# A NEW LINEAR SAMPLING METHOD FOR THE ELECTROMAGNETIC IMAGINING OF BURIED OBJECTS

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We present a new linear sampling method for determining the shape of scattering objects imbedded in a known inhomogeneous medium from a knowledge of the scattered electromagnetic field due to a point source incident field at fixed frequency. The method does not require any a prior information on the physical properties of the scattering object and, under some restrictions, avoids the need to compute the Green's tensor for the background medium.

## 1. Introduction

The mathematical modelling of the application of scattering of electromagnetic waves in mine detection, medical imagining, nondestructive testing etc. leads to the inverse scattering problem of determining the shape of the scattering object imbedded in a known inhomogeneous background. Typically, in such applications, neither the physical properties of the scatterer object nor geometrical features such as the number of components etc. are known a priori. In particular the scatterer can be a perfect conductor of dielectric, partially coated etc and this information in general is not available. The aim of this paper is to develop a method for solving the inverse problem which does not depend on the physical properties of the scatterer and

*Work partially supported by the Air Force Office of Scientific Research under grant FA9550-05-1-0127.

is easy to implement. The solution method we have in mind is a new version of the linear sampling method based on the reciprocity gap functional which, in certain cases, avoids the need to compute the Green's function for the background medium. For the sake of theoretical justification of the method and in order to present the basic ideas we confine ourselves to the case of scattering by a perfect conductor buried in a known piecewise homogeneous background. However, the method can be applied to other type of scatterer and we refer the reader to [1], [4] for the mathematical justification of the method in the case of anisotropic penetrable objects.

We consider the scattering of a time-harmonic electromagnetic field of frequency $\omega$ by a scattering object embedded in a piecewise homogeneous background in $\mathbb{R}^3$. We assume that the magnetic permeability $\mu_0 > 0$ of the background medium is a positive constant whereas the electric permitivity $\epsilon(x)$ and conductivity $\sigma(x)$ are piecewise constant. Moreover we assume that for $|x| = r > R$, for $R$ sufficiently large, $\sigma = 0$ and $\epsilon(x) = \epsilon_0$. After an appropriate scaling [7] and elimination of the magnetic field we now obtain the following equation for the electric field $E$ in the background medium

$$\operatorname{curl}\operatorname{curl} E - k^2 n(x) E = 0,$$

$k = \epsilon_0 \mu_0 \omega^2$ and $n(x) = \frac{1}{\epsilon_0}\left(\epsilon(x) + i\frac{\sigma(x)}{\omega}\right)$. Note that the piecewise constant function $n(x)$ satisfies $n(x) = 1$ for $r > R$, $\Re(n) > 0$ and $\Im(n) \geq 0$. The surfaces across which $n(x)$ is discontinuous are assumed to be piecewise smooth.

Now let $D$ be the support of a perfect conductor embedded in the above piecewise homogeneous background. We suppose that $\mathbb{R}^3 \setminus \overline{D}$ is connected the boundary $\partial D$ of $D$ is piecewise smooth and denote by $\nu$ the outward unit normal. Furthermore, we suppose that the incident field is an electric dipole located at $x_0 \in \Lambda$ with polarization $p \in \mathbb{R}^3$, where $\Lambda$ is a smooth open surface situated in a layer with constant index of refraction $n_s$, given by

$$E_e(x, x_0, p, k_s) := \frac{i}{k_s}\operatorname{curl}_x \operatorname{curl}_x p \, \frac{e^{ik_s|x-x_0|}}{4\pi|x-x_0|} \tag{1}$$

where $k_s^2 = k^2 n_s$. We denote by $\mathbb{G}(x, x_0)$ the free space Green's tensor of the background medium and define $E^i(x) := E^i(x, x_0, p) = \mathbb{G}(x, x_0)p$ which satisfies

$$\operatorname{curl}\operatorname{curl} E^i(x) - k^2 n(x) E^i(x) = p\,\delta(x - x_0) \qquad \text{in } \mathbb{R}^3, \tag{2}$$

where $\delta$ denotes the Dirac distribution. Note that $E^i$ can be written as

$$E^i(x) = E_e(x, x_0, p, k_s) + E_b^s(x) \tag{3}$$

where $E_b^s = E_b^s(\cdot,\, x_0, p)$ is the electric scattered field due to the background medium. The scattering of the dipole $E_e(x, x_0, p, k_s)$ by the perfect conductor $D$ is described by the following boundary value problem: Given $E^i = E^i(\cdot\,, x_0, p) = \mathbb{G}(\cdot\,, x_0)p$, find $E \in H_{loc}(\text{curl}, \mathbb{R}^3 \setminus \overline{D} \cup \{x_0\})$ satisfying

$$\text{curl}\,\text{curl}\,E - k^2 n(x) E = 0 \qquad \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \cup \{x_0\}, \tag{4}$$

$$\nu \times E = 0 \qquad \text{on} \quad \partial D, \tag{5}$$

$$E^s := (E - E^i) \in H_{loc}(\text{curl}, \mathbb{R}^3 \setminus \overline{D}), \tag{6}$$

$$\lim_{r\to\infty} (\text{curl}\, E^s \times x - ikrE^s) = 0. \tag{7}$$

where

$$H(\text{curl}, D) := \{u \in (L^2(D))^3 \,:\, \nabla \times u \in (L^2(D))^3\}$$

and $H_{loc}(\text{curl}, \mathbb{R}^3 \setminus \overline{D})$ the space of functions $u \in H(\text{curl}, K)$ for all compact sets $K \subset \mathbb{R}^3 \setminus \overline{D}$.

**Remark 1.1.** *It is also possible to consider the problem of objects buried in an unbounded multi-layer medium. In this case, the radiation condition and mathematical analysis of the forward become more complicated (see* [8] *for the case of two layered medium). However the following analysis of the inverse scattering problems remains the same.*

Let $\Omega$ be such that $\overline{D}$ is contained in $\Omega$ and the open surface $\Lambda$ is contained in $\mathbb{R}^3 \setminus \overline{\Omega}$. Let $\Gamma$ denote the piecewise smooth boundary of $\Omega$. Note that $\Lambda$ may be a subset of $\Gamma$. The *inverse scattering problem* we are interested in is to determine $D$ from a knowledge of the tangential components $\nu \times E$ and $\nu \times H$ of the total electric field $E = E(\cdot\,, x_0, p)$ and magnetic field $H = \frac{1}{ik}\text{curl}\, E$ measured on $\Gamma$ for all point sources $x_0 \in \Lambda$ and two linearly independent polarizations $p$ tangent to $\Lambda$ at $x_0$. Here $\nu$ denotes the outward unit normal to $\Gamma$. The *linear sampling method* can be use to solve the inverse scattering problem. [1] (for a scholarly review of the this method we direct the reader to [2], [3], [5]). In particular, the linear sampling method is based on finding a tangential field $\varphi_z \in L^2_t(\Lambda)$ that satisfies the following integral equation of the first kind referred to as the *near field equation*:

$$(\mathcal{F}\varphi_z)(x) := \int_\Lambda \nu(x) \times E^s(x, y, \varphi_z(y))\, ds(y) = \nu(x) \times \mathbb{G}(x, z)\, q \tag{8}$$

for all $x \in \Gamma$, where $z \in \Omega$ and $q \in \mathbb{R}^3$ is an artificial polarization. Note that since $E^s$ depends linearly on the polarization $p$, the *near field operator* $\mathcal{F} : L_t^2(\Lambda) \to L_t^2(\Gamma)$ is linear. Assuming that $k$ is not a Maxwell eigenvalue, i.e. the interior boundary value problem

$$\operatorname{curl}\operatorname{curl} E - k^2 n(x) E = 0 \quad \text{in } D \tag{9}$$

$$\nu \times E = -\nu \times \mathbb{G}(\cdot, z) q \quad \text{on } \partial D \tag{10}$$

has a unique solution, one can prove that

(1) For $z \in D$ and a given $\epsilon > 0$, there exists a $\varphi_z^\epsilon \in L_t^2(\Lambda)$ such that

$$\|\mathcal{F}\varphi_z^\epsilon - \nu \times \mathbb{G}(\cdot\,, z) q\|_{L_t^2(\Gamma)} < \epsilon$$

and the corresponding potential $\mathcal{S}\varphi_z^\epsilon$ converges to the solution of (9)-(10) in $H(curl, D)$ as $\epsilon \to 0$.

(2) For a fixed $\epsilon > 0$, we have that

$$\lim_{z \to \partial D} \|S\varphi_z^\epsilon\|_{H(curl,D)} = \infty \quad \text{and} \quad \lim_{z \to \partial D} \|\varphi_z^\epsilon\|_{L_t^2(\Lambda)} = \infty.$$

(3) For $z \in \mathbb{R}^3 \setminus \overline{D}$ and a given $\epsilon > 0$, every $\varphi_z^\epsilon \in L_t^2(\Lambda)$ that satisfies

$$\|\mathcal{F}\varphi_z^\epsilon - \nu \times \mathbb{G}(\cdot\,, z) q\|_{L_t^2(\Gamma)} < \epsilon$$

is such that

$$\lim_{\epsilon \to 0} \|\mathcal{S}\varphi_z^\epsilon\|_{H(curl,D)} = \infty \quad \text{and} \quad \lim_{\epsilon \to 0} \|\varphi_z^\epsilon\|_{L_t^2(\Lambda)} = \infty.$$

The above result provides a characterization for the boundary $\partial D$ of the scattering object $D$. Unfortunately, since the behavior of $S\varphi_z^\epsilon$ is described in terms of a norm depending on the unknown region $D$, $S\varphi_z^\epsilon$ can not be used to characterize $D$. Instead the linear sampling method characterizes the obstacle by the behavior of $\varphi_z^\epsilon$. In particular, given a discrepancy $\epsilon > 0$ and $\varphi_z^\epsilon$ the $\epsilon$-approximate solution of the near field equation, the boundary of the scatterer is reconstructed as the set of points $z$ where the $L_t^2(\Lambda)$ norm of $\varphi_z^\epsilon$ becomes large.

Even though the linear sampling method, in principle, can be used in the case of a quite general inhomogeneous background, the main drawback of the method in this case is the need to compute the background Green's function. This job can be numerically very costly for complex background geometries. The main goal of this paper is to show how, at the expense of additional data and restrictions, one can avoid the need to compute the background Green's function.

## 2. The Reciprocity Gap Functional

We make two additional assumptions. First, we assume that the medium inside the domain $\Omega$ containing the scattering object $D$ is homogeneous with constant index of refraction $n_b$ and define $k_b^2 = k^2 n_b$. Second, we assume that *both* the tangential components $\nu \times E$ and $\nu \times H$ of the total electric field $E = E(\cdot, x_0, p)$ and magnetic field $H = \frac{1}{ik}\text{curl}\, E$, respectively, are known on $\Gamma$ for all point sources $x_0 \in \Lambda$. In other words we assume that we know $\nu \times E|_\Gamma$ and $\nu \times \text{curl} E|_\Gamma$ for all $x_0 \in \Lambda$. Furthermore, without loss of generality, we assume that $\Lambda$ is a closed surface surrounding $\Omega$ situated in a layer with index of refraction $n_s$. By an analyticity argument the following analysis also holds true if the point sources are located on an open analytic surface provided it can be extended to a closed (analytic) surface as above.

We need to recall the definition of the following trace spaces

$$H_{div}^{-\frac{1}{2}}(\partial D) := \left\{ u \in (H^{-\frac{1}{2}}(\partial D))^3, \quad \nu \cdot u = 0, \quad \text{div}_{\partial D}\, u \in H^{-\frac{1}{2}}(\partial D) \right\}$$
$$H_{curl}^{-\frac{1}{2}}(\partial D) := \left\{ u \in (H^{-\frac{1}{2}}(\partial D))^3, \quad \nu \cdot u = 0, \quad \text{curl}_{\partial D}\, u \in H^{-\frac{1}{2}}(\partial D) \right\}$$

with $\text{curl}_{\partial D}$ denoting the surface curl. It is known that traces $\nu \times u|_{\partial D}$ and $\nu \times (u \times \nu)|_{\partial D}$ of $u \in H(\text{curl}, D)$ (or $u \in H_{loc}(\text{curl}, D)$) are in $H_{div}^{-\frac{1}{2}}(\partial D)$ and $H_{curl}^{-\frac{1}{2}}(\partial D)$ respectively. Note that by an integration by parts we can define a duality relation between $H_{div}^{-\frac{1}{2}}(\partial D)$ and $H_{curl}^{-\frac{1}{2}}(\partial D)$.
Let $E = E(\cdot, x_0, p) = E^s(\cdot, x_0, p) + \mathbb{G}(\cdot, x_0)p$ and $H = 1/ik\, \text{curl}\, E$ be the total electric and magnetic fields, respectively, corresponding to the scattering problem (4)-(7). Then for any function $W \in H(\text{curl}, \Omega)$, we can define the *gap reciprocity functional* by

$$\mathcal{R}(E, W) = \int_\Gamma (\nu \times E) \cdot \text{curl}\, W - (\nu \times W) \cdot \text{curl}\, E\, ds. \tag{11}$$

Since $E \in H(\text{curl}, \Omega)$, the integral is interpreted in the sense of the duality between $H_{div}^{-\frac{1}{2}}(\Gamma)$ and $H_{curl}^{-\frac{1}{2}}(\Gamma)$. Note that $E$ depends on $x_0$ and hence so does $\mathcal{R}$. Next we define the subspace $\mathbb{H}(\Omega) \subset H(\text{curl}, \Omega)$ by

$$\mathbb{H}(\Omega) := \left\{ W \in H(\text{curl}, \Omega) : \ \ \text{curl}\,\text{curl}\, W - k_b^2 W = 0 \right\}.$$

The reciprocity gap functional restricted to $\mathbb{H}(\Omega)$ can be seen as an operator $R : \mathbb{H}(\Omega) \to L_t^2(\Lambda)$ defined by

$$R(W)(x_0) \cdot p(x_0) = \mathcal{R}(E(\cdot, x_0, p(x_0)), W) \tag{12}$$

for all $x_0 \in \Lambda$ and $p(x_0)$ a tangent vector to $\Lambda$ at $x_0$. Now, we consider a subset $\{A\varphi \in \mathbb{H}(\Omega) : \varphi \in L^2_{div}(\tilde{\Lambda})\}$ of $\mathbb{H}(\Omega)$, where $A\varphi$ is the single layer potential defined by

$$(A\varphi)(x) := \operatorname{curl}\operatorname{curl}\int_{\tilde{\Lambda}} \varphi(y)\Phi(x,y,k_b)\,ds, \qquad \varphi \in L^2_{div}(\tilde{\Lambda}) \tag{13}$$

where

$$\Phi(x,y,k_b) := \frac{1}{4\pi}\frac{e^{ik_b|x-y|}}{|x-y|}, \quad x \neq y,$$

$\tilde{\Lambda}$ is a regular part of the boundary of some simply connected domain containing $\Omega$ in its interior, and $L^2_{div}(\tilde{\Lambda})$ is the space of vector functions $u \in (L^2(\tilde{\Lambda}))^3$ such that $\nu \cdot u = 0$ and $\operatorname{div}_{\partial D} u \in L^2(\tilde{\Lambda})$. Next, letting

$$E_e(x,z,q,k_b) = \frac{i}{k}\operatorname{curl}_x \operatorname{curl}_x q\,\Phi(x,z,k_b), \qquad q \in \mathbb{R}^3 \tag{14}$$

denote the electric dipole corresponding to $k_b$ we look for a solution $\varphi \in L^2_{div}(\tilde{\Lambda})$ of

$$\mathcal{R}(E, A\varphi) = \mathcal{R}(E, E_e(\cdot,\, z, q, k_b)). \tag{15}$$

Then, the linear sampling method based on the reciprocity gap functional characterizes $D$ from the behavior of $\varphi$ for different sampling points $z \in \Omega$. In the rest of the paper we investigate the solvability of (15) with respect to $\varphi$. To this end we prove the following important lemmas.

**Lemma 2.1.** *Assume that $k_b$ is not a Maxwell eigenvalue for $D$. Then the operator $R : \mathbb{H}(\Omega) \to L^2_t(\Lambda)$ defined by (12) is injective.*

**Proof**. $RW = 0$ means $\mathcal{R}(E(\cdot, x_0, p(x_0)), W) = 0$ for all $(x_0, p(x_0))$ as in (12). Since both $E$ and $W$ satisfy Maxwell's equation in $\Omega \setminus \overline{D}$, we have, using the boundary condition on $\partial D$,

$$0 = \int_\Gamma (\nu \times E)\cdot \operatorname{curl} W - (\nu \times W)\cdot \operatorname{curl} E\, ds = -\int_{\partial D} (\nu \times W)\cdot \operatorname{curl} E\, ds.$$

It suffices to show that the set $L := \{(\operatorname{curl} E(\cdot,\, x_0, p(x_0)))_\top :\, x_0 \in \Lambda\}$ is dense in $H^{-\frac{1}{2}}_{curl}(\partial D)$. Indeed, this fact implies that $\nu \times W = 0$ on $\partial D$ and from the uniqueness of the solution to (9)-(10) we have that $W = 0$ in $D$, whence from the unique continuation principle we obtain $W = 0$ in $\Omega$.
To prove the denseness property, let $f \in H^{-\frac{1}{2}}_{div}(\partial D)$ and assume that

$$\int_{\partial D} f \cdot (\nu \times \operatorname{curl} E)\, ds = 0$$

for all total fields $E$ such that $(\operatorname{curl} E)_\top \in L$. Let $\tilde{E}$ be the unique solution to

$$\begin{aligned} \operatorname{curl}\operatorname{curl}\tilde{E} - k^2 n(x)\tilde{E} = 0 \qquad & \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \\ \nu \times \tilde{E} = f \qquad & \text{on} \quad \partial D \\ \lim_{r\to\infty}\left(\operatorname{curl}\tilde{E} \times x - ikr\tilde{E}\right) = 0. & \end{aligned}$$

By a duality argument we have that

$$\begin{aligned} 0 &= \int_{\partial D} f \cdot (\nu \times \operatorname{curl} E)\, ds = \int_{\partial D} \tilde{E} \cdot [\nu \times \operatorname{curl}(E^s + \mathbb{G}(\cdot,\, x_0)p)]\, ds \\ &= \int_{\partial D} \tilde{E} \cdot (\nu \times \operatorname{curl} E^s)\, ds + \int_{\partial D} \tilde{E} \cdot (\nu \times \operatorname{curl} \mathbb{G}(\cdot,\, x_0)p)\, ds. \end{aligned} \tag{16}$$

Since both $E^s$ and $\tilde{E}$ are radiating solutions to $\operatorname{curl}\operatorname{curl} E - k^2 n(x)E = 0$ outside $D$, by applying the vector Green's formula we have that

$$\int_{\partial D} \tilde{E} \cdot (\nu \times \operatorname{curl} E^s)\, ds = \int_{\partial D} E^s \cdot (\nu \times \operatorname{curl} \tilde{E})\, ds. \tag{17}$$

Substituting (17) into (16) and using the boundary condition $\nu \times E^s = -\nu \times \mathbb{G}(\cdot,\, x_0)p$ on $\partial D$ we have that

$$0 = \int_{\partial D} \tilde{E} \cdot (\nu \times \operatorname{curl} \mathbb{G}(\cdot,\, x_0)p)\, ds + \int_{\partial D} E^s \cdot (\nu \times \operatorname{curl} \tilde{E})\, ds = p \cdot \tilde{E}(x_0).$$

Since $p$ is an arbitrary polarization in the tangent plane to $\Lambda$ at $x_0$, we obtain $\nu \times \tilde{E}(x_0) = 0$ for $x_0 \in \Lambda$. Furthermore, since $\tilde{E}$ is a radiating solution to Maxwell's equations outside the domain bounded by $\Lambda$, we conclude by the uniqueness theorem for the scattering problem for a perfect conductor (c.f. [7]) that $\tilde{E} = 0$ outside the domain bounded by $\Lambda$. Then the unique continuation principle implies that $\tilde{E} = 0$ outside $D$, whence $f = 0$, which proves the lemma. □

**Lemma 2.2.** *Assume that $k_b$ is not a Maxwell eigenvalue for $D$. Then the operator $R : \mathbb{H}(\Omega) \to L^2_t(\Lambda)$ defined by (12) has dense range.*

**Proof**. Consider $\alpha \in L^2_t(\Lambda)$ and assume that

$$(RW, \alpha)_{L^2_t(\Lambda)} = 0 \;\; \text{for all } W \in \mathbb{H}(\Omega).$$

From (12) and the bi-linearity of $\mathcal{R}$ one has

$$(RW, \alpha)_{L^2_t(\Lambda)} = \int_\Lambda \mathcal{R}(E(\cdot, x_0, \alpha(x_0)), W)\, ds(x_0) = \mathcal{R}(\mathcal{E}, W),$$

where

$$\mathcal{E}(x) = \int_\Lambda E(x, x_0, \alpha(x_0))\, ds(x_0). \tag{18}$$

Using Green's vector formulas and the boundary condition on $\partial D$ one concludes that

$$0 = \mathcal{R}(\mathcal{E}, W) = -\int_{\partial D} \operatorname{curl} \mathcal{E} \cdot (\nu \times W)\, ds \tag{19}$$

for all $W \in \mathbb{H}(\Omega)$. Since $\mathbb{H}(\Omega)$ contains the Herglotz wave functions, from [6] one has that the set of $(\nu \times W)|_{\partial D}$ is dense in $H_{div}^{-\frac{1}{2}}(\partial D)$. Therefore

$$\operatorname{curl} \mathcal{E} \times \nu = 0 \quad \text{on } \partial D.$$

Since $\mathcal{E} \times \nu = 0$ on $\partial D$ as well, the extension of $\mathcal{E}$ by 0 inside $D$ satisfies Maxwell's equations inside the domain bounded by $\Lambda$ with the index $n$ set equal to $n_b$ inside $D$. From the unique continuation principle one has that $\mathcal{E}$ is 0 inside the domain bounded by $\Lambda$ and outside $D$. Noting that

$$\mathcal{E}(x) = \int_\Lambda (E^s(x, x_0, \alpha(x_0)) + \mathbb{G}(x, x_0)\alpha(x_0))\, ds(x_0)$$

one concludes that $\mathcal{E} \times \nu$ is continuous across $\Lambda$. The uniqueness theorem of the exterior problem for Maxwell's equations with boundary data $\nu \times \mathcal{E} = 0$ on $\Lambda$ implies that $\mathcal{E} = 0$ outside the domain bounded by $\Lambda$ as well. Finally, from the jump relations of the vector potential across $\Lambda$ [7] we have that

$$0 = \operatorname{curl} \mathcal{E}|_{\Lambda^+} - \operatorname{curl} \mathcal{E}|_{\Lambda^-} = -\alpha \quad \text{on} \quad \Lambda$$

which ends the proof. □

**Lemma 2.3.** *Assume that $k$ is not a Maxwell eigenvalue for $D$. Then the set $\{A\varphi,\ \varphi \in H_{div}^{-\frac{1}{2}}(\tilde{\Lambda})$ is dense in $H(curl, D)$.*

**Proof**. Making use of the well-posedness of

$$\operatorname{curl}\operatorname{curl} W - k^2 n_b W = 0 \quad \text{in } D \tag{20}$$

$$\nu \times W = f \quad \text{on } \partial D \tag{21}$$

with $f \in H_{div}^{-\frac{1}{2}}(\partial D)$, it suffices to show that $\nu \times A\varphi|_{\partial D}$ for all $\varphi \in H_{div}^{-\frac{1}{2}}(\tilde{\Lambda})$ is dense in $H_{div}^{-\frac{1}{2}}(\partial D)$. To this end, let $\psi \in H_{curl}^{-\frac{1}{2}}(\partial D)$ and look at the dual operator $A^* : H_{curl}^{-\frac{1}{2}}(\partial D) \to H_{div}^{-\frac{1}{2}}(\Gamma)$ such that

$$\langle \nu \times A\varphi,\ \psi \rangle_{\partial D} = \langle \varphi,\ A^* \psi \rangle_{\tilde{\Lambda}}$$

where $\langle\cdot,\,\cdot\rangle$ denotes the $H_{div}^{-\frac{1}{2}}$, $H_{curl}^{-\frac{1}{2}}$ duality pairing. By changing the order of integration one can show that for $\psi \in H_{div}^{-\frac{1}{2}}(\partial D)$

$$(A^*\psi)(y) = \nu(y) \times \left( \mathrm{curl}_y\, \mathrm{curl}_y \int_{\partial D} \psi(x)\Phi(x,y)\, ds(x) \right) \times \nu(y), \qquad y \in \tilde{\Lambda}$$

where $\nu$ is the unit outward normal to $\tilde{\Lambda}$. Now, since $k_b$ is not a Maxwell eigenvalue for $D$, we conclude that $A^*$ is injective, whence $\left\{ \nu \times A\varphi|_{\partial D} :\ \varphi \in H_{div}^{-\frac{1}{2}}(\tilde{\Lambda}) \right\}$ is dense in $H_{div}^{-\frac{1}{2}}(\partial D)$. □ Now we are at the position to prove the main result of this paper.

**Theorem 2.1.** *Assume that $k$ is not a Maxwell eigenvalue for $D$ and let $E = E(\cdot\,, x_0, p)$ and $H = 1/ik\, curl E$ be the total electric and magnetic fields, respectively, corresponding to the scattering problem (4)-(7). Then*

*(1) For $z \in D$ and a given $\epsilon > 0$, there exists a $\varphi_z^\epsilon \in H_{div}^{-\frac{1}{2}}(\tilde{\Lambda})$ such that*

$$\|\mathcal{R}(E, A\varphi_z^\epsilon) - \mathcal{R}(E, E_e(\cdot,\, z, q, k_b))\|_{L_t^2(\Lambda)} < \epsilon$$

*and the corresponding potential $A\varphi_z^\epsilon$ converges to the solution of*

$$curl\, curl W - k^2 n_b W = 0 \quad \text{in } D \tag{22}$$

$$\nu \times W = E_e(\cdot,\, z, q, k_b) \quad \text{on } \partial D \tag{23}$$

*in $H(\mathrm{curl}, D)$ as $\epsilon \to 0$.*

*(2) For a fixed $\epsilon > 0$, we have that*

$$\lim_{z\to\partial D} \|A\varphi_z^\epsilon\|_{H(curl,D)} = \infty \quad \text{and} \quad \lim_{z\to\partial D} \|\varphi_z^\epsilon\|_{H_{div}^{-\frac{1}{2}}(\tilde{\Lambda})} = \infty.$$

*(3) For $z \in \mathbb{R}^3 \setminus \overline{D}$ and a given $\epsilon > 0$, every $\varphi_z^\epsilon \in H_{div}^{-\frac{1}{2}}(\tilde{\Lambda})$ that satisfies*

$$\|\mathcal{R}(E, A\varphi_z^\epsilon) - \mathcal{R}(E, E_e(\cdot,\, z, q, k_b))\|_{L_t^2(\Lambda)} < \epsilon$$

*is such that*

$$\lim_{\epsilon\to 0} \|A\varphi_z^\epsilon\|_{H(curl,D)} = \infty \quad \text{and} \quad \lim_{\epsilon\to 0} \|\varphi_z^\epsilon\|_{H_{div}^{-\frac{1}{2}}(\tilde{\Lambda})} = \infty.$$

**Proof**. Let $z \in D$. Since $W \in \mathbb{H}(\Omega)$ and $E_e(\cdot,\, z, q, k_b)$ satisfy $\mathrm{curl}\,\mathrm{curl}\, W - k_b W = 0$ in $\Omega \setminus \overline{D}$, integrating by parts and using the boundary condition for the total field we have that

$$\mathcal{R}(E,W) - \mathcal{R}(E, E_e(\cdot,\, z, q, k_b)) = -\int_{\partial D} (\nu\times W - \nu\times E_e(\cdot,\, z, q, k_b))\cdot \mathrm{curl}\, E\, ds.$$

From the proof of Lemma 2.1 we see that $\mathcal{R}(E,W) = \mathcal{R}(E, E_e(\cdot,\, z, q, k_b))$ has a unique solution $W$ if and only if there exists a $W \in \mathbb{H}(\Omega)$ such that $\nu\times$

$W = \nu \times E_e(\cdot,\, z, q, k_b) = 0$ on $\partial D$ which is in general not true. However from Lemma 2.3 we have that the family $\{A\varphi, \in H^{-\frac{1}{2}}_{div}(\tilde{\Lambda})\}$ is dense in $H(\text{curl}, \Omega)$. Hence, from the trace theorem, for every $\epsilon > 0$ there exists a potential $A\varphi_z^\epsilon$ such that $\nu \times A\varphi_z^\epsilon$ approximates $\nu \times E_e(\cdot,\, z, q)$ with respect to the $H^{-\frac{1}{2}}_{div}(\partial D)$ norm. In particular, $\varphi_z^\epsilon$ is an approximate solution to (15) and $\nu \times A\varphi_z^\epsilon$ converges to the solution of (22)-(23) in the $H(\text{curl}, D)$ norm as $\epsilon \to 0$. Next, since $\|\nu \times E_e(\cdot,\, z, q)\|_{H^{-\frac{1}{2}}_{div}(\partial D)}$ blows up as $z$ approaches the boundary, we obtain that, for a fixed $\epsilon > 0$, $\lim_{z \to \partial D} \|\nu \times A\varphi_z^\epsilon\|_{H^{-\frac{1}{2}}_{div}(\partial D)} = \infty$ and consequently $\lim_{z \to \partial D} \|A\varphi_z^\epsilon\|_{H(curl,D)} = \infty$ and $\lim_{z \to \partial D} \|g_z^\epsilon\|_{L^2_t(S^2)} = \infty$. Now we consider $z \in \Omega \setminus \overline{D}$ and let $g_z^\epsilon$ and its corresponding Herglotz function $A\varphi_z^\epsilon$ be such that

$$\|\mathcal{R}(E, A\varphi_z^\epsilon) - \mathcal{R}(E, E_e(\cdot,\, z, q, k_b))\|_{L^2(\Lambda)} < \epsilon. \tag{24}$$

Note that from Lemma 2.2 we can always find such a $A\varphi_z^\epsilon$. Assume to the contrary that $\|A\varphi_z^\epsilon\|_{H(curl,D)} < C$ where the positive constant $C$ is independent of $\epsilon$. From the trace theorem we have that $\nu \times A\varphi_z^\epsilon$ is also bounded in the $H^{-\frac{1}{2}}_{div}(\partial D)$ norm. Noting that the total field can be written as $E(\cdot,\, x_0, p) = E^s(\cdot,\, x_0, p) + \mathbb{G}(\cdot, x_0)p$ and integrating by parts, we obtain that

$$\begin{aligned}
\mathcal{R}(E, E_e(x, z, q, k_b)) = & \int_\Gamma (\nu \times E^s(x, x_0, p)) \cdot \text{curl}\, E_e(x, z, q, k_b)\, ds_x \\
& - \int_\Gamma (\nu \times E_e(x, z, q, k_b)) \cdot \text{curl}\, E^s(x, x_0, p)\, ds_x \\
& + \int_\Gamma (\nu \times \mathbb{G}(x, x_0)p) \cdot \text{curl}\, E_e(x, z, q, k_b)\, ds_x \\
& - \int_\Gamma (\nu \times E_e(x, z, q, k_b)) \cdot \text{curl}\, \mathbb{G}(x, x_0)p\, ds_x.
\end{aligned}$$

Due to the symmetry of the background Green's function, $E^s(x, x_0, p)$ as a function of $x_0$ solves $\text{curl}_{x_0} \text{curl}_{x_0} E^s(x, x_0, p) - k^2 n(x_0) E^s(x, x_0, p) = 0$ in the domain bounded by $\Lambda$ and $\partial D$. Hence the first two integrals in the above equation give a solution $W(x_0)$ to the same equation as $E^s(\cdot,\, x_0, p)$, while the last two integrals add up to $-\mathbb{G}(z, x_0)p$ by the Stratton-Chu formula and the fact that $E_e(x, z, q, k_b)$ is the fundamental solution of $\text{curl}\,\text{curl}\, E - k_b^2 E = 0$. On the other hand it is easy to see that

$$\mathcal{R}(E, A\varphi_z^\epsilon) = -\int_{\partial D} (\nu \times A\varphi_z^\epsilon) \cdot \text{curl}\, E\, ds.$$

Combining the above results we finally have that

$$\mathcal{R}(E, A\varphi_z^\epsilon) - \mathcal{R}(E, E_e(\cdot, z, q, k_b)) \tag{25}$$
$$= -\int_{\partial D} (\nu \times A\varphi_z^\epsilon) \cdot \operatorname{curl} E \, ds - W(x_0) + \mathbb{G}(z, x_0)p.$$

Now since $\|A\varphi_z^\epsilon\|_{H_{div}^{-\frac{1}{2}}(\partial D)} < C$ there exists a subfamily, still denoted by $A\varphi_z^\epsilon$, that converges weakly to a $V \in H_{div}^{-\frac{1}{2}}(\partial D)$ in the duality pairing between $H_{div}^{-\frac{1}{2}}(\partial D)$ and $H_{curl}^{-\frac{1}{2}}(\partial D)$ as $\epsilon \to 0$. Let us set

$$\tilde{W}(x_0) = \lim_{\epsilon \to 0} \mathcal{R}(E, A\varphi_z^\epsilon) = -\int_{\partial D} (\nu \times V) \cdot \operatorname{curl} E(\cdot, x_0, p) \, ds, \qquad x_0 \in \Lambda.$$

From (24) we now have that

$$\tilde{W}(x_0) = W(x_0) + \mathbb{G}(z, x_0)p \qquad x_0 \in \Lambda. \tag{26}$$

Since $\tilde{W}(x_0)$ and $W(x_0)$ can be continued as radiating solutions to

$$\operatorname{curl}_{x_0} \operatorname{curl}_{x_0} E^s(x, x_0, p) - k^2 n(x_0) E^s(x, x_0, p) = 0$$

outside the domain bounded by $\Lambda$ we deduce by uniqueness and the unique continuation principle that (26) holds true in $\mathbb{R}^3 \setminus (\overline{D} \cup \{z_0\})$. We now arrive at a contradiction by letting $x_0 \to z$. Hence $A\varphi_z^\epsilon$ is unbounded in the $H(D, \operatorname{curl})$ norm as $\epsilon \to 0$, which proves the theorem. □

The above theorem, provides a characterization of the boundary $\partial D$ of the scattering objects. In particular, $\partial D$ is the set of points where the $L_t^2(\tilde{\Lambda})$-norm of the regularized approximate solution $\varphi_z^\epsilon$ of the equation (15). becomes large. For a detailed discussion on the numerical implementation of both the classical linear sampling method and the linear sampling method based on the reciprocity gap functional we refer the reader to [1]. Numerical examples comparing the performance of the linear sampling method based on the reciprocity gap functional to the classical linear method for solving the inverse scattering problem for objects imbedded in a two layered medium are also presented in [1].

## References

1. F. Cakoni, MB. Fares and H. Haddar, The electromagnetic scattering problem for buried objects in an inhomogeneous background *Inverse Problems* (to appear).
2. F. Cakoni Recent developments in the qualitative approach to inverse scattering theory, *J. Comp. Appl. Math* (to appear).

3. F. Cakoni and D. Colton, *Qualitative Methods in Inverse Scattering Theory*, Springer Verlag, (to appear).
4. D. Colton and H. Haddar, An application of the reciproccity gap functional to inverse scattering theory, *Inverse Problems* **21**, 383-398 (2005).
5. D. Colton, H. Haddar and M. Piana, The linear sampling method in inverse electromagnetic scattering theory. *Inverse Problems* **19**, S105-S137 (2003).
6. D. Colton and R. Kress, On the denseness of Herglotz wave functions and electromagnetic Herglotz pairs in Sobolev spaces., *Math. Meth. Appl. Sci.*, **24**, 1289-1303, (2001)
7. D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory, 2nd Edn.*, Springer Verlag, (1998).
8. P. Cutzach and C. Hazard, Existence, uniqueness and analyticity properties for electromagnetic scattering in a two layered medium., *Math. Meth. Appl. Sci.*, **21**, 433-461, (1998).