$ to the Maxwell equations in an inhomogeneous isotropic medium such that $E$ has the form

$$E(x) = e^{iz \cdot x} \left( \eta + r(x) \right)$$

where $z, \eta \in \mathbb{C}^3, \eta \cdot z = 0$ and $z \cdot z = k^2$. In contrast to the case of acoustic waves, it is no longer true that $r(x)$ decays to zero as $|\operatorname{Re} z|$ tends to infinity. Finally, the generalization of Theorem 3.4 to the case of the Maxwell equations in an anisotropic media has been done by Cakoni and Colton [6].

## 3.3 Historical remarks

As previously mentioned, the first uniqueness theorem for the acoustic inverse obstacle problem was given by Schiffer in 1967 for the case of a sound–soft obstacle [62] whereas in 1988 Nachman [69], Novikov [71] and Ramm [80] established a uniqueness result for the inverse scattering problem for an inhomogeneous medium. In 1990 Isakov [41, 42] proved a series of uniqueness theorems for the tranmission problem with discontinuities of $u$ across $\partial D$. His ideas were subsequently utilized by Kirsch, Kress and their co-workers to establish uniqueness theorems for a variety of inverse scattering problems for both acoustic and electromagnetic waves (for references see [23]). In particular the proofs of Theorems 3.1 and 3.2 are based on the ideas of Kirsch and Kress [23, 55].

A global uniqueness theorem for the Maxwell equations in an isotropic inhomogeneous medium was first established in 1992 by Colton and Päivärinta [28] (see also [38]). The results of [28] and [38] are for the case when the magnetic permeability $\mu$ is constant. For uniqueness results in the case when $\mu$ is no longer constant we refer to [72] and [73].

# 4 Iterative and decomposition methods in inverse scattering

## 4.1 Newton iterations in inverse obstacle scattering

We now turn to reconstruction methods for the inverse scattering problem for sound-soft scatterers and as a first group we describe iterative methods. Here the inverse problem is interpreted as a nonlinear ill-posed operator equation which is solved by iteration methods such as regularized Newton methods, Landweber iterations or conjugate gradient methods. For a fixed incident field $u^i$ the solution to the direct scattering problem defines the *boundary to far field operator* $\mathcal{F} : \partial D \mapsto u_\infty$ which maps the boundary $\partial D$ of the scatterer $D$ onto the far field pattern $u_\infty$ of the scattered wave $u^s$. In particular, $\mathcal{F}$ is the imaging operator that takes the scattering object $D$ into its image $u_\infty$ via the scattering process. In terms of this imaging operator, i.e., the boundary to far field operator, given a far field pattern $u_\infty$, the inverse problem consists in solving the operator equation

$$\mathcal{F}(\partial D) = u_\infty \tag{4.1}$$

for the unknown boundary $\partial D$. As opposed to the direct obstacle scattering problem which is linear and well-posed, the operator equation (4.1), i.e., the inverse obstacle scattering problem, is nonlinear and ill-posed. It is nonlinear since the solution to the direct scattering problem depends nonlinearly on the boundary and it is ill-posed because the far field mapping is extremely smoothing due to the analyticity of the far field pattern.

In order to define the operator $\mathcal{F}$ properly, the most appropriate approach is to choose a fixed reference domain $D$ and consider a family of scatterers $D_h$ with boundaries represented in the form $\partial D_h = \{x + h(x) : x \in \partial D\}$ where $h : \partial D \to \mathbb{R}^3$ is of class $C^2$ and is sufficiently small in the $C^2$ norm on $\partial D$. Then we may consider the operator $\mathcal{F}$ as a mapping from a ball

$$V := \{h \in C^2(\partial D) : \|h\|_{C^2} < a\} \subset C^2(\partial D)$$

with sufficiently small radius $a > 0$ into $L^2(S^2)$. However, for ease of presentation, we proceed differently and restrict ourselves to boundaries $\partial D$ that can be parameterized by mapping them globally onto the unit sphere $S^2$, i.e.,

$$\partial D = \{p(\hat{x}) : \hat{x} \in S^2\} \tag{4.2}$$

for some injective $C^2$ function $p : S^2 \to \mathbb{R}^3$. As simple example, the reader should consider the case of star-like domains where

$$p(\hat{x}) = r(\hat{x})\hat{x}, \quad \hat{x} \in S^2, \tag{4.3}$$

with a radial distance function $r : S^2 \to (0, \infty)$. Then, with some appropriate subspace $W \subset C^2(S^2)$, we may interpret the operator $\mathcal{F}$ as a mapping

$$\mathcal{F} : W \to L^2(S^2), \quad \mathcal{F} : p \mapsto u_\infty,$$

and consequently the inverse obstacle scattering problem consists in solving

$$\mathcal{F}(p) = u_\infty \tag{4.4}$$

for the unknown function $p$.

Since $\mathcal{F}$ is nonlinear we may linearize

$$\mathcal{F}(p + q) = \mathcal{F}(p) + \mathcal{F}'(p)q + O(q^2)$$

in terms of a Fréchet derivative $\mathcal{F}'$. Then, given an approximation $p$ for the solution of (4.4), in order to obtain an update $p+q$ we solve the approximate linear equation

$$\mathcal{F}(p) + \mathcal{F}'(p)q = u_\infty \tag{4.5}$$

for $q$. We note that the linearized equation inherits the ill-posedness of the nonlinear equation and therefore regularization is required. As in the classical Newton iterations, this linearization procedure is iterated until some stopping criteria is satisfied.

In principle the parameterization of the update $\partial D_{p+q} = \{p(\hat{x}) + q(\hat{x}) : \hat{x} \in S^2\}$ is not unique. To cope with this ambiguity the simplest possibility is to allow only perturbations of the form

$$q(\hat{x}) = z(\hat{x})\, \nu(p(\hat{x})), \quad x \in S^2, \tag{4.6}$$

with a scalar function $z$. We denote the corresponding linear space of normal $L^2$ vector fields by $L^2_{\text{normal}}(S^2)$.

The Fréchet differentiabitlity of the operator $\mathcal{F}$ is addressed in the following theorem:

**Theorem 4.1** *The boundary to far field mapping $\mathcal{F} : p \mapsto u_\infty$ is Fréchet differentiable. The derivative is given by*

$$\mathcal{F}'(p)q = v_{q,\infty},$$

*where $v_{q,\infty}$ is the far field pattern of the radiating solution $v_q$ to Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}$ satisfying the Dirichlet boundary condition*

$$v_q = -\nu \cdot q \, \frac{\partial u}{\partial \nu} \quad \text{on } \partial D \tag{4.7}$$

*in terms of the total field $u = u^i + u^s$.*

The boundary condition (4.7) for the derivative can be obtained formally by using the chain rule to differentiate the boundary condition $u = 0$ on $\partial D$ with respect to the boundary. Extensions of Theorem 4.1 to the Neumann boundary condition, the perfect conductor boundary condition and to the impedance boundary condition in acoustics and electromagnetics are also available.

To justify the application of regularization methods for stabilizing (4.5), injectivity and dense range of the operator $\mathcal{F}'(p) : L^2_{\text{normal}}(S^2) \to L^2(S^2)$ needs to be established. This is settled for the Dirichlet condition and, for $\lambda$ sufficiently large, for the impedance boundary condition and remains an open problem for the Neumann boundary condition. In the classical Tikhonov regularization, (4.5) is replaced by

$$\alpha q + [\mathcal{F}'(p)]^* \mathcal{F}'(p) q = [\mathcal{F}'(p)]^* \{u_\infty - \mathcal{F}(p)\} \tag{4.8}$$

with some positive regularization parameter $\alpha$ and the $L^2$ adjoint $[\mathcal{F}'(p)]^*$ of $\mathcal{F}'(p)$. For details on the numerical implementation we refer to [23] and the references therein. The numerical examples strongly indicate that it is advantageous to use some Sobolev norm instead of the $L^2$ norm as the penalty term in the Tikhonov regularization. Numerical examples in three dimensions have been reported by Farhat et al [32] and by Harbrecht and Hohage [39].

In closing this section on Newton iterations we note as their main advantages that this approach is conceptually simple and, as the numerical examples in the literature indicate, leads to highly accurate reconstructions with reasonable stability against errors in the far field pattern. On the other hand, it should be noted that for the numerical implementation an efficient forward solver is needed and good a priori information is required in order to ensure convergence. On the theoretical side the convergence of regularized Newton iterations for inverse obstacle scattering problems has not been completely settled, although some progress has been made through the work of Hohage [40] and Potthast [78].

Newton type iterations can also be employed for the simultaneous determination of the boundary shape and the impedance function $\lambda$ in the impedance boundary condition (1.8) [59].

## 4.2 Decomposition methods

The main idea of decomposition methods is to break up the inverse obstacle scattering problem into two parts: the first part deals with the ill-posedness by constructing the scattered wave $u^s$ from its far field pattern $u_\infty$ and the second part deals with the nonlinearity by determining the unknown boundary $\partial D$ of the scatterer as the location where the boundary condition for the total field $u^i + u^s$ is satisfied in a least-squares sense. In the *potential method*, for the first part enough a priori information on the unknown scatterer $D$ is assumed so one can place a closed surface $\Gamma$

inside $D$. Then the scattered field $u^s$ is sought as a single-layer potential

$$u^s(x) = \int_\Gamma \varphi(y)\Phi(x,y)\,ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{4.9}$$

with an unknown density $\varphi \in L^2(\Gamma)$. In this case the far field pattern $u_\infty$ has the representation

$$u_\infty(\hat{x}) = \frac{1}{4\pi}\int_\Gamma e^{-ik\,\hat{x}\cdot y}\varphi(y)\,ds(y), \quad \hat{x} \in S^2.$$

Given the far field pattern $u_\infty$ the density $\varphi$ is now found by solving the integral equation of the first kind

$$S_\infty \varphi = u_\infty \tag{4.10}$$

with the compact integral operator $S_\infty : L^2(\Gamma) \to L^2(S^2)$ given by

$$(S_\infty\varphi)(\hat{x}) := \frac{1}{4\pi}\int_\Gamma e^{-ik\,\hat{x}\cdot y}\varphi(y)\,ds(y), \quad \hat{x} \in S^2.$$

Due to the analytic kernel of $S_\infty$ the integral equation (4.10) is severely ill-posed. For a stable numerical solution of (4.10) Tikhonov regularization can be applied, i.e., the ill-posed equation (4.10) is replaced by

$$\alpha\varphi_\alpha + S_\infty^* S_\infty \varphi_\alpha = S_\infty^* u_\infty \tag{4.11}$$

with some positive regularization parameter $\alpha$ and the adjoint $S_\infty^*$ of $S_\infty$.

Given an approximation of the scattered wave $u^s_\alpha$ obtained by inserting a solution $\varphi_\alpha$ of (4.11) into the potential (4.9), the unknown boundary $\partial D$ is then determined by requiring the sound-soft boundary condition

$$u^i + u^s = 0 \quad \text{on } \partial D \tag{4.12}$$

to be satisfied in a least-squares sense, i.e., by minimizing the $L^2$ norm of the defect

$$\left\|u^i + u^s_\alpha\right\|^2_{L^2(\Lambda)} \tag{4.13}$$

over a suitable set of admissible surfaces $\Lambda$. Instead of solving this minimization problem one can also visualize $\partial D$ by color coding the values of the modulus $|u|$ of the total field $u \approx u^i + u^s_\alpha$ on a sufficiently fine grid over some domain containing the scatterer.

Clearly we can expect (4.10) to have a solution $\varphi \in L^2(\Gamma)$ if and only if $u_\infty$ is the far field of a radiating solution to the Helmholtz equation in the exterior of $\Gamma$ with sufficiently smooth boundary values on $\Gamma$. Hence the solvability of (4.10) is related to the regularity properties of the scattered wave which in general cannot be known in advance for the unknown scatterer $D$. Nevertheless it is possible to provide a solid theoretical foundation to the above procedure [23, 54]. This is achieved by combining the minimization of the Tikhonov functional for (4.10) and the defect minimization for (4.13) into one cost functional

$$\|S_\infty\varphi - u_\infty\|^2_{L^2(S^2)} + \alpha\|\varphi\|^2_{L^2(\Gamma)} + \gamma\left\|u^i + u^s_\alpha\right\|^2_{L^2(\Lambda)}. \tag{4.14}$$

Here $\gamma > 0$ denotes a coupling parameter which has to be chosen appropriately for the numerical implementation in order to make the two terms in (4.14) of the same magnitude, for example $\gamma = \|u_\infty\|_{L^2(S^2)}/\|u^i\|_\infty$.

Note that the potential approach can also be employed for the inverse problem to recover the impedance given the shape of the scatterer. In this case the far field equation (4.10) is solved with $\Gamma$ replaced by the known boundary $\partial D$. After the density $\varphi$ is obtained $\lambda$ can be determined in a least-squares sense from the impedance boundary condition (1.8) after evaluating the trace and the normal derivative of the single-layer potential (4.9) on $\partial D$.

The *point source method* of Potthast [77] can also be interpreted as a decomposition method. Its motivation is based on Huygens' principle from Theorem 2.6, i.e., the scattered field representation

$$u^s(x) = -\int_{\partial D} \frac{\partial u}{\partial \nu}(y)\, \Phi(x,y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{4.15}$$

and the far field representation

$$u_\infty(\hat{x}) = -\frac{1}{4\pi}\int_{\partial D} \frac{\partial u}{\partial \nu}(y)\, e^{-ik\,\hat{x}\cdot y}\, ds(y), \quad \hat{x} \in S^2. \tag{4.16}$$

For $z \in \mathbb{R}^3 \setminus \bar{D}$ we choose a domain $B_z$ such that $z \notin B_z$ and $\bar{D} \subset B_z$ and approximate the point source $\Phi(\cdot, z)$ by a *Herglotz wave function*, i.e., a superposition of plane waves such that

$$\Phi(y,z) \approx \int_{S^2} e^{ik\,y\cdot d} g_z(d)\, ds(d), \quad y \in B_z, \tag{4.17}$$

for some $g_z \in L^2(S^2)$. Under the assumption that there does not exist a nontrivial solution to the Helmholtz equation in $B_z$ with homogeneous Dirichlet boundary condition on $\partial B_z$, the Herglotz wave functions are dense in $H^{1/2}(\partial B_z)$ [24, 31] and consequently the approximation (4.17) can be achieved uniformly with respect to $y$ on compact subsets of $B_z$. We can now insert (4.17) into (2.20) and use (2.21) to obtain

$$u^s(z) \approx 4\pi \int_{S^2} g_z(\hat{x}) u_\infty(-\hat{x})\, ds(\hat{x}) \tag{4.18}$$

as an approximation for the scattered wave $u^s$. Knowing an approximation for the scattered wave the boundary $\partial D$ can be found as above from the boundary condition (4.12).

The approximation (4.17) can in practice be obtained by solving the ill-posed linear integral equation

$$\int_{S^2} e^{ik\,y\cdot d} g_z(d)\, ds(d) = \Phi(y,z), \quad y \in \partial B_z, \tag{4.19}$$

via Tikhonov regularization and the Morozov discrepancy principle. Note that although the integral equation (4.19) is in general not solvable, the approximation property (4.18) is ensured through the above denseness result on Herglotz wave functions.

An advantage of decomposition methods is that the separation of the ill-posedness and the nonlinearity is conceptually straightforward. A second and main advantage consists in the fact that their numerical implementation does not require a forward solver. As a disadvantage, as in the Newton method of the previous section, if we go beyond visualization of the level surfaces of $|u|$ and proceed with the minimization a good a priori information on the unknown scatterer is needed for the iterative solution of the optimization problem. The accuracy of the reconstructions using decomposition methods is slightly inferior to that using Newton iterations.

## 4.3 Iterative methods based on Huygens' principle

We recall Huygens' principle (4.15) and (4.16). In view of the sound-soft boundary condition, from (4.15) we conclude that

$$u^i(x) = \int_{\partial D} \frac{\partial u}{\partial \nu}(y)\, \Phi(x,y)\, ds(y), \quad x \in \partial D. \tag{4.20}$$

Now we can interpret (4.16) and (4.20) as a system of two integral equations for the unknown boundary $\partial D$ of the scatterer and the induced surface flux

$$\varphi := -\frac{\partial u}{\partial \nu} \quad \text{on } \partial D.$$

It is convenient to call (4.16) the *data equation* since it contains the given far field for the inverse problem and (4.20) the *field equation* since it represents the boundary condition. Both equations are linear with respect to the flux and nonlinear with respect to the boundary. Equation (4.16) is severely ill-posed whereas (4.20) is only mildly ill-posed.

Obviously there are three options for an iterative solution of (4.16) and (4.20). In a first method, given an approximation for the boundary $\partial D$ one can solve the mildly ill-posed integral equation of the first kind (4.20) for $\varphi$. Then, keeping $\varphi$ fixed, equation (4.16) is linearized with respect to $\partial D$ to update the boundary approximation. This approach has been proposed by Johansson and Sleeman [46]. In a second approach, following ideas first developed for the Laplace equation by Kress and Rundell [60], one also can solve the system (4.16) and (4.20) simultaneously for $\partial D$ and $\varphi$ by Newton iterations, i.e., by linearizing both equations with respect to both unknowns. This idea has been analyzed by Ivanyshyn and Kress [43, 44]. Whereas in the first method the burden of the ill-posedness and nonlinearity is put on one equation, in a third method a more even distribution of the difficulties is obtained by reversing the roles of (4.16) and (4.20), i.e., by solving the severely ill-posed equation (4.16) for $\varphi$ and then linearizing (4.20) to obtain the boundary update. With a slight modification this approach may also be interpreted as a decomposition method since to some extent it separates the ill-posedness and the nonlinearity. It combines the decomposition method from the previous Section 4.2 with elements of Newton iterations from Section 4.1. Therefore it has also been termed as a *hybrid method* and as such was analyzed by Kress and Serranho [58, 86].

For a more detailed description of these three methods, using the parameterization (4.2), we introduce the parameterized single-layer operator and far field operator $A, A_\infty : C^2(S^2) \times L^2(S^2) \to L^2(S^2)$ by

$$A(p,\psi)(\hat{x}) := \int_{S^2} \Phi(p(\hat{x}), p(\hat{y}))\psi(\hat{y})\, ds(\hat{y}), \quad \hat{x} \in S^2,$$

and

$$A_\infty(p,\psi)(\hat{x}) := \frac{1}{4\pi} \int_{S^2} e^{-ik\,\hat{x}\cdot p(\hat{y})}\psi(\hat{y})\, ds(\hat{y}), \quad \hat{x} \in S^2.$$

Then (4.16) and (4.20) can be written in the operator form

$$A_\infty(p,\psi) = u_\infty \tag{4.21}$$

and

$$A(p,\psi) = -u^i \circ p \tag{4.22}$$

where we have incorporated the surface element into the density function via

$$\psi(\hat{x}) := J(\hat{x})\, \varphi(p(\hat{x})) \tag{4.23}$$

with the Jacobian $J$ of the mapping $p$. The linearization of these equations requires the Fréchet derivatives of the operators $A$ and $A_\infty$ with respect to $p$. These can be obtained by formally differentiating their kernels with respect to $p$, i.e.,

$$(A'(p,\psi)q)(\hat{x}) = \int_{S^2} \operatorname{grad}_x \Phi(p(\hat{x}), p(\hat{y})) \cdot [q(\hat{x}) - q(\hat{y})]\, \psi(\hat{y})\, ds(\hat{y}), \quad x \in S^2,$$

and

$$(A'_\infty(p,\psi)q)(\hat{x}) = -\frac{ik}{4\pi}\int_{S^2} e^{-ik\,\hat{x}\cdot p(\hat{y})}\,\hat{x}\cdot q(\hat{y})\,\psi(\hat{y})\,ds(\hat{y}), \quad x \in S^2.$$

For fixed $p$, provided $k^2$ is not a Dirichlet eigenvalue of the negative Laplacian in $D$, both in a Hölder space setting $A(p,\cdot) : C^{0,\alpha}(S^2) \to C^{1,\alpha}(S^2)$ or in a Sobolev space setting $A(p,\cdot) : H^{-1/2}(S^2) \to H^{1/2}(S^2)$, the operator $A(p,\cdot)$ is a homeomorphism [23]. In this case, given an approximation to the boundary parameterization $p$, the field equation (4.22) can be solved for the density $\psi$. Then, keeping $\psi$ fixed, linearizing the data equation (4.21) with respect to $p$ leads to the linear equation

$$A'_\infty(p, \underbrace{[A(p,\cdot)]^{-1}(u^i\circ p)}_{-\psi})q = -u_\infty - A_\infty(p, \underbrace{[A(p,\cdot)]^{-1}(u^i\circ p)}_{-\psi}) \tag{4.24}$$

for $q$ to update the parameterization $p$ via $p+q$. This procedure can be iterated.

For fixed $p$ the operator $A'_\infty(p,[A(p,\cdot)]^{-1}(u^i\circ p))$ has a smooth kernel and therefore is severely ill-posed. This requires stabilization, for example via Tikhonov regularization. The following theorem ensures injectivity and dense range as prerequesits for Tikhonov regularization. We recall the form (4.6) introduced for uniqueness of the parameterization of the update and the corresponding linear space $L^2_{\text{normal}}(S^2)$ of normal $L^2$ vector fields.

**Theorem 4.2** *Assume that $k^2$ is not a Neumann eigenvalue of the negative Laplacian in $D$. Then the operator*

$$A'_\infty(p,[A(p,\cdot)]^{-1}(u^i\circ p)) : L^2_{\text{normal}}(S^2) \to L^2(S^2)$$

*is injective and has dense range.*

One can relate this approach to the Newton iterations for the nonlinear equation (4.1) for the boundary to far field operator of Section 4.1. In the case when $k^2$ is not a Dirichlet eigenvalue of the negative Laplacian in $D$ one can write

$$\mathcal{F}(p) = -A_\infty(p,[A(p,\cdot)]^{-1}(u^i\circ p)).$$

By the product and chain rule this implies

$$\begin{aligned}\mathcal{F}'(p)q = &-A'_\infty(p,[A(p,\cdot)]^{-1}(u^i\circ p))q \\ &+A_\infty(p,[A(p,\cdot)]^{-1}A'(p,[A(p,\cdot)]^{-1}(u^i\circ p))q) \\ &-A_\infty(p,[A(p,\cdot)]^{-1}((\operatorname{grad} u^i)\circ p)\cdot q).\end{aligned} \tag{4.25}$$

Hence we observe a relation between the above iterative scheme and the Newton iterations for the boundary to far field map as expressed by the following theorem:

**Theorem 4.3** *The iteration scheme given by* (4.24) *can be interpreted as Newton iterations for* (4.1) *with the derivative of* $\mathcal{F}$ *approximated by the first term in the representation* (4.25).

As to be expected from this close relation to Newton iterations for (4.1), the quality of the reconstructions via (4.24) can compete with those of Newton iterations with the benefit of reduced computational costs.

The second approach for iteratively solving the system (4.21) and (4.22) consists in simultaneously linearizing both equations with respect to both unknowns. In this case, given approximations $p$ and $\psi$ both for the boundary parameterization and the density, the system of linear equations

$$A'_\infty(p,\psi)q + A_\infty(p,\chi) = -A_\infty(p,\psi) + u_\infty \tag{4.26}$$

and

$$A'(p,\psi)q + ((\operatorname{grad} u^i)\circ p)\cdot q + A(p,\chi) = -A(p,\psi) - u^i \circ p \tag{4.27}$$

has to be solved for $q$ and $\chi$ in order to obtain updates $p+q$ for the boundary parameterization and $\psi + \chi$ for the density. This procedure again is iterated and coincides with Newton's method for the system (4.21) and (4.22).

For uniqueness reasons the updates must be restricted, for example to normal fields of the form (4.6). Due to the smoothness of the kernels both equations (4.26) and (4.27) are severely ill-posed and require regularization with respect to both unknowns. In particular for the parameterization update it is appropriate to incorporate penalties for Sobolev norms of $q$ to guarantee smoothness of the boundary whereas for the density $L^2$ penalty terms on $\chi$ are sufficient.

The simultaneous iterations (4.26) and (4.27) again exhibit connections to the Newton iteration for (4.1):

**Theorem 4.4** *Assume that* $k^2$ *is not a Dirichlet eigenvalue of the negative Laplacian in* $D$ *and set* $\psi := -[A(p,\cdot)]^{-1}(u^i\circ p)$. *If* $q$ *satisfies the linearized boundary to far field equation* (4.5) *then* $q$ *and*

$$\chi := -[A(p,\cdot)]^{-1}(A'(p,\psi)q + ((\operatorname{grad} u^i)\circ p)\cdot q)$$

*satisfy the linearized data and field equations* (4.26) *and* (4.27). *Conversely, if* $q$ *and* $\chi$ *satisfy* (4.26) *and* (4.27) *then* $q$ *satisfies* (4.5).

Theorem 4.4 illustrates the difference between the iteration method based on (4.26) and (4.27) and the Newton iterations for (4.1). In general when performing (4.26) and (4.27) in the sequence of updates the relation $A(p,\psi) = -(u^i\circ p)$ between the approximations $p$ and $\psi$ for the parameterization and the density will not be satisfied. This observation also indicates a possibility to use (4.26) and (4.27) for implementing a Newton scheme for (4.1). It is only necessary to replace the update $\psi+\chi$ for the density by $-[A(p+q,\cdot)]^{-1}(u^i\circ(p+q))$, i.e., at the expense of throwing away $\chi$ and solving a boundary integral equation for a new density. For a numerical implementation and three dimensional examples we refer to [45].

In a third method, in order to evenly distribute the burden of the ill-posedness and the nonlinearity of the inverse obstacle scattering problem, instead of solving the field equation (4.22) for the density and then linearizing the data equation one can also solve the severely ill-posed data equation (4.21) for the density and then linearize the mildly ill-posed field equation (4.22) to update the boundary. In this case, given an approximation for the boundary parameterization $p$, first the data equation (4.21) is solved for the density $\psi$. Then, keeping $\psi$ fixed, the field equation (4.22) is linearized to obtain the linear equation

$$A'(p,\psi)\,q + ((\operatorname{grad} u^i)\circ p)\cdot q = -A(p,\psi) - u^i\circ p \tag{4.28}$$

for $q$ to update the parameterization $p$ via $p+q$. This procedure of alternatingly solving (4.21) and (4.28) can be iterated. To some extent this procedure mimics a decomposition method in the sense that it decomposes the inverse problem into a severely ill-posed linear problem and a nonlinear problem.

The hybrid method suggested by Kress and Serranho [58, 86] can be considered as a slight modification of the above procedure. In this method, given an approximation $p$ for the parameterization of the boundary, the data equation (4.21) is solved for the density $\psi$ via regularization. Injectivity and dense range of the operator $A_\infty(p,\cdot): L^2(S^2)\to L^2(S^2)$ are guaranteed provided $k^2$ is not a Dirichlet eigenvalue for the negative Laplacian in $D$ [23] . Then one can define the single-layer potential

$$u^s(x) = \int_{S^2}\Phi(x,p(\hat y))\psi(\hat y)\,ds(\hat y)$$

and evaluate the boundary values of $u := u^i + u^s$ and its derivatives on the surface represented by $p$ via the jump relations. Finally an update $p+q$ is found by linearizing the boundary condition $u\circ(p+q)=0$, i.e., by solving the linear equation

$$u\circ p + ((\operatorname{grad} u)\circ p)\cdot q = 0 \tag{4.29}$$

for $q$. For uniqueness of the update representation the simplest possibility is to allow only perturbations of the form (4.6). Then injectivity for the linear equation (4.29) can be established for the exact boundary.

After introducing the operator

$$(\widetilde{A}(p,\psi)\,q)(\hat x) := \int_{S^2}\operatorname{grad}_x\Phi(p(\hat x),p(\hat y))\cdot q(\hat x)\,\psi(\hat y)\,ds(\hat y) - \frac{1}{2}\,\frac{\psi(\hat x)\,[\nu(p(\hat x))\cdot q(\hat x)]}{J(\hat x)}$$

and observing the jump relations for the single-layer potential and (4.23), the equation (4.29) can be rewritten as

$$\widetilde{A}(p,\psi)\, q + ((\operatorname{grad} u^i) \circ p) \cdot q = -A(p,\psi) - u^i \circ p. \tag{4.30}$$

Comparing this with (4.28) we discover a relation between solving the data and field equation iteratively via (4.21) and (4.28) and the hybrid method of Kress and Serranho. In the hybrid method the Fréchet derivative of $A$ with respect to $p$ is replaced by the operator $\widetilde{A}$ where one linearizes only with respect to the evaluation surface for the single-layer potential but not with respect to the integration surface. For the numerical implementation of the hybrid method and numerical examples in three dimensions we refer to [87].

All three methods of this section can be applied to the Neumann boundary condition, the perfect conductor boundary condition and to the impedance boundary condition in acoustics and electromagnetics. They also can be employed for the simultaneous reconstruction of the boundary shape and the impedance function $\lambda$ in the impedance boundary condition (1.8) [85].

## 4.4 Newton iterations for the inverse medium problem

Analogously to the inverse obstacle scattering problem, we can reformulate the inverse medium problem as a nonlinear operator equation. To this end we define the *far field operator* $\mathcal{F} : m \mapsto u_\infty$ that maps $m := 1 - n$ to the far field pattern $u_\infty$ for plane wave incidence $u^i(x) = e^{ik\, x\cdot d}$. Since by Theorem 3.3 we know that $m$ is uniquely determined by a knowledge of $u_\infty(\hat{x}, d)$ for all incident and observation directions $\hat{x}, d \in S^2$, we interpret $\mathcal{F}$ as an operator from $C(B)$ into $L^2(S^2 \times S^2)$ for a ball $B$ that contains the unknown support of $m$.

In view of the Lippmann–Schwinger equation (2.23) and the far field representation (2.24) we can write

$$(\mathcal{F}(m))(\hat{x}, d) = -\frac{k^2}{4\pi} \int_B e^{-ik\,\hat{x}\cdot y} m(y) u(y,d)\, dy, \quad \hat{x}, d \in S^2, \tag{4.31}$$

where $u(\cdot, d)$ is the unique solution of

$$u(x,d) + k^2 \int_B \Phi(x,y) m(y) u(y,d)\, dy = u^i(x,d), \quad x \in B. \tag{4.32}$$

From (4.32) it can be seen that the Fréchet derivative $v_q$ of $u$ with respect to $m$ (in direction $q$) satisfies the Lippmann–Schwinger equation

$$v_q(x,d) + k^2 \int_B \Phi(x,y)[m(y) v_q(y,d) + q(y) u(y,d)]\, dy = 0, \quad x \in B. \tag{4.33}$$

From this and (4.31) it follows that the Fréchet derivative of $\mathcal{F}$ is given by

$$(\mathcal{F}'(m)\, q)(\hat{x}, d) = -\frac{k^2}{4\pi} \int_B e^{-ik\,\hat{x}\cdot y} [m(y) v_q(y, d) + q(y) u(y, d)]\, dy, \quad \hat{x}, d \in S^2,$$

which coincides with the far field pattern of the solution $v_q(\cdot, d)$ of (4.33). Hence, we have proven the following theorem:

**Theorem 4.5** *The far field mapping* $\mathcal{F} : m \mapsto u_\infty$ *is Fréchet differentiable. The derivative is given by*

$$\mathcal{F}'(m) q = v_{q,\infty}$$

*where* $v_{q,\infty}$ *is the far field pattern of the radiating solution* $v_q$ *to*

$$\Delta v + k^2 n v = -k^2 u q \quad \textit{in } \mathbb{R}^3. \tag{4.34}$$

This characterization of the Fréchet derivative can be used to establish injectivity of $\mathcal{F}'(m)$. We now have all the prerequisites available for a regularized Newton iteration analogous to (4.8).

A similar approach as that given above is also possible for the electromagnetic inverse medium problem.

## 4.5 Least squares methods for the inverse medium problem

In view of the Lippmann-Schwinger equation (2.23) and the far field representation (2.24) the inverse medium problem is equivalent to solving the system consisting of the *field equation*

$$u(x, d) + k^2 \int_B \Phi(x, y) m(y) u(y, d)\, dy = u^i(x, d), \quad x \in B, d \in S^2, \tag{4.35}$$

and the *data equation*

$$-\frac{k^2}{4\pi} \int_B e^{-ik\,\hat{x}\cdot y} m(y) u(y, d)\, dy = u_\infty(\hat{x}, d), \quad \hat{x}, d \in S^2, \tag{4.36}$$

where $B$ is a ball containing the support of $m$. In principle one can first solve the ill-posed linear equation (4.36) to determine the source $mu$ from the far field pattern and then solve the nonlinear equation (4.35) to construct the contrast $m$. After defining the volume potential operator $T : L^2(B \times S^2) \to L^2(B \times S^2)$ and the far field operator $F : L^2(B \times S^2) \to L^2(S^2 \times S^2)$ by

$$(Tv)(x, d) := -k^2 \int_B \Phi(x, y) v(y, d)\, dy, \quad x \in B,\ d \in S^2,$$

and

$$(Fv)(\hat{x}, d) := -\frac{k^2}{4\pi} \int_B e^{-ik\,\hat{x}\cdot d} v(y, d)\, dy, \quad \hat{x}, d \in S^2,$$

we rewrite the field equation (4.35) as

$$u^i + Tmu = u \tag{4.37}$$

and the data equation (4.36) as

$$Fmu = u_\infty. \tag{4.38}$$

We can now define the cost function

$$\mu(m, u) := \frac{\|u^i + Tmu - u\|^2_{L^2(B\times S^2)}}{\|u^i\|^2_{L^2(B\times S^2)}} + \frac{\|u_\infty - Fmu\|^2_{L^2(S^2\times S^2)}}{\|u_\infty\|^2_{L^2(S^2\times S^2)}} \tag{4.39}$$

and reformulate the inverse medium problem as the optimization problem to minimize $\mu$ over the contrast $m \in V$ and the fields $u \in W$ where $V$ and $W$ are appropriately chosen admissible sets. The weights in the cost function are chosen such that the two terms are of the same magnitude.

This optimization problem is similar in structure to that used in (4.14) in connection with the decomposition method for the inverse obstacle scattering problem. However, since by Theorem 3.3 all incident directions are required, the discrete versions of the optimization problem suffer from a large number of unknowns. Analogous to the two step approaches of Sections 4.2 and 4.3 for the inverse obstacle scattering problem, one way to reduce the computational complexity is to treat the fields and the contrast separately, for example by a modified conjugate gradient method as proposed by Kleinman and van den Berg [56]. In a modified version of this approach van den Berg and Kleinman [4] transformed the Lippmann–Schwinger equation (4.37) into the equation

$$mu^i + mTw = w \tag{4.40}$$

for the contrast sources $w := mu$ and instead of simultaneously updating the contrast $m$ and the fields $u$ the contrast is updated together with the contrast source $w$. The cost function (4.39) is now changed to

$$\mu(m, w) := \frac{\|mu^i + mTw - w\|^2_{L^2(B\times S^2)}}{\|u^i\|^2_{L^2(B\times S^2)}} + \frac{\|u_\infty - Fmu\|^2_{L^2(S^2\times S^2)}}{\|u_\infty\|^2_{L^2(S^2\times S^2)}} \,.$$

The above approach for the acoustic inverse medium problem can be adapted to the case of electromagnetic waves.

## 4.6 Born approximation

The Born approximation turns the inverse medium scattering problem into a linear problem and therefore is often employed in practical applications. In view of (2.24), for plane wave incidence we have the linear integral equation

$$-\frac{k^2}{4\pi}\int_{\mathbb{R}^3} e^{-ik\,(\hat{x}-d)\cdot y} m(y)\,dy = u_\infty(\hat{x},d) \quad \hat{x},d\in S^2. \tag{4.41}$$

Solving (4.41) for the unknown $m$ corresponds to inverting the Fourier transform of $m$ restricted to the ball of radius $2k$ centered at the origin, i.e., only incomplete data is available. This causes uniqueness ambiguities and leads to severe ill-posedness of the inversion. Thus the ill-posedness which seemed to have disappeared through the inversion of the Fourier transform is back on stage. For details we refer to [61].

A counterpart of the Born approximation in inverse obstacle scattering starts from the far field of the physical optics approximation (2.22) for a convex sound-soft scatterer $D$ in the *back scattering* direction, i.e.,

$$u_\infty(-d;d) = -\frac{1}{4\pi}\int_{\nu(y)\cdot d<0} \frac{\partial}{\partial\nu(y)}\, e^{2ik\,d\cdot y}\,ds(y).$$

Analogously, replacing $d$ by $-d$, we have

$$u_\infty(d;-d) = -\frac{1}{4\pi}\int_{\nu(y)\cdot d>0} \frac{\partial}{\partial\nu(y)}\, e^{-2ik\,d\cdot y}\,ds(y).$$

Combining the last two equations and using Green's integral theorem we find

$$\int_{\mathbb{R}^3} \chi(y) e^{2ik\,d\cdot y}\,dy = \frac{\pi}{k^2}\,\left\{u_\infty(-d;d) + \overline{u_\infty(d;-d)}\right\}, \quad d\in S^2, \tag{4.42}$$

with the characteristic function $\chi$ of the scatterer $D$. Equation (4.42) is known as the *Bojarski identity*. Hence, in the physical optics approximation, the Fourier transform has again to be inverted from incomplete data since the physical optics approximation is valid only for large wave numbers $k$. For details we refer to [61].

## 4.7 Historical remarks

The boundary condition (4.7) was obtained by Roger [82] who first employed Newton type iterations for the approximate solution of inverse obstacle scattering problems. Rigorous foundations for the Fréchet differentiability were given by Kirsch [48] in the sense of a domain derivative via variational methods and by Potthast [76] via boundary integral equation techniques. The potential method as a prototype of decomposition methods has been proposed by Kirsch and Kress [54]. The point source method has been suggested by Potthast [77]. The iterative methods based on Huygens' principle were introduced by Johansson and Sleeman [46], by Ivanyshyn and Kress [44] (extending a method proposed by Kress and Rundel [60] from potential theory to acoustics) and by Kress [58] and Serranho [86]. The methods described in Sections 4.4–4.6 have been investigated by numerous researchers over the past thirty years.

# 5 Qualitative methods in inverse scattering

## 5.1 The far field operator and its properties

A different approach to solving inverse scattering problems than the use of iterative methods is the use of qualitative methods [7]. These methods have the advantage of requiring less a priori information than iterative methods (e.g. it is not necessary to know the topology of the scatterer or the boundary conditions satisfied by the total field) and in addition reduces a nonlinear problem to a linear problem. On the other hand the implementation of such methods often requires more data than iterative methods do and in the case of a penetrable inhomogeneous medium only recovers the support of the scatterer together with some estimates on its material properties.

We begin by considering the scattering problem for a sound-soft obstacle (3.1)–(3.2). The *far field operator* $F : L^2(S^2) \to L^2(S^2)$ for this problem is defined by

$$(Fg)(\hat{x}) := \int_{S^2} u_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in S^2, \tag{5.1}$$

where $u_\infty$ is the far field pattern associated with (3.1)–(3.2). By superposition $Fg$ is seen to be the far field pattern corresponding to the Herglotz wave function

$$v_g(x) := \int_{S^2} e^{ikx\cdot d} g(d)\, ds(d), \quad x \in \mathbb{R}^3, \tag{5.2}$$

as incident field. The function $g \in L^2(S^2)$ is known as the kernel of the Herglotz wave function. The far field operator $F$ is compact. It can also be shown that for the case of scattering by a sound-soft obstacle the far field operator is normal [7]. Of basic importance to us is the following theorem [23]:

**Theorem 5.1** *The far field operator $F$ corresponding to* (3.1)–(3.2) *is injective with dense range if and only if there does not exist a Dirichlet eigenfunction for $D$ which is a Herglotz wave function.*

*Proof.* The proof is based on the reciprocity relation (3.4). In particular, for the $L^2$ adjoint $F^* : L^2(S^2) \to L^2(S^2)$, the reciprocity relation implies that

$$F^* g = \overline{RFR\overline{g}} \tag{5.3}$$

where $R : L^2(S^2) \to L^2(S^2)$ is defined by $(Rg)(d) := g(-d)$. Hence, the operator $F$ is injective if and only if its adjoint $F^*$ is injective. Recalling that the denseness of the range of $F$ is equivalent to the injectivity of $F^*$, by (5.3) we need only to show the injectivity of $F$. To this end, we note that $Fg = 0$ is equivalent to the existence of a Herglotz wave function $v_g$ with kernel $g$ for which the far field pattern of the corresponding scattered field $v^s$ is $v_\infty = 0$. By Rellich's lemma this implies that $v^s = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ and the boundary condition $v_g + v^s = 0$ on $\partial D$ now shows that $v_g = 0$ on $\partial D$. Since by hypothesis $v_g$ is not a Dirichlet eigenfunction, we can conclude that $v_g = 0$ in $D$ and hence $g = 0$. □

We will now turn our attention to the far field operator associated with the inhomogeneous medium problems (3.8) and (3.11). In both cases we again define the far field operator by (5.1) where $u_\infty$ is now the far field pattern corresponding to (3.8) or (3.11). We first consider equation (3.8) which corresponds to scattering by an inhomogeneous medium. The analogue of Theorem 5.1 is the following [23]:

**Theorem 5.2** *The far field operator $F$ corresponding to* (3.8) *is injective with dense range if and only if there does not exist a solution $v, w \in L^2(D), v - w \in H^2(D)$ of the* interior transmission problem

$$\Delta v + k^2 v = 0 \quad \text{in } D \tag{5.4}$$

$$\Delta w + k^2 n w = 0 \quad \text{in } D \tag{5.5}$$

$$v = w \quad \text{on } \partial D \tag{5.6}$$

$$\frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} \quad \text{on } \partial D \tag{5.7}$$

*such that $v$ is a Herglotz wave function. Values of $k > 0$ for which there exists a nontrivial solution of* (5.4)–(5.7) *are called* transmission eigenvalues.

A similar theorem holds for equation (3.11) which corresponds to scattering by an anisotropic medium where now (5.5) is replaced by

$$\nabla \cdot A \nabla w + k^2 n w = 0 \quad \text{in } D \tag{5.8}$$

and in (5.7) the normal derivative $\frac{\partial w}{\partial \nu}$ is replaced by $\nu \cdot A \nabla w$. If the coefficients in (5.5) or (5.8) are real valued then the far field operator is normal.

In the case of electromagnetic waves, the far field operator becomes

$$(Fg)(\hat{x}) := \int_{S^2} E_\infty(\hat{x}, d, g(d))\, ds(d), \quad \hat{x} \in S^2, \tag{5.9}$$

where now $g \in L_t^2(S^2)$, the space of square integrable tangential vector fields defined on $S^2$, and $E_\infty$ is the electric far field pattern defined by (3.7). Theorems analogous to Theorems 5.1 and 5.2 are also valid in this case [23].

## 5.2 The linear sampling method

The linear sampling method is a non-iterative method for solving the inverse scattering problem that was first introduced by Colton and Kirsch [20] and Colton, Piana and Potthast [30]. To describe this method we first consider the case of scattering by a sound–soft obstacle, i.e., (3.1)–(3.2), and assume that for every $z \in D$ there exists a solution $g = g(\cdot, z) \in L^2(S^2)$ to the *far field equation*

$$Fg = \Phi_\infty(\cdot, z) \tag{5.10}$$

where

$$\Phi_\infty(\hat{x}, z) = \frac{1}{4\pi} e^{-ik\hat{x}\cdot z}, \quad \hat{x} \in S^2.$$

Since the right hand side of (5.10) is the far field pattern of the fundamental solution (2.1), it follows from Rellich's lemma that

$$\int_{S^2} u^s(x, d) g(d)\, ds(d) = \Phi(x, z)$$

for $x \in \mathbb{R}^3 \setminus D$. From the boundary condition $u = 0$ on $\partial D$ we see that

$$v_g(x) + \Phi(x, z) = 0 \quad \text{for } x \in \partial D \tag{5.11}$$

where $v_g$ is the Herglotz wave function with kernel $g$. We can now conclude from (5.11) that $v_g$ becomes unbounded as $z \to x \in \partial D$ and hence

$$\lim_{\substack{z \to \partial D \\ z \in D}} \|g(\cdot, z)\|_{L^2(S^2)} = \infty,$$

i.e., $\partial D$ is characterized by points $z$ where the solution of (5.10) becomes unbounded.

Unfortunately, in general the far field equation (5.10) does not have a solution nor does the above analysis say anything about what happens when $z \in \mathbb{R}^3 \setminus D$. To address these issues we first define the single-layer operator $S : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ by

$$(S\varphi)(x) := \int_{\partial D} \varphi(y) \Phi(x, y)\, ds(y), \quad x \in \partial D,$$

define the Herglotz operator operator $H : L^2(\partial D) \to H^{-1/2}(\partial D)$ as the operator mapping $g$ to the trace of the Herglotz wave function (5.2) on $\partial D$ and let $\mathcal{F} : H^{-1/2}(\partial D) \to L^2(S^2)$ be defined by

$$(\mathcal{F}\varphi)(\hat{x}) := \int_{\partial D} \varphi(y) e^{-ik\hat{x}\cdot y}\, ds(y), \quad \hat{x} \in S^2.$$

Then, using on the one hand the fact that Herglotz wave functions are dense in the space of solutions to the Helmholtz equation in $D$ with respect to the norm in the Sobolev space $H^1(D)$ and on the other the factorization of the far field operator $F$ as

$$F = -\frac{1}{4\pi}\, \mathcal{F} S^{-1} H$$

one can prove the following result [7, 53]:

**Theorem 5.3** *Assume that $k^2$ is not a Dirichlet eigenvalue of the negative Laplacian for $D$ and let $F$ be the far field operator corresponding to (3.1)–(3.2). Then*
*1. For $z \in D$ and a given $\epsilon > 0$ there exists $g_{z,\epsilon} \in L^2(S^2)$ such that*

$$\|Fg_{z,\epsilon} - \Phi_\infty(\cdot, z)\|_{L^2(S^2)} < \epsilon$$

*and the corresponding Herglotz wave function $v_{g_{z,\epsilon}}$ converges to a solution of*

$$\Delta u + k^2 u = 0 \quad \textit{in } D$$

$$u = -\Phi(\cdot, z) \quad \textit{on } \partial D$$

*in $H^1(D)$ as $\epsilon \to 0$.*
*2. For $z \in \mathbb{R}^3 \setminus D$ and a given $\epsilon > 0$, every $g_{z,\epsilon} \in L^2(S^2)$ that satisfies*

$$\|Fg_{z,\epsilon} - \Phi_\infty(\cdot, z)\|_{L^2(S^2)} < \epsilon$$

*is such that* $\lim_{\epsilon\to 0} \|v_{g_{z,\epsilon}}\|_{H^1(D)} = \infty$.

We note that the difference between cases 1) and 2) of this theorem is that for $z \in D$ the far field pattern $\Phi_\infty(\cdot, z)$ is in the range of $\mathcal{F}$ whereas for $z \in \mathbb{R}^3 \setminus D$ this is no longer true. The *linear sampling method* is based on attempting to compute the function $g_{z,\epsilon}$ in the above theorem by using Tiknonov regularization to solve $Fg = \Phi_\infty(\cdot, z)$. In particular, one expects that the regularized solution will be relatively smaller for $z$ in $D$ than $z$ in $\mathbb{R}^3 \setminus \bar{D}$ and this behavior can be visualized by color coding the values of the regularized solution on a grid over some domain containing $D$. A more precise statement of this observation will be made in the next section after we have discussed the factorization method for solving the inverse scattering problem. Further discussion of why linear sampling works if regularization methods are used to solve (5.10) can be found in [2, 3]. In addition to the inverse scattering problem (3.1)–(3.2) it is also possible to treat mixed boundary value problems as well as scattering by both isotropic and anisotropic inhomogeneous media where in the latter case we must assume that $k$ is not a transmission eigenvalue. For full details we refer the reader to [7]. Note that in each case it is not necessary to know the material properties of the scatterer in order to determine the support of the scatterer from a knowledge of the far field pattern via solving the far field equation $Fg = \Phi_\infty(\cdot, z)$.

The linear sampling method can also be extended to the case of electromagnetic waves where the far field equation (5.10) is now replaced by

$$\int_{S^2} E_\infty\left(\hat{x}, d, g(d)\right)\, ds(d) = E_{e,\infty}(\hat{x}, z, q)$$

where $E_\infty(\hat{x}, d, p)$ is the electric far field pattern corresponding to the incident field (2.38), $g \in L^2_t(S^2)$ and $E_{e,\infty}$ is the electric far field pattern of the electric dipole

$$E_e(x, z, q) := \frac{i}{k}\ \mathrm{curl}_x\, \mathrm{curl}_x\, q\Phi(x, z), \quad H_e(x, z, q) := \mathrm{curl}_x\, q\Phi(x, z). \tag{5.12}$$

Full details can be found in the lecture notes [13].

We close this section by briefly describing a version of the linear sampling method based on the reciprocity gap functional which is applicable to objects situated in a piecewise homogeneous background medium. Assume that an unknown scattering object is embedded in a portion $B$ of a piecewise inhomogeneous medium where the index of refraction is constant with wave number $k$. Let $B_0 \subset B$ be a domain in $B$ having a smooth boundary $\partial B_0$ such that the scattering obstacle $D$ satisfies $D \subset B_0$ and let $\nu$ be the unit outward normal to $\partial B_0$. We now define the *reciprocity gap functional* by

$$R(u, v) := \int_{\partial B_0} \left( u\, \frac{\partial v}{\partial \nu} - v\, \frac{\partial u}{\partial \nu} \right) ds$$

where $u$ and $v$ are solutions of the Helmholtz equation in $B_0 \setminus \bar{D}$ and $u, v \in C^1(\overline{B_0} \setminus \bar{D})$. In particular, we want $u$ to be the total field due to a point source situated at $x_0 \in B \setminus \overline{B_0}$ and $v = v_g$ to be a Herglotz wave function with kernel $g$. We then consider the integral equation

$$R(u, v_g) = R(u, \Phi_z)$$

where $\Phi_z := \Phi(\cdot, z)$ is the fundamental solution (2.1) and $u = u(\cdot, x_0)$ where $x_0$ is now assumed to be on a smooth surface $C$ in $B \setminus \overline{B_0}$ that is homotopic to $\partial B_0$. If $D$ is a sound-soft obstacle, we assume that $k^2$ is not a Dirichlet eigenvalue of the negative Laplacian in $D$ and if $D$ is an isotropic inhomogeneous medium we assume that $k$ is not a transmission eigenvalue. We then have the following theorem [18]:

**Theorem 5.4** *Assume that the above assumptions on $D$ are satisfied. Then*
*1. If $z \in D$ then there exists a sequence $\{g_n\}$ in $L^2(S^2)$ such that*

$$\lim_{n\to\infty} R(u, v_{g_n}) = R(u, \Phi_z), \quad x_0 \in C,$$

*and $v_{g_n}$ converges in $L^2(D)$.*
*2. If $z \in B_0 \setminus D$ then for every sequence $\{g_n\}$ in $L^2(S^2)$ such that*

$$\lim_{n\to\infty} R(u, v_{g_n}) = R(u, \Phi_z), \quad x_0 \in C,$$

*we have that* $\lim_{n\to\infty} \|v_{g_n}\|_{L^2(D)} = \infty$.

In particular, Theorem 5.4 provides a method for determining $D$ from a knowledge of the Cauchy data of $u$ on $\partial B_0$ in a manner analogous to that of the linear sampling method. Numerical examples using this method can be found in [18]. The extension of Theorem 5.4 to the Maxwell equations, together with numerical examples, can be found in [14].

## 5.3 The factorization method

The linear sampling method is complicated by the fact that in general $\Phi_\infty(\cdot, z)$ is not in the range of the far field operator $F$ for either $z \in D$ or $z \in \mathbb{R}^3 \setminus \bar{D}$. For the case of acoustic waves when $F$ is normal (e.g. the scattering problem corresponding to (3.1)–(3.2) or (3.8) for $n$ real valued) the problem was resolved by Kirsch in [49] and [50] who proposed replacing the far field equation $Fg = \Phi_\infty(\cdot, z)$ by

$$(F^*F)^{1/4} g = \Phi_\infty(\cdot, z) \tag{5.13}$$

where $F^*$ is again the adjoint of $F$ in $L^2(S^2)$. In particular, if $G : H^{1/2}(\partial D) \to L^2(S^2)$ is defined by $Gf = v_\infty$ where $v_\infty$ is the far field pattern of the solution to the radiating exterior Dirichlet problem (see Theorem 2.5) with boundary data $f \in L^2(\partial D)$, then the following theorem is valid [49]:

**Theorem 5.5** *Assume that $k^2$ is not a Dirichlet eigenvalue of the negative Laplacian for $D$. Then the ranges of $G : H^{1/2}(\partial D) \to L^2(S^2)$ and $(F^*F)^{1/4} : L^2(S^2) \to L^2(S^2)$ coincide.*

A result analogous to Theorem 5.5 is also valid for the scattering problem corresponding to (3.8) for $n$ real valued where we now must assume the $k$ is not an interior transmission eigenvalue [50]. Note that Theorem 5.5 provides an alternate method to the linear sampling method for solving the inverse scattering problem corresponding to the scattering of acoustic waves by a sound-soft obstacle. This follows from the fact that $\Phi_\infty(\cdot, z)$ is in the range of $G$ if and only if $z \in D$, i.e., equation (5.13) is solvable if and only if $z \in D$. This is an advantage over the linear sampling method since if (5.13) is solved by using Tikhonov regularization then as the noise level on $u_\infty$ tends to zero the norm of the regularized solution remains bounded if and only if $z \in D$. A similar statement cannot be made if regularization methods are used to solve $Fg = \Phi_\infty(\cdot, z)$. However, using Theorem 5.5, the following theorem has been established by Arens and Lechleiter [3] (see also [53]):

**Theorem 5.6** *Let $F$ be the far field operator associated with the scattering problem* (3.1)–(3.2) *and assume that $k^2$ is not a Dirichlet eigenvalue of the negative Laplacian for $D$. For $z \in D$ let $g_z \in L^2(S^2)$ be the solution of $(F^*F)^{1/4} g_z = \Phi_\infty(\cdot, z)$ and for every $z \in \mathbb{R}^3$ and $\epsilon > 0$ let $g_{z,\epsilon}$ be the solution of $Fg = \Phi_\infty(\cdot, z)$ obtained by Tikhonov regularization, i.e., the unique solution of $\epsilon g + F^*Fg = F^*\Phi_\infty$. Then the following statements are valid:*
*1. Let $v_{g_{z,\epsilon}}$ be the Herglotz wave function with kernel $g_{z,\epsilon}$. Then for every $z \in D$ the limit $\lim_{\epsilon\to 0} v_{g_{z,\epsilon}}(z)$ exists. Furthermore, there exists $c > 0$, depending only on $F$, such that for every $z \in D$ we have that*

$$c\|g_z\|^2_{L^2(S^2)}\| \leq \lim_{\epsilon\to 0} \left|v_{g_{z,\epsilon}}(z)\right| \leq \|g_z\|^2_{L^2(S^2)}.$$

*2. For $z \notin D$ we have that $\lim_{\epsilon\to 0} v_{g_{z,\epsilon}}(z) = \infty$.*

Using Theorem 5.5 to solve the inverse scattering problem associated with the scattering problem (3.1)–(3.2) is called the *factorization method*. This method has been extended to a wide variety of scattering problems for both acoustic and electromagnetic waves and for details we refer the reader to [53]. Since this method and its generalizations are fully discussed in another chapter of this book, we will not pursue the topic further here. A drawback of both the linear sampling method and the factorization method is the large amount of data needed for the inversion procedure. In particular, although the linear sampling method can be applied for limited aperture far field data, one still needs multistatic data defined on an open subset of $S^2$.

## 5.4 Lower bounds for the surface impedance

One of the advantages that the linear sampling method has over other qualitative methods in inverse scattering theory is that the far field equation can not only be used to determine the support of the scatterer but in some circumstances can also be used to obtain lower bounds on the constitutive parameters of the scattering object. In this section we will consider two such problems, the determination of the surface impedance of a partially coated object and the determination of the index of refraction of a non-absorbing scatterer. In the first case we will need to consider a mixed boundary value problem for the Helmholtz equation whereas in the second case we will need to investigate the spectral properties of the interior transmission problem introduced in Theorem 5.2 of the previous section.

Mixed boundary value problems typically model the scattering by objects that are coated by a thin layer of material on part of the boundary. In the study of inverse problems for partially coated obstacles, it is important to mention that, in general, it is not known a priori whether or not the scattering object is coated and if so what is the extent of the coating. We will focus our attention in this section on the special case when on the coated part of the boundary the total field satisfies an impedance boundary condition and on the remaining part of the boundary the total field (or the tangential component in the case of electromagnetic waves) vanishes. This corresponds to the case when a perfect conductor is partially coated by a thin dielectric layer. For other mixed boundary value problems in scattering theory and their associated inverse problems we refer the reader to [7] and the references contained therein.

Let $D \subset \mathbb{R}^3$ be as described in Section 1 and let $\partial D$ be dissected as $\partial D = \Gamma_D \cup \Pi \cup \Gamma_I$ where $\Gamma_D$ and $\Gamma_I$ are disjoint, relatively open subsets of $\partial D$ having $\Pi$ as their common boundary. Let $\lambda \in L_\infty(\Gamma_I)$ be such that $\lambda(x) \geq \lambda_0 > 0$ for all $x \in \Gamma_I$. We consider the scattering problem for the Helmholtz equation (3.1) where $u = u^i + u^s$ satisfies the boundary condition

$$\begin{aligned} u = 0 \quad \text{on } \Gamma_D, \\ \frac{\partial u}{\partial \nu} + i\lambda u = 0 \quad \text{on } \Gamma_I, \end{aligned} \tag{5.14}$$

$u^i(x) = e^{ik\,x\cdot d}$ and $u^s$ is a radiating solution. It can be shown that this direct scattering problem has a unique solution in $H_{\text{loc}}(\mathbb{R}^3 \setminus \bar{D})$ [7] . We again define the far field operator by (5.1) where $u_\infty$ is now the far field pattern corresponding to the boundary condition (5.14).

In [7] it is shown that there exists a unique solution $u_z \in H^1(D)$ of the interior mixed boundary value problem

$$\Delta u_z + k^2 u_z \;=\; 0 \quad \text{in } D \tag{5.15}$$

$$u_z + \Phi(\cdot, z) \;=\; 0 \quad \text{on } \Gamma_D \tag{5.16}$$

$$\frac{\partial}{\partial \nu}\left(u_z + \Phi(\cdot, z)\right) + i\lambda\left(u_z + \Phi(\cdot, z)\right) \;=\; 0 \quad \text{on } \Gamma_I \tag{5.17}$$

for $z \in D$ where $\Phi$ is the fundamental solution to the Helmholtz equation. Then if $\Phi_\infty(\cdot, z)$ is the far field pattern of $\Phi(\cdot, z)$ we have the following theorem [7]:

**Theorem 5.7** *Let $\epsilon > 0$, $z \in D$, and $u_z$ be the unique solution of* (5.15)–(5.17). *Then there exists a Herglotz wave function $v_{g_{z,\epsilon}}$ with kernel $g_{z,\epsilon} \in L^2(S^2)$ such that*

$$\|u_z - v_{g_{z,\epsilon}}\|_{H^1(D)} \leq \epsilon.$$

*Moreover, there exists a positive constant $c$ independent of $\epsilon$ such that*

$$\|Fg_{z,\epsilon} - \Phi_\infty(\cdot, z)\|_{L^2(S^2)} \leq c\epsilon.$$

We can now use Green's formula to show that [8]

$$\int_{\partial D} \lambda |u_z + \Phi(\cdot, z)|^2 \, ds = -\frac{k}{4\pi} - \operatorname{Im} u_z(z).$$

From this we immediately deduce the inequality

$$\|\lambda\|_{L^\infty(\Gamma_I)} \geq \frac{-k/4\pi - \operatorname{Im} u_z(z)}{\|u_z + \Phi(\cdot, z)\|^2_{L^2(\partial D)}}\;. \tag{5.18}$$

How is the inequality (5.18) of practical use? To evaluate the right hand side of (5.18) we need to know $\partial D$ and $u_z$. Both are determined by solving the far field equation $Fg = \Phi_\infty(\cdot, z)$ using Tikhonov regularization and then using the linear sampling method to determine $\partial D$ and the regularized solution $g \in L^2(S^2)$ to construct the Herglotz wave function $v_g$. By Theorem 5.7 we expect that $v_g$ is an approximation to $u_z$. However, at this time, there is no analogue of Theorem 5.6 for the mixed boundary value problem and hence this is not guaranteed. Nevertheless in all numerical experiments to date this approximation appears to be remarkably accurate and thus allows us to obtain a lower bound for $\|\lambda\|_{L^\infty(\Gamma_I)}$ via (5.18).

The corresponding scattering problem for the Maxwell equations is to find a solution $E = E^i + E^s$ to (3.5) satisfying the mixed boundary condition

$$\begin{aligned} \nu \times E = 0 \quad \text{on } \Gamma_D \\ \nu \times \operatorname{curl} E - i\lambda(\nu \times E) \times \nu = 0 \quad \text{on } \Gamma_I \end{aligned} \tag{5.19}$$

where $E^i$ is the plane wave (2.38) and $E^s$ is radiating. The existence of a unique solution $E$ in an appropriate Sobolev space is shown in [12]. We again define the far field operator by (5.9) where $E_\infty$ is now the electric far field pattern corresponding to (5.19). Analogous to (5.15)–(5.17) we now have the interior mixed boundary value problem

$$\operatorname{curl}\operatorname{curl} E_z - k^2 E_z = 0 \quad \text{in } D \tag{5.20}$$

$$\nu \times [E_z + E_e(\cdot, z, q)] = 0 \quad \text{on } \Gamma_D \tag{5.21}$$

$$\nu \times \operatorname{curl}[E_z + E_e(\cdot, z, q)] - i\lambda\,[\nu \times (E_z + E_e(\cdot, z, q))] = 0 \quad \text{on } \Gamma_I \tag{5.22}$$

where $z \in D$ and $E_e$ is the electric dipole defined by (5.12). The existence of a unique solution to (5.20)–(5.22) in an appropriate Sobolev space is established in [12]. From the analysis in [8] we have the inequality

$$\|\lambda\|_{L^\infty(\Gamma_I)} \geq \frac{-k^2|q|^2/6\pi + k\,\mathrm{Re}(q \cdot E_z)}{\|E_z + E_e(\cdot, z, q)\|^2_{L^2_t(\partial D)}} \tag{5.23}$$

analogous to (5.18) for the Helmholtz equation. For numerical examples using (5.23) we refer the reader to [27].

Similar inequalities as those derived above for the impedance boundary value problem can also be obtained for the *conductive boundary value problem*, i.e., the case when a dielectric is partially coated by a thin, highly conducting layer [7, 27].

## 5.5 Transmission eigenvalues

We have previously encountered transmission eigenvalues in Theorem 5.2 where they were connected with the injectivity and dense range of the far field operator. In this section we shall examine transmission eigenvalues and the interior transmission problem in more detail. This investigation is particularly relevant to the inverse scattering problem since transmission eigenvalues can be determined from the far field pattern [11] and, as will be seen, can be used to obtain lower bounds for the index of refraction.

We begin by considering the interior transmission problem (5.4)–(5.7) from Theorem 5.2. and will be concerned with the existence and countability of transmission eigenvalues. The existence of transmission eigenvalues was first established by Päivärinta and Sylvester [74] and their results were strengthened by Cakoni, Gintides and Haddar [15].

**Theorem 5.8** *Assume that* $n$ *is real valued such that* $n(x) > 1$ *for all* $x \in \bar{D}$ *or* $0 < n(x) < 1$ *for all* $x \in \bar{D}$. *Then there exist an infinite number of transmission eigenvalues.*

We note that it can be shown that as $\sup_{x\in D} |n(x) - 1| \to 0$ then the first transmission eigenvalue tends to infinity, i.e., in the Born approximation transmission eigenvalues do not exist [29].

Similar results as in Theorem 5.8 can be obtained for an anisotropic medium and for the Maxwell equations [15].

By Theorem 5.8 the existence of transmission eigenvalues is established. It can also be shown that the set of transmission eigenvalues is discrete [16, 21, 51, 84]. The following theorem [29] establishes a lower bound for the first transmission eigenvalue which is reminiscent of the famous *Faber–Krahn inequality* for the first Dirichlet eigenvalue for the negative Laplacian (which we denote by $\lambda_1$).

**Theorem 5.9** *Assume that* $n(x) > 1$ *for* $x \in \overline{D}$ *and let* $k_1 > 0$ *be the first transmission eigenvalue for the interior transmission problem* (5.4)–(5.7). *Then*

$$k_1^2 \geq \frac{\lambda_1(D)}{\sup\limits_{x\in D} n(x)} .$$

Theorem 5.9 has been generalized to the case of anisotropic media and the Maxwell equations [9].

Finally, in the case of the interior transmission problem (5.4)–(5.7) where there are cavities in $D$, i.e., regions $D_0 \subset D$ where $n(x) = 1$ for $x \in D_0$, it can be shown that transmission eigenvalues exist, form a discrete set and the first transmission eigenvalue $k_1$ satisfies [10]

$$k_1^2 \geq \frac{\lambda_1(D)}{\sup\limits_{x\in D\setminus D_0} n(x)} .$$

Note that, since in each of the above cases $D$ can be determined by the linear sampling method, $\lambda_1(D)$ is known and hence given $k_1$ the above inequalities yield a lower bound for the supremum of the index of refraction.

## 5.6 Historical Remarks

The use of qualitative methods to solve inverse scattering problems began with the 1996 paper of Colton and Kirsch [20] and the 1997 paper of Colton, Piana and Potthast [30]. These papers were in turn motivated by the dual space method of Colton and Monk developed in [25] and [26]. Both [20] and [30] were concerned with the case of scattering of acoustic waves. The extension of the linear sampling method to electromagnetic waves was first outlined by Kress [57] and then discussed in more detail by Colton, Haddar and Monk [19] and Haddar and Monk [35]. The factorization method was introduced in 1998 and 1999 by Kirsch [49, 50] for acoustic scattering problems. Attempts to extend the factorization method to the case of electromagnetic waves have been only partly successful. In particular the factorization method for the scattering of electromagnetic waves by a perfect conductor remains an open question.

In addition to the linear sampling and factorization methods there have been a number of other qualitative methods developed primarily by Ikehata and Potthast and their co-workers. Although space is too short to discuss these alternate qualitative methods in this survey, we refer the reader to [53, 79] for details and references.

The countability of transmission eigenvalues for acoustic waves was established by Colton, Kirsch and Päivärinta [21] and Rynne and Sleeman [84] and for the Maxwell equations by Cakoni and Haddar [16] and Kirsch [51]. The existence of transmission eigenvalues for acoustic waves was first given by Päivärinta and Sylvester [74] for the isotropic case and for the anisotropic case by Cakoni and Haddar [17] and Kirsch [52] who also established the existence of transmission eigenvalues for Maxwell's equations. These results were subsequently improved by Cakoni, Gintides and Haddar [15]. Inequalities for the first transmission eigenvalues were first obtained by Colton, Päivärinta and Sylvester [29] and Cakoni, Colton and Haddar [9, 10].

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